HW 2

Math 2243
Linear Algebra and Differential Equations

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Contents

1 Problem 3 section 3.4

Problem

In Problems 1-4, two matrices A and B and two numbers c and d are given. Compute the matrix $cA + dB$

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix}, c = -2, d = 4$$

Solution

$$\begin{aligned} cA + dB &= -2\begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix} + 4\begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 0 \\ 0 & -14 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -16 & 20 \\ 12 & 8 \\ 28 & 16 \end{bmatrix} \\ &= \begin{bmatrix} -26 & 20 \\ 12 & -6 \\ 22 & 18 \end{bmatrix} \end{aligned}$$

2 Problem 5 section 3.4

Problem

In Problems 5-12, two matrices A and B are given. Calculate whichever of the matrices AB and BA is defined.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

A dimension is 2×2 and B dimension is 2×2. So inner dimensions agree. Both AB and BA are defined. Using definition of matrix multiplication we obtain

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 1 \\ -10 & 12 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} BA &= \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 8 \\ 11 & 5 \end{bmatrix} \end{aligned}$$

3 Problem 8 section 3.4

Problem

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$$

Solution

A dimension is 2×3 and B dimension is 3×2. Hence AB is $(2 \times 3)(3 \times 2) = 2 \times 2$ matrix. Therefore inner dimensions agree. And BA is define since $(3 \times 2)(2 \times 3) = 3 \times 3$. Therefore inner dimensions agree.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 15 \\ 35 & 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} BA &= \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{bmatrix} \end{aligned}$$

4 Problem 11 section 3.4

Problem

$$A = \begin{bmatrix} 3 & -5 \end{bmatrix}, B = \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix}$$

Solution

A dimension is 1×2 and B dimension is 2×4. Hence AB is $(1 \times 2)(2 \times 4) = 1 \times 4$ matrix. Therefore inner dimensions agree. And BA is not defined since $(2 \times 4)(1 \times 2)$. Therefore inner dimensions do not agree. So only AB is defined here.

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 1 & 5 & 3 \end{bmatrix} \end{aligned}$$

5 Problem 3 section 3.5

Problem

In Problems 1-8, first apply the formulas in (9) to find A^{-1}. Then use A^{-1} (as in Example 5) to solve the system $Ax = b$.

$$A = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solution

Formula (9) is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{36 - 35}\begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}$$

Hence

$$x = A^{-1}b = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 33 \\ -28 \end{bmatrix}$$

6 Problem 10 section 3.5

Problem

In Problems 9-22, use the method of Example 7 to find the inverse A^{-1} of each given matrix A.

$$A = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$$

Solution

The augmented matrix is

$$\begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow -4R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix}$$

Since the left side of the augments matrix is now the identity matrix, then we read A^{-1} from the right side. Hence

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix} \end{aligned}$$

7 Problem 16 section 3.5

Problem

$$A = \begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 2 \\ 2 & -3 & -3 \end{bmatrix}$$

Solution

The augmented matrix is

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_1 + R_2$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow -(2)R_1 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow 3R_2$ and $R_3 \rightarrow 2R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & -3 & 3 & 3 & 0 \\ 0 & 6 & 6 & -4 & 0 & 2 \end{bmatrix}$$

$R_3 \rightarrow R_2 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & -3 & 3 & 3 & 0 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_3$ gives

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 3 & 2 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 - \frac{1}{2}R_2$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -6 & 0 & 2 & 6 & 2 \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 \rightarrow \frac{-1}{6}R_2$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & \frac{-1}{3} \\ 0 & 0 & 3 & -1 & 3 & 2 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{3}R_3$ gives

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & \frac{-1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

Since the left side of the augments matrix is now the identity matrix, then we read A^{-1} from the right side. Hence

$$\begin{aligned} A^{-1} &= \begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{3} & -1 & \frac{-1}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -3 & 0 & 3 \\ -1 & -3 & -1 \\ -1 & 3 & 2 \end{bmatrix} \end{aligned}$$

8 Problem 4 section 3.6

Problem

Use cofactor expansions to evaluate the determinants in Problems 1-6. Expand along the row or column that minimizes the

amount of computation that is required.

$$A = \begin{bmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{bmatrix}$$

Solution

Row 4 has most zeros. Hence expansion is on row 4.

$$\begin{aligned} |A| &= (-)(-3)\begin{vmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{vmatrix} \\ &= 3\begin{vmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{vmatrix} \end{aligned}$$

For $\begin{vmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{vmatrix}$ we expand on 3rd row. The above becomes

$$\begin{aligned} |A| &= 3\left((-)4\begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix}\right) \\ &= -12\begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix} \\ &= -12(30 - 24) \end{aligned}$$

Therefore

$$|A| = -72$$

9 Problem 9 section 3.6

Problem

In Problems 7-12, evaluate each given determinant after first simplifying the computation (as in Example 6) by adding an appropriate multiple of some row or column to another.

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{bmatrix}$$

Solution

Adding multiple of some row to another row does not change the determinant of a matrix. Same for adding multiple of some column to another column. We can take advantage of this to add more zeros to the matrix before applying the cofactor method to reduce the computation needed.

Let $R_3 \to -2R_1 + R_3$ gives

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix}$$

Expansion on third row now gives

$$\begin{aligned} |A| &= (+)2 \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} \\ &= 2(15) \end{aligned}$$

Therefore

$$|A| = 30$$

10 Problem 21 section 3.6

Problem

Use Cramer's rule to solve the systems in Problems 21-32.

$$3x + 4y = 2$$
$$5x + 7y = 1$$

Solution

The system in matrix form is

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence using Cramer's rule

$$x = \frac{\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}} = \frac{14 - 4}{21 - 20} = 10$$

And

$$y = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}} = \frac{3 - 10}{21 - 20} = -7$$

Hence the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$$

11 Additional problem 1

Problem

Give an example of matrices A and B where $AB = BA$

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then

$$\begin{aligned} AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \end{aligned} \tag{1}$$

And

$$\begin{aligned} BA &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{bmatrix} \end{aligned} \tag{2}$$

For (1,2) to be equal implies that

$$\begin{aligned} ae+bg &= ae+cf \\ af+bh &= be+df \\ ce+dg &= ag+ch \\ cf+dh &= bg+dh \end{aligned}$$

Simplifying gives

$$\begin{aligned} bg &= cf \\ af+bh &= be+df \\ ce+dg &= ag+ch \\ cf &= bg \end{aligned}$$

First equation is the same as the fourth. Hence the above becomes

$$\begin{aligned} bg &= cf \\ af+bh &= be+df \\ ce+dg &= ag+ch \end{aligned}$$

Let $a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$. The above becomes

$$\begin{aligned} 2g &= 18 \\ 6+2h &= 10+24 \\ 15+4g &= g+3h \end{aligned}$$

or

$$\begin{aligned} g &= 9 \\ h &= 14 \end{aligned}$$

Hence and example is

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ B &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix} \end{aligned}$$

To verify

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 51 & 74 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 51 & 74 \end{bmatrix}$$

12 Additional problem 2

Problem

Give an example of matrices C and D where $CD \neq DC$.

Solution

From the last problem, we found a solution that makes $CD = DC$ to be

$$g = 9$$
$$h = 14$$

So any other value will make $CD \neq DC$. Hence an example is

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$D = \begin{bmatrix} e & f+1 \\ g & h \end{bmatrix} = \begin{bmatrix} 5 & 6+1 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix}$$

To verify

$$CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 23 & 35 \\ 51 & 77 \end{bmatrix}$$

But

$$DC = \begin{bmatrix} 5 & 7 \\ 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 26 & 38 \\ 51 & 74 \end{bmatrix}$$

Hence $CD \neq DC$

13 Additional problem 3

Problem

Let $A;B$, and C be invertible $n\times n$ matrices. Is the product ABC invertible? If it is invertible, what is $(ABC)^{-1}$?

Solution

Let $ABC = D$. Premultiplying both sides by A^{-1} gives

$$\begin{aligned} A^{-1}ABC &= A^{-1}D \\ BC &= A^{-1}D \end{aligned}$$

Premultiplying both sides by B^{-1} gives

$$\begin{aligned} B^{-1}BC &= B^{-1}A^{-1}D \\ B &= B^{-1}A^{-1}D \end{aligned}$$

Premultiplying both sides by C^{-1} gives

$$I = \left(C^{-1}B^{-1}A^{-1}\right)D \tag{1}$$

Starting with $ABC = D$ again, but now post multiplying both sides by C^{-1} gives

$$\begin{aligned} ABCC^{-1} &= DC^{-1} \\ AB &= DC^{-1} \end{aligned}$$

Post multiplying both sides by B^{-1} gives

$$\begin{aligned} ABB^{-1} &= DC^{-1}B^{-1} \\ A &= DC^{-1}B^{-1} \end{aligned}$$

Post multiplying both sides by A^{-1} gives

$$I = D\left(C^{-1}B^{-1}A^{-1}\right) \tag{2}$$

Comparing (1,2) we see that

$$\left(C^{-1}B^{-1}A^{-1}\right)D = D\left(C^{-1}B^{-1}A^{-1}\right) = I \tag{3}$$

This means $C^{-1}B^{-1}A^{-1}$ is the inverse of D by definition (page 177 of book) which says if $AB = BA = I$ then B is the inverse of A.

But D is the product of ABC. Hence the product is invertible. And from (3), its inverse is given by

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

14 Additional problem 4

Problem

Let $T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ be diagonal matrix. What is $\det(T)$?

Solution

The determinant of a diagonal matrix is the product of the elements on the diagonal. Hence

$$\det(T) = t_1 t_2 t_3$$

This comes from expansion over any row or column. For example, expansion along row 1 gives

$$\begin{aligned} \det(T) &= t_1 \begin{vmatrix} t_2 & 0 \\ 0 & t_3 \end{vmatrix} \\ &= t_1 t_2 \det(\begin{bmatrix} t_3 \end{bmatrix} \\ &= t_1 t_2 t_3 \end{aligned}$$

Note that the sign of the elements are all positive for 3×3 since n is odd here.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

15 Additional problem. Optional

Problem

Optional: Consider an $n \times n$ diagonal matrix T. What is $det(T)$? The required part of this problem asks you to answer this question for the case where $n = 3$.

Solution

$$T = \begin{bmatrix} t_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & t_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & t_3 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & t_n \end{bmatrix}$$

det(T) is the product of all elements on the diagonal. This comes from expansion over any row. For example, expansion on row 1 gives

$$\begin{aligned} \det(T) &= t_1 \begin{vmatrix} t_2 & 0 & \cdots & \cdots & 0 \\ 0 & t_3 & \cdots & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & t_n \end{vmatrix} \\ &= t_1 t_2 \begin{vmatrix} t_3 & \cdots & \cdots & 0 \\ \cdots & t_4 & \cdots & 0 \\ \cdots & \cdots & \ddots & 0 \\ \cdots & \cdots & \cdots & t_n \end{vmatrix} \\ &= t_1 t_2 t_3 \begin{vmatrix} t_4 & \cdots & 0 \\ \cdots & \ddots & 0 \\ \cdots & \cdots & t_n \end{vmatrix} \end{aligned}$$

And so on until the last entry

$$\begin{aligned} \det(T) &= t_1 t_2 t_3 \cdots t_n \\ &= \prod_{i=1}^{n} t_n \end{aligned}$$

Note on the sign. In expansion, we have to take account of sign changes. If n is odd, then the sign of the elements are all positive on the diagonal as in case $n = 3$ above. So we do not need to worry about this case.

For even n, the sign on diagoanl also remains positive, since the formula is $(-1)^{i+j}$ where i, j are the index of the diagonal elements, and this always adds to even number since $i = j$ on the diagonal. For an example for $n = 4$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

We see that product on the diagonal always has positive signs.