HW 10

Math 2243
Linear Algebra and Differential Equations

Fall 2020
University of Minnesota, Twin Cities

Nasser M. Abbasi

December 6, 2020 Compiled on December 6, 2020 at 5:17am

Contents

1 Problem 4, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$
\begin{aligned}
x_1'(t) &= 4x_1(t) + x_2(t) \\
x_2'(t) &= 6x_1(t) - x_2(t)
\end{aligned}
$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$
\begin{aligned}
\lambda^2 - 3\lambda - 10 &= 0 \\
(\lambda - 5)(\lambda + 2) &= 0
\end{aligned}
$$

The roots are therefore

$$
\begin{aligned}
\lambda_1 &= 5 \\
\lambda_2 &= -2
\end{aligned}
$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue -2

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $6v_1 = -1$ or $v_1 = \frac{-1}{6}$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{-1}{6} \\ 1 \end{bmatrix}$$

normalizing the eigenvector gives

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

<u>eigenvalue 5</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $-v_1 = -1$ or $v = 1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= e^{5t}\,\vec{v}_1\\ &= e^{5t}\begin{bmatrix}1\\1\end{bmatrix}\end{aligned}$$

Since eigenvalue −2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= e^{-2t}\,\vec{v}_2\\ &= e^{-2t}\begin{bmatrix}-1\\6\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = c_1e^{5t}\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{-2t}\begin{bmatrix}-1\\6\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}c_1e^{5t} - c_2e^{-2t}\\c_1e^{5t} + 6c_2e^{-2t}\end{bmatrix}$$

2 Problem 6, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$\begin{aligned} x_1'(t) &= 9x_1(t) + 5x_2(t) \\ x_2'(t) &= -6x_1(t) - 2x_2(t) \end{aligned}$$

With initial conditions

$$x_1(0) = 1, x_2(0) = 0$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\begin{aligned} \lambda^2 - 7\lambda + 12 &= 0 \\ (\lambda - 3)(\lambda - 4) &= 0 \end{aligned}$$

The roots of the above are therefore

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= 4 \end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 3

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $6v_1 = -5$ or $v_1 = \frac{-5}{6}$. Hence the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \frac{-5}{6} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_1 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

<u>eigenvalue 4</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using first row, and letting $v_2 = 1$ gives $5v_1 = -5$ or $v_1 = -1$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} -5 \\ 6 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{4t}\\ &= e^{4t}\begin{bmatrix}-1\\1\end{bmatrix}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t}\\ &= e^{3t}\begin{bmatrix}-5\\6\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = c_1 e^{4t}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}-5\\6\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}-c_1 e^{4t} - 5c_2 e^{3t}\\ c_1 e^{4t} + 6c_2 e^{3t}\end{bmatrix} \tag{1}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix}x_1(0) = 1\\x_2(0) = 0\end{bmatrix}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-c_1 - 5c_2\\ c_1 + 6c_2\end{bmatrix}$$

Adding first equation to second gives $1 = c_2$. From second equation this gives $0 = c_1 + 6$ or $c_1 = -6$

$$\begin{aligned}c_1 &= -6\\ c_2 &= 1\end{aligned}$$

Substituting these constants back in Eq. (1) gives

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}6e^{4t} - 5e^{3t}\\ -6e^{4t} + 6e^{3t}\end{bmatrix}$$

3 Problem 8, section 7.3

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution.

$$\begin{aligned} x_1'(t) &= x_1(t) - 5x_2(t) \\ x_2'(t) &= x_1(t) - x_2(t) \end{aligned}$$

<u>Solution</u>

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\begin{aligned} \lambda^2 + 4 &= 0 \\ \lambda^2 &= -4 \end{aligned}$$

Therefore the roots are

$$\begin{aligned} \lambda_1 &= 2i \\ \lambda_2 &= -2i \end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

<u>eigenvalue $2i$</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 - 2i & -5 \\ 1 & -1 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From first row, letting $v_2 = 1$ then $(1 - 2i)v_1 = 5$ or $v_1 = \frac{5}{1-2i} = \frac{(1+2i)5}{(1-2i)(1+2i)} = \frac{(1+2i)5}{1-4i^2} = \frac{(1+2i)5}{5} = 1+2i$. Hence

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 + 2i \\ 1 \end{bmatrix}$$

eigenvalue $-2i$

The second eigenvector is complex conjugate of the first. Therefore

$$\vec{v}_2 = \begin{bmatrix} 1 - 2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} 1 + 2i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} 1 - 2i \\ 1 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective.

Since eigenvalue $2i$ is complex, then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2it} \\ &= e^{2it} \begin{bmatrix} 1 + 2i \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue $-2i$ is complex, then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2it}\\ &= e^{-2it}\begin{bmatrix} 1-2i \\ 1 \end{bmatrix}\end{aligned}$$

The complex eigenvectors found above, which are complex conjugate of each others, are now converted to real basis as follows (Using Euler relation that $e^{i\theta} = \cos\theta + i\sin\theta$).

First we break $\vec{x}_1(t)$ into real part and imaginary part (we could also have done this using $\vec{x}_2(t)$, either one will work.

$$\begin{aligned}\vec{x}_1(t) &= e^{2it}\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}\\ &= \begin{bmatrix} e^{2it}(1+2i) \\ e^{2it} \end{bmatrix}\\ &= \begin{bmatrix} (\cos 2t + i\sin 2t)(1+2i) \\ \cos 2t + i\sin 2t \end{bmatrix}\\ &= \begin{bmatrix} (\cos 2t + i\sin 2t) + 2i(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \end{bmatrix}\\ &= \begin{bmatrix} \cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{bmatrix}\\ &= \begin{bmatrix} (\cos 2t - 2\sin 2t) + i(2\cos 2t + \sin 2t) \\ \cos 2t + i(\sin 2t) \end{bmatrix} \qquad (1)\end{aligned}$$

Therefore, using

$$\begin{aligned}\vec{x}_1(t) &= \operatorname{Re}\left(\vec{x}_1(t)\right)\\ \vec{x}_2(t) &= \operatorname{Im}\left(\vec{x}_1(t)\right)\end{aligned}$$

From (1) this gives

$$\begin{aligned}\vec{x}_1(t) &= \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix}\\ \vec{x}_2(t) &= \begin{bmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}\end{aligned}$$

The final solution becomes

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2\cos(2t) + \sin(2t) \\ \sin(2t) \end{bmatrix}$$

Hence

$$\begin{aligned} x_1(t) &= c_1 \cos(2t) - 2c_1 \sin(2t) + 2c_2 \cos(2t) + c_2 \sin(2t) \\ x_2(t) &= c_1 \cos(2t) + c_2 \sin(2t) \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= (c_1 + 2c_2) \cos(2t) + (c_2 - 2c_1) \sin(2t) \\ x_2(t) &= c_1 \cos(2t) + c_2 \sin(2t) \end{aligned}$$

4 Problem 18, section 7.3

In Problems 17 through 25, the eigenvalues of the coefficient matrix can be found by inspection and factoring. Apply the eigenvalue method to find a general solution of each system

$$
\begin{aligned}
x_1'(t) &= x_1 + 2x_2 + 2x_3 \\
x_2'(t) &= 2x_1 + 7x_2 + x_3 \\
x_3'(t) &= 2x_1 + x_2 + 7x_3
\end{aligned}
$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or in matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Or

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{bmatrix} \right) = 0$$

Expanding along first row gives

$$
\begin{aligned}
(1-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 2 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 7-\lambda \\ 2 & 1 \end{vmatrix} &= 0 \\
(1-\lambda)\left((7-\lambda)^2 - 1\right) - 2(2(7-\lambda)-2) + 2(2-2(7-\lambda)) &= 0 \\
(1-\lambda)(7-\lambda)^2 - (1-\lambda) - 4(7-\lambda) + 4 + 4 - 4(7-\lambda) &= 0 \\
(1-\lambda)(7-\lambda)^2 - 1 + \lambda - 28 + 4\lambda + 8 - 28 + 4\lambda &= 0 \\
(1-\lambda)(7-\lambda)^2 + 9\lambda - 49 &= 0 \\
(1-\lambda)\left(\lambda^2 - 14\lambda + 49\right) + 9\lambda - 49 &= 0 \\
-\lambda^3 + 15\lambda^2 - 63\lambda + 49 + 9\lambda - 49 &= 0 \\
-\lambda^3 + 15\lambda^2 - 63\lambda + 9\lambda &= 0 \\
-\lambda^3 + 15\lambda^2 - 54\lambda &= 0 \\
\lambda^3 - 15\lambda^2 + 54\lambda &= 0 \\
\lambda\left(\lambda^2 - 15\lambda + 54\right) &= 0 \\
\lambda(\lambda-6)(\lambda-9) &= 0
\end{aligned}
$$

Therefore the roots are

$$
\begin{aligned}
\lambda_1 &= 9 \\
\lambda_2 &= 6 \\
\lambda_3 &= 0
\end{aligned}
$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 0

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 7 & 1 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]
$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variables gives $v_1 = -4t, v_2 = t$. Hence the solution is

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

<u>eigenvalue 6</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & 2 & 2 & 0 \\ 0 & \frac{9}{5} & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system in Echelon form is

$$\begin{bmatrix} -5 & 2 & 2 \\ 0 & \frac{9}{5} & \frac{9}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = -t$. Hence the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

eigenvalue 9

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} - (9)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -8 & 2 & 2 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & 2 & 2 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = \frac{t}{2}, v_2 = t$. Hence the eigenvector is

$$\vec{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the normalized eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{6t} \\ &= e^{6t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\
&= e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Or

$$
\begin{aligned}
x_1(t) &= -4c_1 + c_3 e^{9t} \\
x_2(t) &= c_1 - c_2 e^{6t} + 2c_3 e^{9t} \\
x_3(t) &= c_1 + c_2 e^{6t} + 2c_3 e^{9t}
\end{aligned}
$$

5 Problem 38, section 7.3

For each matrix A given in Problems 38 through 40, the zeros in the matrix make its characteristic polynomial easy to calculate. Find the general solution of $\vec{x}'(t) = A\vec{x}(t)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

Solution

This is a system of linear ODE's which can be written as

$$\vec{x}'(t) = A\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 3 & 3-\lambda & 0 \\ 0 & 0 & 4 & 4-\lambda \end{bmatrix}\right) = 0$$

Since this is a lower triangular matrix, then the determinant is the product of the entries on the diagonal Hence the characteristic equation is

$$(1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda) = 0$$

Therefore the roots are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \\ \lambda_4 &= 4 \end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue 1

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 1 with row 2 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right]$$

current pivot $A(3,3)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system is now in Echelon form

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = -\frac{t}{4}, v_2 = \frac{t}{2}, v_3 = -\frac{3t}{4}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ -\frac{3t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

By letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

<u>eigenvalue 2</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with a non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(3,3)$ is still zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in Echelon form. Hence

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = 0, v_2 = \frac{t}{6}, v_3 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{6} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$$

<u>eigenvalue 3</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{array}\right]$$

current pivot $A(3,3)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 3 with row 4 gives

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in Echelon form. Hence

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = 0, v_2 = 0, v_3 = -\frac{t}{4}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

By letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Normalizing gives

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$$

eigenvalue 4

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 + 4R_3 \implies \left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_4 and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = 0, v_3 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis.

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{t} = e^{t}\begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t} = e^{2t}\begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{3t} = e^{3t}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_4(t) = \vec{v}_4 e^{4t} = e^{4t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 e^{t}\begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix} + c_4 e^{4t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Or

$$\begin{aligned} x_1(t) &= -c_1 e^{t} \\ x_2(t) &= 2c_1 e^{t} + c_2 e^{2t} \\ x_3(t) &= -3c_1 e^{t} - 3c_2 e^{2t} - c_3 e^{3t} \\ x_4(t) &= 4c_1 e^{t} + 6c_2 e^{2t} + 4c_3 e^{3t} + c_4 e^{4t} \end{aligned}$$

6 Additional problem 1

Consider the differential equation $x'''(t) + x''(t) - 2x'(t) = 0$. (a) Transform this into an equivalent system of first-order differential equations. (b) Write the system from (a) as $\vec{x}'(t) = A\vec{x}(t)$ (the matrix A should be 3×3). (c) Use the eigenvalue method to solve the system. (d) Using your solution to (c), what is the general solution $x(t)$ to the given differential equation?

Solution

6.1 Part (a)

Since this is a third order ODE, we need three state variables. Let

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= x'' \end{aligned}$$

Taking derivatives of the above gives

$$\begin{aligned} x_1' &= x' \\ &= x_2 \\ x_2' &= x'' \\ &= x_3 \\ x_3' &= x''' \\ &= -x'' + 2x' \\ &= -x_3 + 2x_2 \end{aligned}$$

Therefore the equations are

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_2 - x_3 \end{aligned}$$

6.2 Part (b)

The equations in part (a) in matrix form are

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

6.3 Part (c)

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}0&1&0\\0&0&1\\0&2&-1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda&1&0\\0&-\lambda&1\\0&2&-1-\lambda\end{bmatrix}\right)=0$$

Expansion along first column gives

$$\begin{aligned}-\lambda\begin{vmatrix}-\lambda&1\\2&-1-\lambda\end{vmatrix}&=0\\-\lambda((-\lambda)(-1-\lambda)-2)&=0\\-\lambda\left(\lambda^2+\lambda-2\right)&=0\\-\lambda(\lambda+2)(\lambda-1)&=0\end{aligned}$$

Hence the roots are $\lambda_1=0,\lambda_2=-2,\lambda_3=1$. For each eigenvalue we find the corresponding eigenvector.

$\underline{\lambda=0}$

We need now to determine the eigenvector $\vec{v}$ where

$$\begin{aligned}A\vec{v}&=\lambda\vec{v}\\A\vec{v}-\lambda\vec{v}&=\vec{0}\\(A-\lambda I)\vec{v}&=\vec{0}\\\left(\begin{bmatrix}0&1&0\\0&0&1\\0&2&-1\end{bmatrix}-(0)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}&=\begin{bmatrix}0\\0\\0\end{bmatrix}\\\left(\begin{bmatrix}0&1&0\\0&0&1\\0&2&-1\end{bmatrix}-\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}&=\begin{bmatrix}0\\0\\0\end{bmatrix}\\\begin{bmatrix}0&1&0\\0&0&1\\0&2&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}&=\begin{bmatrix}0\\0\\0\end{bmatrix}\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \implies \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_1 and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_2 = 0, v_3 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda = -2}$

We need now to determine the eigenvector $\vec{v}$ where

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of free variable gives equations $v_1 = \frac{t}{4}, v_2 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ -\frac{t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$\underline{\lambda = 1}$

We need now to determine the eigenvector $\vec{v}$ where

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variable gives equations $v_1 = t, v_2 = t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarizes the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
−2	1	1	No	$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvectors, then the solution is

$$\begin{aligned} \vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) \\ &= c_1 e^{\lambda_1 t}\vec{v}_1(t) + c_2 e^{\lambda_2 t}\vec{v}_2(t) + c_3 e^{\lambda_3 t}\vec{v}_3(t) \\ &= c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

Or

$$\begin{aligned} x_1(t) &= c_1 + c_2 e^{-2t} + c_3 e^{t} \\ x_2(t) &= -2c_2 e^{-2t} + c_3 e^{t} \\ x_3(t) &= 4c_2 e^{-2t} + c_3 e^{t} \end{aligned}$$

6.4 Part (d)

From the solution we found in part(c), which is the general solution in vector form, this part is asking what is the solution to $x'''(t) + x''(t) - 2x'(t) = 0$. Since the solution to this ode is $x(t)$, and this is the same as $x_1(t)$, then the solution to the ODE is

$$x(t) = c_1 + c_2 e^{-2t} + c_3 e^{t}$$

Which is the first row in the vector solution found in part(c).

7 Additional problem 2

Consider the following system of brine tanks: There are three tanks. Tank 1 contains 20L of water, tank 2 contains 30L of water, and tank 3 contains 60L of water. Fresh water is pumped into tank 1 at a rate of 120 L/min. The well-mixed solution is pumped from tank 1 to tank 2, from tank 2 to tank 3, and out of tank 3 all at a rate of 120 L/min.

(a) Draw and label a diagram describing the system (b) Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ -x_2(t) \\ x_3(t) \end{bmatrix}$ be the vector function of the amount of salt in each tank at time t. Write a differential equation $\vec{x}'(t) = A\vec{x}(t)$ describing the system.(c) Find the general solution to the differential equation you wrote in (b) using the eigenvalue method. (d) Initially, there is 100 kg of salt in tank 1 and 20 kg of salt in tank 2. Find the particular solution corresponding to these initial conditions.

Solution

7.1 Part (a)

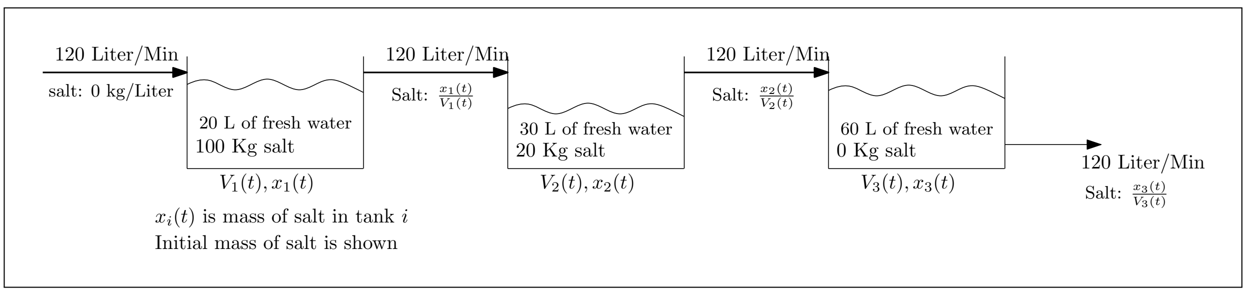

Figure 1: Diagram description of the problem

7.2 Part (b)

We notice that, since the rate of flow in and out from each tank is the same, then volume of water mix is constant and remain the same all the time in each tank. Hence

$$\begin{aligned} x_1'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)0\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= -120\frac{x_1(t)}{20(t)} \end{aligned} \tag{1}$$

And

$$\begin{aligned} x_2'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= 120\frac{x_1(t)}{20} - 120\frac{x_2(t)}{30} \end{aligned} \tag{2}$$

And

$$\begin{aligned} x_3'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(120\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \\ &= 120\frac{x_2(t)}{30} - 120\frac{x_3(t)}{60} \end{aligned} \tag{3}$$

Therefore the differential equations are

$$\begin{aligned} x_1'(t) &= -6x_1(t) \\ x_2'(t) &= 6x_1(t) - 4x_2(t) \\ x_3'(t) &= 4x_2(t) - 2x_3(t) \end{aligned}$$

In Matrix form

$$\begin{aligned} \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} &= \begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \\ \vec{x}' &= A\vec{x} \end{aligned}$$

7.3 Part (c)

We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & 0 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Since this is lower triangle matrix, then the determinant is the product of the elements along the diagonal. Hence

$$(-6-\lambda)(-4-\lambda)(-2-\lambda)=0$$

Hence the roots are

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -6\\ \lambda_3 &= -4\end{aligned}$$

Next, the eigenvectors for each eigenvalue are found.

eigenvalue −6

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-6 & 0 & 0\\ 6 & -4 & 0\\ 0 & 4 & -2\end{bmatrix}-(-6)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 6 & 2 & 0\\ 0 & 4 & 4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 6 & 2 & 0 & 0\\ 0 & 4 & 4 & 0\end{array}\right]$$

current pivot $A(1,1)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 1 with row 2 gives

$$\left[\begin{array}{ccc|c}6 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 4 & 4 & 0\end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with non-zero row. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c}6 & 2 & 0 & 0\\ 0 & 4 & 4 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system is now in Echelon form

$$\begin{bmatrix}6 & 2 & 0\\ 0 & 4 & 4\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equations for the leading variables in terms of the free variables gives equations $v_1 = \frac{t}{3}, v_2 = -t$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\begin{aligned} \vec{v}_1(t) &= \begin{bmatrix} \frac{1}{3} \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \end{aligned}$$

<u>eigenvalue −4</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned} \left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

current pivot $A(2,2)$ is zero. Hence we need to replace current pivot row with one non-zero. Replacing row 2 with row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = -\frac{t}{2}$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\vec{v}_2(t) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

<u>eigenvalue -2</u>

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 0 \\ 6 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now apply forward elimination to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variable is v_3 and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of the free variable gives equations $v_1 = 0, v_2 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Letting $t = 1$ the eigenvector is

$$\vec{v}_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue.

If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$
-6	1	1	No	$\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$

Now that the we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis.

Since eigenvalue −2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t}\\ &= e^{-2t}\begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$

Since eigenvalue −4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-4t}\\ &= e^{-4t}\begin{bmatrix}0\\-1\\2\end{bmatrix}\end{aligned}$$

Since eigenvalue −6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-6t}\\ &= e^{-6t}\begin{bmatrix}1\\-3\\3\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix} = c_1 e^{-2t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_2 e^{-4t}\begin{bmatrix}0\\-1\\2\end{bmatrix} + c_3 e^{-6t}\begin{bmatrix}1\\-3\\3\end{bmatrix}$$

7.4 Part (d)

Initial conditions are

$$\begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix} = \begin{bmatrix}100\\20\\0\end{bmatrix}$$

Hence the solution found in part(c) at $t = 0$ becomes

$$\begin{bmatrix}100\\20\\0\end{bmatrix} = c_1\begin{bmatrix}0\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\2\end{bmatrix} + c_3\begin{bmatrix}1\\-3\\3\end{bmatrix}$$

Or in matrix form

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix}$$

From first row, $c_3 = 100$. From second row $-c_2 - 3c_3 = 20$ or $c_2 = -20 - 3c_3 = -20 - 300 = -320$ and from last row $c_1 + 2c_2 + 3c_3 = 0$ or $c_1 = -2c_2 - 3c_3 = -2(-320) - 3(100) = 340$. Hence

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 340 \\ -320 \\ 100 \end{bmatrix}$$

And the solution found at end of part (c) becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 340e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 320e^{-4t}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 100e^{-6t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Or

$$\begin{aligned} x_1(t) &= 100e^{-6t} \\ x_2(t) &= 320e^{-4t} - 300e^{-6t} \\ x_3(t) &= 340e^{-2t} - 640e^{-4t} + 300e^{-6t} \end{aligned}$$

We see that as $t \to \infty$ then there will be no salt left in any tank, since each $x_i(t) \to 0$, and therefore only fresh water will remain in each tank, as expected.