Homework 12 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

1.4.4 We separate variables and then integrate.

$$\begin{aligned}(1+x)\frac{dy}{dx} &= 4y \\ \frac{1}{y}\,dy &= \frac{4}{1+x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{4}{1+x}\,dx \\ \ln y &= 4\ln(1+x) + C\end{aligned}$$

Now we exponentiate both sides to solve for y.

$$\begin{aligned}y &= e^{4\ln(1+x)+C} \\ y &= C_1(1+x)^4\end{aligned}$$

1.4.17 We can factor to write the differential equation as $y' = (1+x)(1+y)$. Now we separate variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= (1+x)(1+y) \\ \frac{1}{1+y}\,dy &= (1+x)\,dx \\ \int \frac{1}{1+y}\,dy &= \int (1+x)\,dx \\ \ln(1+y) &= x + \frac{x^2}{2} + C\end{aligned}$$

We exponentiate to solve for y.

$$\begin{aligned}1+y &= e^{x+\frac{1}{2}x^2+C} \\ y &= C_1 e^{x+\frac{1}{2}x^2} - 1\end{aligned}$$

In this case, solving for y gives us a bit of a mess, so it would be acceptable to leave it in the implicit form found above.

1.4.19 We separate variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= ye^x \\ \frac{1}{y}\,dy &= e^x\,dx \\ \int \frac{1}{y}\,dy &= \int e^x\,dx \\ \ln y &= e^x + C\end{aligned}$$

We have the initial condition $y(0) = 2e$, so we find the constant C.

$$\begin{aligned}\ln(2e) &= e^0 + C \\ \ln 2 + \ln e &= 1 + C \\ \ln 2 &= C\end{aligned}$$

Now we exponentiate to solve for y.

$$\begin{aligned}y &= e^{e^x + \ln 2} \\ y &= 2e^{e^x}\end{aligned}$$

1.4.33 Our population is modeled by $\frac{dP}{dt} = kP$, so that $P(t) = Ce^{kt}$. Let t be the years since 1960 and $P(t)$ the population in thousands. Then our initial conditions are $P(0) = 25$ and $P(10) = 30$. This lets us solve for the constants C and k:

$$\begin{aligned}25 &= Ce^{0k} \\ 25 &= C\end{aligned}$$

$$\begin{aligned}30 &= 25e^{10k} \\ \frac{6}{5} &= e^{10k} \\ 10k &= \ln(6/5) \\ k &= \frac{\ln(6/5)}{10} \approx 0.0182\end{aligned}$$

So $P(t) = 25e^{0.0182t}$ and in 2000 we predict $P(40) = 25e^{k \cdot 40} = 25(6/5)^4 \approx 51.8$ thousand residents.

1.4.43 The temperature is modeled by $\frac{dT}{dt} = k(0 - T) = -kT$ so that $T(t) = Ce^{-kt}$. Our initial conditions are $T(0) = 25$ and $T(20) = 15$. We can solve for the constants now.

$$25 = Ce^0 = C$$

$$\begin{aligned} 15 &= 25e^{-20k} \\ \frac{3}{5} &= e^{-20k} \\ -20k &= \ln(3/5) \\ k &= -\frac{\ln(3/5)}{20} \approx 0.0255 \end{aligned}$$

We want to know when $T(t) = 5$.

$$\begin{aligned} 5 &= 25e^{-kt} \\ \frac{1}{5} &= e^{-kt} \\ -kt &= \ln(1/5) \\ t &= -\frac{\ln(1/5)}{k} = 20\frac{\ln(1/5)}{\ln(3/5)} \approx 63.01 \end{aligned}$$

So it will take about 63 minutes for the buttermilk to cool to $5°$.

1.5.3 We have $y'+3y = 2xe^{-3x}$, so that $P(x) = 3$. Our integrating factor is $\exp(\int 3\ dx) = e^{3x}$. After multiplying by e^{3x}, we have

$$\begin{aligned} \frac{d}{dx}\left[ye^{3x}\right] &= 2x \\ ye^{3x} &= x^2 + C \\ y &= x^2e^{-3x} + Ce^{-3x} \end{aligned}$$

1.5.17 We have $(1+x)y' + y = \cos x$, which after dividing by $1+x$ is

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}$$

So $P(x) = \frac{1}{1+x}$ and our integrating factor is $\exp(\int \frac{1}{1+x}\ dx) = \exp(\ln(1+x)) = 1+x$. After multiplication by $1+x$, we have

$$\begin{aligned} \frac{d}{dx}\left[y(1+x)\right] &= \cos x \\ y(1+x) &= \sin x + C \\ y &= \frac{\sin x + C}{1+x} \end{aligned}$$

We are given the initial condition that $y(0) = 1$, so we have

$$\begin{aligned} 1 &= \frac{\sin 0 + C}{1} \\ 1 &= C \end{aligned}$$

So our solution is $y = \frac{\sin x + 1}{1+x}$

1.5.37 After t seconds, the volume of liquid in the tank is $V(t) = 100 + 5t - 3t = 100 + 2t$. The differential equation that describes the amount $x(t)$ of salt in the tank at time t is

$$x' = 5 \cdot 1 - 3 \cdot \frac{x}{100 + 2t}$$

We rewrite this slightly to allow us to find the integrating factor.

$$x' + \frac{3}{100 + 2t} x = 5$$

So $P(t) = \frac{3}{100+2t}$ and our integrating factor is $\exp(\int \frac{3}{100+2t} dt) = \exp(\frac{3}{2} \ln(100 + 2t)) = (100 + 2t)^{3/2}$. After multiplying by the integrating factor, we have

$$\begin{aligned} \frac{d}{dt} \left[x(100 + 2t)^{3/2} \right] &= 5(100 + 2t)^{3/2} \\ x(100 + 2t)^{3/2} &= (100 + 2t)^{5/2} + C \\ x &= (100 + 2t) + \frac{C}{(100 + 2t)^{3/2}} \end{aligned}$$

Initially, we have $x(0) = 50$ pounds of salt. So we can solve for C.

$$\begin{aligned} 50 &= 100 + \frac{C}{100^{3/2}} \\ -50 \cdot 1000 &= C \end{aligned}$$

So, we have

$$x(t) = 100 + 2t - \frac{50,000}{(100 + 2t)^{3/2}}$$

We want the amount of salt when the tank is full. This happens when $V(t) = 400$, so when $t = 150$. At that time, we have

$$x(150) = 400 - \frac{50,000}{400^{3/2}} = 393.75 \text{ pounds of salt}$$

2.1.15 We are given that $\frac{dP}{dt} = aP - bP^2 = bP(\frac{a}{b} - P)$. We are given that the birth rate is aP, which at $t = 0$ is B_0 and the death rate is bP^2, which at $t = 0$ is D_0. Since the initial population is $P(0) = P_0$, this tells us that $aP_0 = B_0$ and $bP_0^2 = D_0$. So we have

$$\begin{aligned} \frac{a}{b} &= \frac{B_0/P_0}{D_0/P_0^2} \\ &= \frac{B_0 P_0}{D_0} \end{aligned}$$

So we have written our differential equation in the form $\frac{dP}{dt} = kP(M - P)$ where $k = b = \frac{D_0}{P_0^2}$ and $M = \frac{a}{b} = \frac{B_0 P_0}{D_0}$. Thus our limiting population is indeed $\frac{B_0 P_0}{D_0}$.

2.1.16 In this case we have $P_0 = 120$, $B_0 = 8$, and $D_0 = 6$. So our differential equation can be written as

$$\frac{dP}{dt} = \frac{6}{120^2}P\left(\frac{8\cdot 120}{6} - P\right) = \frac{1}{2400}P\,(160 - P)$$

Knowing k, M, and P_0, we can use the formula for the general solution of a logistic equation to get

$$\begin{aligned} P(t) &= \frac{160\cdot 120}{120 + (160-120)e^{-\frac{1}{2400}\cdot 160t}} \\ &= \frac{480}{3 + e^{-t/15}} \end{aligned}$$

We wish to know when $P(t) = 0.95\cdot M = 152$. We solve:

$$\begin{aligned} 152 &= \frac{480}{3 + e^{-t/15}} \\ 456 + 152e^{-t/15} &= 480 \\ 152e^{-t/15} &= 24 \\ \frac{-t}{15} &= \ln(3/19) \\ t &= -15\ln(3/19) \approx 27.7 \end{aligned}$$

So it will take nearly 28 months for the population to reach 95% of the limiting population.

2.1.17 In this case, we have $P_0 = 240$, $B_0 = 9$, and $D_0 = 12$. So our differential equation can be written as

$$\frac{dP}{dt} = \frac{12}{240^2}P\left(\frac{9\cdot 240}{12} - P\right) = \frac{1}{4800}P\,(180 - P)$$

Knowing k, M, and P_0, we can use the formula for the general solution of a logistic equation to get

$$\begin{aligned} P(t) &= \frac{180\cdot 240}{240 + (180-240)e^{-\frac{1}{4800}\cdot 180t}} \\ &= \frac{720}{4 - e^{-3t/80}} \end{aligned}$$

We wish to know when $P(t) = 1.05\cdot M = 189$. We solve:

$$\begin{aligned} 189 &= \frac{720}{4 - e^{-3t/80}} \\ 756 - 189e^{-3t/80} &= 720 \\ 189e^{-3t/80} &= 36 \\ \frac{-3t}{80} &= \ln(4/21) \\ t &= \frac{-80\ln(4/21)}{3} \approx 44.2 \end{aligned}$$

So it will take just more than 44 months for the population to fall to 105% of the limiting population.

Additional Problems:

1. (a) $y' + y = 0$ has characteristic equation $r + 1 = 0$ with root $r = -1$. So we have general solution $y = c_1 e^{-x}$.

   (b) Our particular solution has form $y_p = Ae^x$ since there is no repetition. We substitute into the equation to find the value of A:

   $$y_p' + y_p = Ae^x + Ae^x = e^x$$

   So $A = \frac{1}{2}$ and we have particular solution $y_p = \frac{1}{2}e^x$.

   (c) The general solution is $y = y_p + y_c = \frac{1}{2}e^x + c_1 e^{-x}$.

   (d) For $y' + y = e^x$ we have $P(x) = 1$ and $Q(x) = e^x$.

   (e) Our integrating factor is

   $$\begin{aligned} e^{\int P(x)\,dx} &= e^{\int 1\,dx} \\ &= e^x \end{aligned}$$

   So we have

   $$\begin{aligned} y'e^x + ye^x &= e^{2x} \\ \frac{d}{dx}\left[y \cdot e^x\right] &= e^{2x} \\ y \cdot e^x &= \int e^{2x}\,dx \\ y \cdot e^x &= \frac{1}{2}e^{2x} + C \\ y &= \frac{1}{2}e^x + Ce^{-x} \end{aligned}$$

   (f) The solutions we wrote in (e) and (c) are identical, except for the name of the constants.

2. (a) The initial value problem is

   $$\frac{dP}{dt} = kP(100 - P) \qquad\qquad P(0) = 5$$

   where $P(t)$ is the number of people with Green's disease (in thousands) on day t.

(b) This differential equation is separable:

$$\frac{1}{P(100-P)}dP = k\ dt$$

After calculating a partial fraction decomposition, we can integrate both sides:

$$\begin{aligned}
\frac{1}{100}\int\left(\frac{1}{P}+\frac{1}{100-P}\right)dP &= \int k\ dt \\
\frac{1}{100}\left(\ln P - \ln(100-P)\right) &= kt + C \\
\ln\left(\frac{P}{100-P}\right) &= 100kt + C_0 \\
\frac{P}{100-P} &= C_1 e^{100kt}
\end{aligned}$$

At this point, it seems prudent to solve for the constant C_1. With $P(0) = 5$, we compute

$$\begin{aligned}
\frac{5}{100-5} &= C_1 e^0 \\
C_1 &= \frac{5}{95}
\end{aligned}$$

We will leave the symbol C_1 in our calculation for the time being as we solve for P:

$$\begin{aligned}
\frac{P}{100-P} &= C_1 e^{100kt} \\
P &= 100C_1 e^{100kt} - PC_1 e^{100kt} \\
P(1 + C_1 e^{100kt}) &= 100C_1 e^{100kt} \\
P &= \frac{100C_1 e^{100kt}}{1 + C_1 e^{100kt}} \\
P &= \frac{100 \cdot \frac{5}{95} e^{100kt}}{1 + \frac{5}{95} e^{100kt}}
\end{aligned}$$

Multiplying the fraction by $95e^{-100kt}$ in both numerator and denominator, we get a cleaner expression

$$P(t) = \frac{500}{95e^{-100kt} + 5}$$

(c) We are given that $P'(0) = 0.5$ and that $P(0) = 5$. We can put these values into the differential equation to get

$$\begin{aligned}
0.5 &= k(5)(100-5) \\
k &= \frac{0.5}{5 \cdot 95} \approx 0.00105
\end{aligned}$$

(d) We need to know when $P(t) = 50$. So, we solve for t in

$$\begin{aligned}
\frac{500}{95e^{-100kt}+5} &= 50 \\
500 &= 50 \cdot 95e^{-100kt} + 5 \cdot 50 \\
\frac{500 - 5 \cdot 50}{50 \cdot 95} &= e^{-100kt} \\
\frac{1}{19} &= e^{-100kt} \\
-100kt &= \ln(\tfrac{1}{19}) \approx -2.944 \\
t &\approx \frac{-2.944}{-100 \cdot 0.00105} = 28.04
\end{aligned}$$

So it will take about 28 days for half the population to be infected.

(e) One way to view this kind of model is that it tells us what will happen if the system is left to run without intervention. This model will only be accurate if human behavior, the biology of the disease, and our treatment capability all stay the same. Factors like mask wearing, social distancing, curfews, hand washing, business closures, and holiday celebrations can all impact the level of transmission between individuals. There may be changes to the transmissibility of the disease itself, caused by mutations or by the weather. Medical intervention may allow us to make infected individuals no longer contagious or make some people immune through vaccines. There may also be further complicating factors such as travel to and from other cities.

Another relevant saying here is "garbage in, garbage out." This means that our model is only as good as the data we feed into it. If the count of total infections or daily infections is wrong due to inaccurate tests, insufficient testing, or incomplete reporting, our model has no hope of predicting the true numbers.

With all of that said, what utility can we still get from this model? Well, it does tell us about one possible scenario for how disease transmission could evolve. If we adjust our assumptions slightly, we can get other possible scenarios. In reality, most modeling of this kind gives a *range* of possible outcomes, rather than a single prediction. This model is one such possible outcome and is probably most useful when viewed in the context of other possible outcomes.

The predictions of this model also give us a benchmark to compare future data to. If we introduce public health interventions like mask mandates, stay-at-home orders, or messaging about hand washing, we can assess their effectiveness by comparing future data to our predictions. If there are fewer infections than our model predicted, that indicates that the public health interventions may be helping. If infection rates rise above our predictions, we will need to explore possible causes such as disease mutations, weather changes, or "superspreader" events. Having this model helps us understand whether the new data we get each day is as expected, a cause for concern, or a cause for celebration.