Homework 4 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

4.3.9 We need to write $(1, 0, -7)$ as a linear combination of $(5, 3, 4)$ and $(3, 2, 5)$. We set up the augmented matrix and row reduce:

$$\begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix} \xrightarrow{-R_3+R_1} \begin{bmatrix} 1 & -2 & 8 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{bmatrix}$$

$$\xrightarrow[-4R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & -2 & 8 \\ 0 & 8 & -24 \\ 0 & 13 & -39 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{13}R_3]{\frac{1}{8}R_2} \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\xrightarrow[2R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

This system has the unique solution $c_2 = -3$ and $c_1 = 2$, so

$$(1, 0, -7) = 2(5, 3, 4) - 3(3, 2, 5)$$

4.3.17 We determine linear independence by row reduction:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{2R_3+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{bmatrix}$$

Since we have leading entries in all three columns, the homogeneous system has a unique solution and thus the vectors are linearly independent.

4.3.18 We determine linear independence by row reduction:

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \xrightarrow{\frac{3}{2}R_1+R_3} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have c_3 as a free variable. With foresight to prevent fractions, we set $c_3 = 5$. Back substitution gives $c_2 = 1$ and $c_1 = 3$. These vectors are linearly dependent, and we have a nontrivial linear combination equalling zero:

$$3(2, 0, -3) + (4, -5, -6) + 5(-2, 1, 3) = (0, 0, 0)$$

4.4.6 We have 3 vectors in $\mathbb{R}^3$, so it suffices to compute a determinant:

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= -1$$

This determinant is nonzero, so the vectors form a basis.

4.4.16 We do row reduction to our system:

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 8 & 7 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$
$$\xrightarrow[-R_2]{3R_2+R_1} \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 5 \end{bmatrix}$$

We have a free variable $x_3 = t$ and we solve to get $x_2 = -5t$, $x_1 = 11t$. So a typical solution looks like $\vec{x} = t(11, -5, 1)$ and thus a basis for the solution space is $\{(11, -5, 1)\}$.

4.4.20 We do row reduction to our system:

$$\begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 7 & 21 & -7 \\ 0 & 6 & 18 & -6 \end{bmatrix}$$
$$\xrightarrow[\frac{1}{6}R_3]{\frac{1}{7}R_2} \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$
$$\xrightarrow[3R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have free variables $x_3 = s$, $x_4 = t$ and we solve to get $x_2 = -3s + t$, $x_1 = s - 2t$. A typical solution looks like $\vec{x} = s(1, -3, 1, 0) + t(-2, 1, 0, 1)$ so our basis for the solution space is $\{(1, -3, 1, 0), (-2, 1, 0, 1)\}$.

4.5.5 We reduce the given matrix into echelon form (steps omitted):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis for the row space is the nonzero rows of the echelon matrix: $\{(1,1,1,1),(0,1,3,11),(0,0,0,1)\}$.

The pivot columns of the echelon matrix are 1, 2, and 4. So a basis for the column space is the corresponding columns of our original matrix: $\{(1,3,2),(1,1,5),(1,4,12)\}$.

4.5.7 We reduce the given matrix into echelon form (steps omitted):

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space is the nonzero rows of the echelon matrix: $\{(1,1,-1,7),(0,1,2,3)\}$.

The pivot columns of the echelon matrix are 1 and 2. So a basis for the column space is the corresponding columns of our original matrix: $\{(1,1,1,2),(1,4,3,5)\}$.

4.5.15 To find a subset of S that is a basis for $\operatorname{span} S$, we put the vectors in the columns of a matrix and find a basis for the column space. First, we do row reduction (steps omitted):

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns are pivot columns, so vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_4$ make up a basis for $\operatorname{span} S$.

Additional Problems:

1. Suppose that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. To show $\vec{u}_1$ and $\vec{u}_2$ are independent, we set up a homogeneous system:

$$\begin{aligned} c_1\vec{u}_1 + c_2\vec{u}_2 &= \vec{0} \\ c_1(2\vec{v}_1) + c_2(\vec{v}_1 + \vec{v}_2) &= \vec{0} \\ (2c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 &= \vec{0} \end{aligned}$$

This is a linear combination of the $\vec{v}_i$ equal to the zero vector, so since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent we have that $2c_1 + c_2 = 0$ and $c_2 = 0$. From the first equation, we can conclude $c_1 = 0$ as well so the $\vec{u}_i$ must be linearly independent as well.

2. W is the set of solutions to the homogeneous linear equation $x_1 - 5x_2 = 0$. So x_3 and x_2 are free variables and we set $x_2 = s, x_3 = t$. Solving, we get $x_1 = 5s$. So, we have

$$\vec{x} = \begin{bmatrix} 5s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The basis vectors for W are thus $(5, 1, 0)$ and $(0, 0, 1)$.

3. To show S' is linearly independent, we set up the homogeneous linear system

$$c\vec{v} + c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

If $c \neq 0$, then we can write

$$\vec{v} = -\frac{c_1}{c}\vec{v}_1 - \frac{c_2}{c}\vec{v}_2 - \frac{c_3}{c}\vec{v}_3$$

which would mean that $\vec{v}$ is in the span of S (something we assumed was false). So we must have $c = 0$. Then our system is

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Since we know S is linearly independent, we can conclude that $c_1 = c_2 = c_3 = 0$. So all constants must be 0 and thus S' is linearly independent.