HW 11

Math 2243
Linear Algebra and Differential Equations

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Nasser M. Abbasi

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Contents

1 Problem 6, section 1.2

Solve

$$\begin{aligned}\frac{dy}{dx} &= x\sqrt{x^2+9}\\ y(-4) &= 0\end{aligned}$$

Solution

This is separable ODE. Integrating both sides gives

$$y(x) = \int x\sqrt{x^2+9}\,dx + c \tag{1}$$

Where c is constant of integration. To integrate $\int x\sqrt{x^2+9}\,dx$, let $u = x^2+9$. Hence $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$. Therefore the integral becomes

$$\begin{aligned}\int x\sqrt{x^2+9}\,dx &= \int x\sqrt{u}\,\frac{du}{2x}\\ &= \frac{1}{2}\int u^{\frac{1}{2}}du\\ &= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\\ &= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\\ &= \frac{1}{3}u^{\frac{3}{2}}\end{aligned}$$

But $u = x^2+9$, hence the above becomes

$$\int x\sqrt{x^2+9}\,dx = \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}}$$

Substituting the above in (1) gives

$$y(x) = \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}} + c \tag{2}$$

The constant c is from initial conditions. Since $y(-4) = 0$ then Eq (2) becomes

$$\begin{aligned}0 &= \frac{1}{3}(16+9)^{\frac{3}{2}} + c\\ &= \frac{1}{3}(25)^{\frac{3}{2}} + c\\ &= \frac{1}{3}\left(5^2\right)^{\frac{3}{2}} + c\\ &= \frac{1}{3}(5)^3 + c\\ &= \frac{125}{3} + c\end{aligned}$$

Hence $c = -\frac{125}{3}$. Therefore the solution (2) becomes

$$\begin{aligned} y(x) &= \frac{1}{3}\left(x^2+9\right)^{\frac{3}{2}} - \frac{125}{3} \\ &= \frac{1}{3}\left(\left(x^2+9\right)^{\frac{3}{2}} - 125\right) \end{aligned}$$

2 Problem 8, section 1.2

Solve

$$\begin{aligned}\frac{dy}{dx} &= \cos 2x\\ y(0) &= 1\end{aligned}$$

<u>Solution</u>

This is separable ODE. Integrating both sides gives

$$\begin{aligned}y(x) &= \int \cos 2x dx + c\\ &= \frac{1}{2}\sin(2x) + c \qquad (1)\end{aligned}$$

The constant c is from initial conditions. Since $y(0) = 1$ then (1) becomes

$$\begin{aligned}1 &= \frac{\sin(0)}{2} + c\\ &= c\end{aligned}$$

Hence the solution (1) becomes

$$y(x) = \frac{1}{2}\sin(2x) + 1$$

3 Problem 24, section 1.2

A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

Solution

Let the ground by level 0 (i.e. $y = 0$) and let up be positive and down negative. Therefore $y(0) = 400$ ft and assuming initial velocity is zero then $y'(0) = v(0) = 0$.. Therefore

$$v(t) = \int a(t)dt$$

Where $a(t)$ is the acceleration, which in this case is $g = -32$ ft/sec^2. The above becomes

$$\begin{aligned} v(t) &= -32t + v(0) \\ &= -32t \end{aligned} \tag{1}$$

And

$$\begin{aligned} y(t) &= \int v(t)dt \\ &= \int -32t\ dt \\ &= -\frac{32}{2}t^2 + y(0) \end{aligned}$$

But $y(0) = 400\,\text{ft}$. The above becomes

$$y(t) = -16t^2 + 400$$

To find the time it takes to hit the ground, the above is solved for $y(t) = 0$. This gives

$$\begin{aligned} 0 &= -16t^2 + 400 \\ t^2 &= \frac{400}{16} \\ &= 25 \end{aligned}$$

Therefore the time is $t = 5$ seconds. Now we know how long it takes to reach the ground, we can find the velocity when ball strike the ground from (1). Substituting $t = 5$ in (1) gives

$$\begin{aligned} v(5) &= -32(5) \\ &= -160 \text{ ft/sec} \end{aligned}$$

So it strikes the ground with speed 160 ft/sec in the downwards (negative) direction.

4 Problem 26, section 1.2

A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground (b) when it passes the top of the building (c) its total time in the air.

Solution

4.1 Part a

Let the ground be level 0. (i.e. $y = 0$) and let up be positive and down negative. Therefore $y(0) = 20$ m. Initial velocity is 100 m/s, hence $y'(0) = v(0) = 100$. The acceleration due to gravity is $g = -9.8$ m/s^2.

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= -gt + v(0) \\ &= -gt + 100 \end{aligned}$$

When the ball reaches maximum high above the building, it must have zero velocity. From the above this means

$$\begin{aligned} 0 &= -9.8t + 100 \\ t &= \frac{100}{g} \text{ sec} \end{aligned}$$

The above is how long it takes for the ball to reach maximum high. Now

$$\begin{aligned} y(t) &= \int v(t)dt \\ &= \int \left(-gt + 100\right)dt + y(0) \\ y(t) &= -\frac{1}{2}gt^2 + 100t + y(0) \end{aligned}$$

But $y(0) = 20$. Therefore

$$y(t) = -\frac{1}{2}gt^2 + 100t + 20$$

Substituting $t = \frac{100}{g}$ in the above, gives the distance traveled above the ground until the ball reached maximum high. Therefore

$$\begin{aligned} y\left(\frac{100}{g}\right) &= -\frac{1}{2}g\left(\frac{100}{g}\right)^2 + 100\left(\frac{100}{g}\right) + 20 \\ &= -\frac{1}{2}\frac{100^2}{g} + \frac{100^2}{g} + 20 \\ &= \frac{1}{2}\frac{100^2}{g} + 20 \end{aligned}$$

Using $g = 9.8$ the above gives

$$\begin{aligned} y\left(\frac{100}{g}\right) &= \frac{1}{2}\frac{100^2}{9.8} + 20 \\ y_{\max} &= 530.2 \text{ meter} \end{aligned}$$

4.2 Part b

The ball will take the same amount of time to fall down back to top of building, as the time it took to reach the maximum high above the building, since the distance is the same, and the acceleration is the same (gravity acceleration). This time is $t_0 = \frac{100}{g}$ sec found in part (a). Therefore, twice this time gives

$$\begin{aligned} t_{\text{travel}} &= \frac{200}{g} \\ &= \frac{200}{9.8} \\ &= 20.408 \text{ sec} \end{aligned}$$

4.3 Part c

Now we find the time it take to reach the ground. We now take initial velocity as $v(0) = 0$, which is when the ball was at its maximum high above the building. And initial position is from part (a) was found $y_{\max} = 530.2$ meter. Hence $y(0) = 530.2$ m. Now we will find the time to reach the ground, starting from the maximum high.

$$\begin{aligned} v(t) &= \int g dt \\ &= gt + v(0) \\ &= gt \end{aligned}$$

And

$$\begin{aligned} y(t) &= \int v(t) dt \\ &= \int gt\ dt + y(0) \\ &= \frac{1}{2} g t^2 + 530.2 \end{aligned}$$

When it hits the ground $y(t) = 0$,. Hence we now have an equation to solve for time

$$0 = \frac{1}{2} g t^2 + 530.2$$

But $g = -9.8$. The above becomes

$$\begin{aligned} 0 &= \frac{1}{2}(-9.8) t^2 + 530.2 \\ t^2 &= \frac{2(530.2)}{9.8} \end{aligned}$$

Hence $t = \sqrt{\frac{2(530.2)}{9.8}} = 10.402$ sec. This is the time it takes to fall to the ground, starting from maximum high. Adding to this time, the time it took to reach maximum high from top of building, which is $\frac{100}{g}$ sec as found from part (a), gives total time in air

$$\begin{aligned} t_{\text{total}} &= 10.402 + \frac{100}{9.8} \\ &= 20.606 \text{ sec} \end{aligned}$$

5 Problem 5, section 1.3

Solution

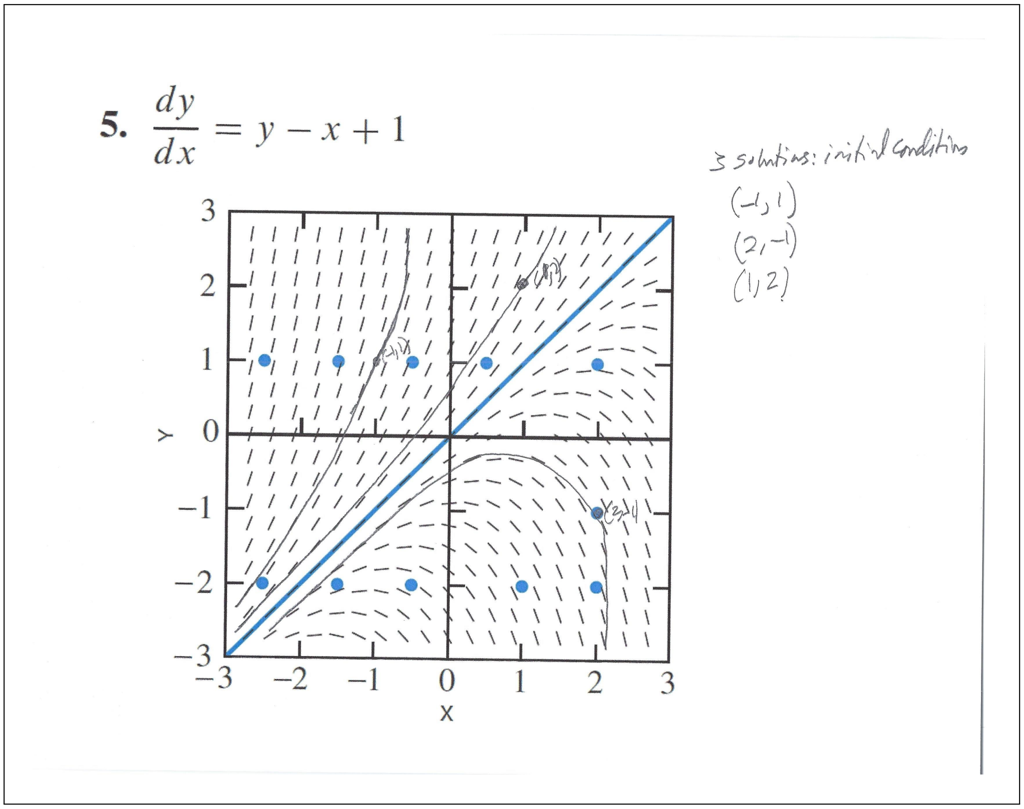

Figure 1: Shoiwng 3 solution curves with different initial conditions

6 Problem 9, section 1.3

Solution

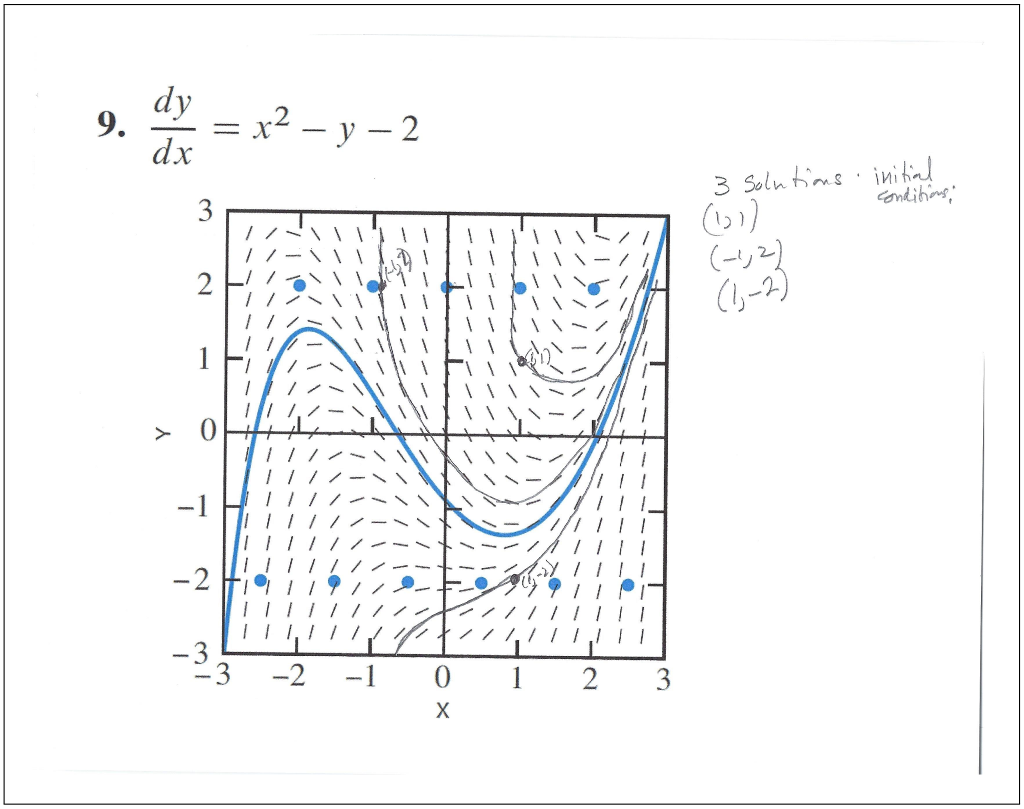

Figure 2: Shoiwng 3 solution curves with different initial conditions

7 Additional problem 1

A racecar accelerates from stationary at a rate of 14 m/s^2. How long does it take the car to reach its top speed of 300 km/h? How far does the car travel in that time?

Solution

Let $x(0) = 0, v(0) = 0$ and $a = 14$ m/s^2.

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int 14dt \\ &= 14t + v(0) \\ &= 14t \end{aligned}$$

Since we want to find time to reach $v_{\max} = 300$ km/h which in SI units is $\frac{(300)(1000)}{(60)(60)} = \frac{250}{3}$ m/sec. Substituting this in the above gives

$$\begin{aligned} \frac{250}{3} &= 14t_{\max} \\ t_{\max} &= \frac{250}{3(14)} \\ &= \frac{125}{21} \\ &= 5.95 \text{ seconds} \end{aligned}$$

To find the distance traveled in this time, since

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int 14tdt \\ &= \frac{14}{2}t^2 + x(0) \\ &= 7t^2 \end{aligned}$$

When $t = t_{\max}$ the above gives

$$\begin{aligned} x(t_{\max}) &= 7\left(\frac{125}{21}\right)^2 \\ &= 248.02 \text{ meters} \end{aligned}$$

8 Additional problem 2

The car is approaching a tight turn at 300 km/h. In order to safely make the corner, it must be traveling at 80 km/h when it enters the corner. The brakes on the car cause a deceleration of 39 m/s^2. How far away from the corner must the driver begin braking to make the corner?

Solution

In SI units 300 km/h is $\frac{(300)(1000)}{(60)(60)} = \frac{250}{3}$ m/s. And 80 km/h is $\frac{80(1000)}{(60)(60)} = \frac{200}{9}$ m/s. Therefore we have initial velocity $v(0) = \frac{250}{3}$ m/s and final velocity $v_f(t) = \frac{200}{9}$ m/s and have acceleration of -39 m/s^2.

We first find the time it takes to go from $v(0)$ to $v_f(t)$. Since

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int -39dt \\ &= -39t + v(0) \end{aligned}$$

Therefore we have the equation

$$\begin{aligned} v_f(t) &= -39t + v(0) \\ \frac{200}{9} &= -39t + \frac{250}{3} \\ 39t &= \frac{250}{3} - \frac{200}{9} \\ t_f &= \frac{550}{351} \\ &= 1.567 \text{ sec} \end{aligned}$$

This is the time needed to decelerate from 300 km/h to 80 km/h. Now we find the distance traveled during this time. Since

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int -39t + v(0)dt \\ &= \int -39t + \frac{250}{3}dt \\ &= -\frac{39}{2}t^2 + \frac{250}{3}t + x(0) \end{aligned}$$

Let $x(0) = 0$, by taking initial position as zero. Replacing t in the above with t_f found earlier gives

$$\begin{aligned} x(t) &= -\frac{39}{2}\left(1.567^2\right) + \frac{250}{3}(1.567) \\ &= 82.7 \text{ meter} \end{aligned}$$

Therefore the car needs to be 82.7 meter away from corner to begin the braking.

9 Additional problem 3

At the exit of the corner, two cars are traveling at 100 km/h, with car A being 10 m behind car B. Out of the corner, car A accelerates at 14 m/s^2 and car B accelerates at 13 m/s^2. How much time does it take for car A to be right next to car B? How fast are the cars going when this happens? How far from the corner exit have they traveled?

Solution

Using SI units, 100 km/h is $\frac{(100)(1000)}{(60)(60)} = \frac{250}{9}$ m/s. Let at $t = 0$, $x_A(0) = 0$ and therefore $x_B(0) = 10$, since car B is ahead by 10 meters initially. Let $v_A(0) = \frac{250}{9}$ m/s and also $v_B(0) = \frac{250}{9}$ m/s. We now need to determine the time, say t_f, where $x_A\left(t_f\right) = x_B\left(t_f\right)$. But for car A we have

$$\begin{aligned} v_A(t) &= \int a_A(t)dt \\ &= \int 14dt \\ &= 14t + v_A(0) \\ &= 14t + \frac{250}{9} \end{aligned}$$

And

$$\begin{aligned} x_A(t) &= \int v_A(t)dt \\ &= \int\left(14t + \frac{250}{9}\right)dt \\ &= \frac{14}{2}t^2 + \frac{250}{9}t + x_A(0) \\ &= 7t^2 + \frac{250}{9}t \end{aligned} \tag{1}$$

Since $x_A(0) = 0$. Now we do the same for car B

$$\begin{aligned} v_B(t) &= \int a_B(t)dt \\ &= \int 13dt \\ &= 13t + v_A(0) \\ &= 13t + \frac{250}{9} \end{aligned}$$

And

$$
\begin{aligned}
x_B(t) &= \int v_B(t)dt \\
&= \int\left(13t + \frac{250}{9}\right)dt \\
&= \frac{13}{2}t^2 + \frac{250}{9}t + x_B(0) \\
&= \frac{13}{2}t^2 + \frac{250}{9}t + 10 \qquad (2)
\end{aligned}
$$

Since $x_B(0) = 10$ m. Now we solve for t by equating (1) and (2)

$$
\begin{aligned}
7t^2 + \frac{250}{9}t &= \frac{13}{2}t^2 + \frac{250}{9}t + 10 \\
7t^2 + &= \frac{13}{2}t^2 + 10 \\
7t^2 - \frac{13}{2}t^2 &= 10 \\
\frac{1}{2}t^2 &= 10 \\
t^2 &= 20 \\
t &= \sqrt{20} \\
t_f &= 4.47 \text{ sec}
\end{aligned}
$$

So it takes 4.47 sec for car A to be be next to car B. To find the speed at this time, we substitute this value of time back in the velocity equation above. For car A

$$
\begin{aligned}
v_A(t) &= 14t + \frac{250}{9} \\
v_A\left(t_f\right) &= 14(4.47) + \frac{250}{9} \\
&= 90.36 \text{ m/s} \qquad (3)
\end{aligned}
$$

And for car B

$$
\begin{aligned}
v_B(t) &= 13t + \frac{250}{9} \\
v_B\left(t_f\right) &= 13(4.47) + \frac{250}{9} \\
&= 85.89 \text{ m/s}
\end{aligned}
$$

To find the distance traveled during this time, we substitute this time in the position equation. For car A, from Eq (1)

$$
\begin{aligned}
x_A(t) &= 7t^2 + \frac{250}{9}t \\
x_A\left(t_f\right) &= 7(4.47)^2 + \frac{250}{9}(4.47) \\
&= 264.03 \text{ meter}
\end{aligned}
$$

The distance traveled by car B is 10 meters less than this value, since it was ahead by 10 meters at the start at time $t = 0$.