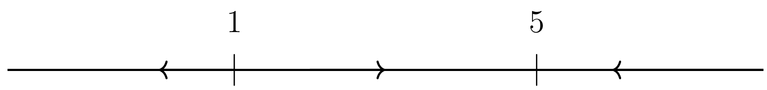
These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Additional Problems:

1. We need our function $f(x)$ to have zeros at $x = 1, 5$ and we need the phase diagram to be



We take as a first guess $f(x) = (x-1)(x-5)$. We find then that $f(0) = (-1)(-5) = 5 > 0$, which would be the wrong sign. So we take as a next guess $f_1(x) = -(x-1)(x-5)$. Then we have $f_1(0) = -5$, $f_1(3) = 4$, and $f_1(6) = -5$ which is exactly what is needed. So our differential equation is

$$\frac{dx}{dt} = -(x-1)(x-5)$$

2. We need our function to have zeros at $x = 1, 5, 7$ and we need the phase diagram to be

The problem here is that we have two conflicting signs needed in the interval between 5 and 7. To rectify this, we will introduce another critical point at 6 where the sign of $f(x)$ will change. This means we are now looking to get the phase diagram

We take as a guess $f(x) = (x-1)(x-5)(x-6)(x-7)$. We see that $f(0) = (-1)(-5)(-6)(-7) > 0$, so the sign is wrong again. Our next guess is $f_1(x) = -(x-1)(x-5)(x-6)(x-7)$. You can check that now all of the signs are correct, so our differential equation is

$$\frac{dx}{dt} = -(x-1)(x-5)(x-6)(x-7)$$

If we add the additional requirement that $f(x)$ is continuous on all of $\mathbb{R}$, you can show that the only way to have both $x = 5$ and $x = 7$ as stable equilibria is if there is another equilibrium solution between 5 and 7. If we don't require that $f(x)$ is continuous, we can use a function like

$$f(x) = -\frac{(x-1)(x-5)(x-7)}{x-6}$$

which has only the three required equilibria and no others. This discontinuity at $x = 6$ makes this differential equation quite unpleasant to work with, however.

3. (a) We are solving $\frac{dv}{dt} = a - \rho v$ where a and ρ are some unknown constants. This is a separable equation, and we can write it as

$$\frac{1}{a - \rho v} dv = dt$$

Integrating both sides and solving for v, we get

$$\begin{aligned} -\frac{1}{\rho} \ln(a - \rho v) &= t + C_0 \\ \ln(a - \rho v) &= -\rho t + C_1 \\ a - \rho v &= C_2 e^{-\rho t} \\ -\rho v &= C_2 e^{-\rho t} - a \\ v &= C_3 e^{-\rho t} + \frac{a}{\rho} \end{aligned}$$

With the initial condition $v(0) = v_0$, we have

$$v_0 = C_3 + \frac{a}{\rho}$$

So our particular solution is

$$v(t) = \left(v_0 - \frac{a}{\rho}\right) e^{-\rho t} + \frac{a}{\rho}$$

Since ρ is a positive constant, the top speed is

$$\lim_{t \to \infty} v(t) = \frac{a}{\rho}$$

(b) If the top speed is 85 m/s and our acceleration is 14 m/s^2, then we can solve for ρ.

$$\begin{aligned} 85 &= \frac{14}{\rho} \\ \rho &= \frac{14}{85} \approx 0.165 \end{aligned}$$

In the case where DRS is active, we have a top speed of 90 m/s, so

$$\begin{aligned} 90 &= \frac{14}{\rho_{DRS}} \\ \rho_{DRS} &= \frac{14}{90} \approx 0.156 \end{aligned}$$

(c) Car A begins 10 meters behind car B so $x_A(0) = -10$ and $x_B(0) = 0$. Both cars have initial velocity 25 m/s, so using our solution to (a) we can write down the two velocity functions in terms of the drag coefficients.

$$\begin{aligned} v_A(t) &= \left(25 - \frac{14}{\rho_{DRS}}\right) e^{-\rho_{DRS} t} + \frac{14}{\rho_{DRS}} \\ v_B(t) &= \left(25 - \frac{14}{\rho}\right) e^{-\rho t} + \frac{14}{\rho} \end{aligned}$$

Integrating with respect to t, we get the position functions

$$\begin{aligned} x_A(t) &= \left(-\frac{25}{\rho_{DRS}} + \frac{14}{\rho_{DRS}^2}\right) e^{-\rho_{DRS} t} + \frac{14}{\rho_{DRS}} t + C_A \\ x_B(t) &= \left(-\frac{25}{\rho} + \frac{14}{\rho^2}\right) e^{-\rho t} + \frac{14}{\rho} t + C_B \end{aligned}$$

Our initial conditions now let us solve for the constants.

$$\begin{aligned} -10 &= \left(-\frac{25}{\rho_{DRS}} + \frac{14}{\rho_{DRS}^2}\right) + C_A \\ C_A &= \frac{25}{\rho_{DRS}} - \frac{14}{\rho_{DRS}^2} - 10 \\ 0 &= \left(-\frac{25}{\rho} + \frac{14}{\rho^2}\right) + C_B \\ C_B &= \frac{25}{\rho} - \frac{14}{\rho^2} \end{aligned}$$

We now want to know when $x_A(t) = x_B(t)$. So we set these equal to each other, substitute the values we found for ρ and ρ_{DRS}, and solve for t.

$$\begin{aligned} 417.857 e^{\frac{-14t}{90}} + 90t - 427.857 &= 364.286 e^{\frac{-14t}{85}} + 85t - 364.286 \\ 417.857 e^{\frac{-14t}{90}} - 364.286 e^{\frac{-14t}{85}} + 5t &= 63.571 \\ t &\approx 8.306 \end{aligned}$$

So it will take about 8.3 seconds for the two cars to be side by side. At time $t = 8.306$, we have $x_A(t) = x_B(t) = 434.472$ meters. So by the time the cars have driven 500 meters, car A should be ahead.

(d) Setting $a = 0$ m/s^2 and $v_0 = 80$ m/s in our solution found in (a), we have

$$v(t) = 80 e^{-\rho t}$$

Given that $\rho = \frac{14}{85}$, we want to find when $v(t) = 10$.

$$\begin{aligned} 10 &= 80e^{\frac{-14t}{85}} \\ \frac{-14t}{85} &= \ln\left(\frac{10}{80}\right) \\ t &= -\frac{85}{14}\ln\left(\frac{10}{80}\right) \approx 12.625 \end{aligned}$$

The position function is obtained by integrating, so that

$$x(t) = -\frac{80}{\rho}e^{-\rho t} + C$$

Taking the place where we begin coasting as position 0, we have $x(0) = 0$ and $C = \frac{80}{\rho}$. Now we evaluate the position function at the calculated time to get

$$\begin{aligned} x\left(-\frac{85}{14}\ln\left(\frac{10}{80}\right)\right) &= -\frac{80}{14/85}e^{\frac{14}{85}\cdot\frac{85}{14}\ln\left(\frac{10}{80}\right)} + \frac{80}{14/85} \\ &= -\frac{80 \cdot 85}{14} \cdot \frac{10}{80} + \frac{80 \cdot 85}{14} = 425 \end{aligned}$$

So the driver will need to coast for 425 meters to reach the safe speed of 10 m/s.

An interesting thing to consider about this exponential decay model is that the car is not predicted to come to a complete stop after any finite amount of time, i.e. $v(t)$ is strictly positive for all $t \geq 0$. This doesn't match our real-world experience, indicating that forces like friction between the asphalt and the tires are playing a role that this model isn't accounting for.

One consequence of this model that does match our real-world experience can be seen if you run the numbers again for a target velocity of 5 m/s. In this case, it takes 16.8 seconds to slow down and we travel 455 meters. It takes more than 30% longer to decrease our speed by 75 m/s than it does to decrease our speed by 70 m/s. Despite the significant increase in time, we only increase distance traveled by 7%. This is significantly different from the behavior of the constant deceleration model we explored in Additional Problem 2 of Homework 11. But if you have played golf or billiards, you will have experienced how long it can take for the ball to slowly roll those last few inches before coming to a stop.