HW 1

Math 2243
Linear Algebra and Differential Equations

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Contents

1 Problem 11 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

Swapping R_2, R_1 gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 3 & 11 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

$R_2 \to (-2)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 3 & 7 & 9 & -12 \end{pmatrix}$$

$R_3 \to (-3)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & -2 & 3 & -18 \end{pmatrix}$$

$R_3 \to 2R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

The leading variables are x, y, z. No free variables. Hence the system is consistent.

The equations after elimination are

$$\begin{aligned} x + 3y + 2z &= 2 \\ y - z &= 7 \\ z &= -4 \end{aligned}$$

Backsubstitution gives

$$z = -4$$

And

$$\begin{aligned} y - (-4) &= 7 \\ y &= 7 - 4 \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} x + 3(3) + 2(-4) &= 2 \\ x &= 2 - 9 + 8 \\ &= 1 \end{aligned}$$

The solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

2 Problem 15 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} x + 3y + 2z &= 5 \\ x - y + 3z &= 3 \\ 3x + y + 8z &= 10 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 1 & -1 & 3 & 3 \\ 3 & 1 & 8 & 10 \end{pmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -4 & 1 & -2 \\ 3 & 1 & 8 & 10 \end{pmatrix}$$

$R_3 \to (-3)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -4 & 1 & -2 \\ 0 & -8 & 2 & -5 \end{pmatrix}$$

$R_3 \to (-2)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -4 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The equations after elimination are

$$\begin{aligned} x + 3y + 2z &= 5 \\ -4y + z &= -2 \\ 0 &= -1 \end{aligned}$$

Therefore the system is inconsistent due to the last row result above which is not valid. No solution exist.

3 Problem 17 section 3.1

Problem

use the method of elimination to determine whether the given linear system is consistent or inconsistent. For each consistent system, find the solution if it is unique; otherwise, describe the infinite solution set in terms of an arbitrary parameter t

$$\begin{aligned} 2x - y + 4z &= 7 \\ 3x + 2y - 2z &= 3 \\ 5x + y + 2z &= 15 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 2 & -1 & 4 & 7 \\ 3 & 2 & -2 & 3 \\ 5 & 1 & 2 & 15 \end{pmatrix}$$

Scaling first row by 3 and second row by 2 gives

$$\begin{pmatrix} 6 & -3 & 12 & 21 \\ 6 & 4 & -4 & 6 \\ 5 & 1 & 2 & 15 \end{pmatrix}$$

$R_2 \to -R_1 + R_2$ gives

$$\begin{pmatrix} 6 & -3 & 12 & 21 \\ 0 & 7 & -16 & -15 \\ 5 & 1 & 2 & 15 \end{pmatrix}$$

Scaling first row by 5 and last row by 6 gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 30 & 6 & 12 & 90 \end{pmatrix}$$

$R_3 \to -R_1 + R_3$ gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 0 & 21 & -48 & -15 \end{pmatrix}$$

$R_3 \to -(3)R_2 + R_3$ gives

$$\begin{pmatrix} 30 & -15 & 60 & 105 \\ 0 & 7 & -16 & -15 \\ 0 & 0 & 0 & 30 \end{pmatrix}$$

The equations after elimination are

$$\begin{aligned} 30x - 15y + 60z &= 105 \\ 7y - 16z &= -15 \\ 0 &= 30 \end{aligned}$$

Therefore the system is inconsistent due to the last row result above which is not valid. No solution exist.

4 Problem 7 section 3.2

Problem

The linear systems in Problems 1–10 are in echelon form. Solve each by back substitution

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_2 - 2x_3 + 7x_4 &= 7 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 4 & -5 & 17 \\ 0 & 1 & -2 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the leading variables are x_1, x_2 and the free variables are x_3, x_4. Let $x_3 = s, x_4 = t$. From second row

$$\begin{aligned} x_2 - 2x_3 + 7x_4 &= 7 \\ x_2 - 2s + 7t &= 7 \\ x_2 &= 7 + 2s - 7t \end{aligned}$$

And from first row

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 5x_4 &= 17 \\ x_1 + 2(7 + 2s - 7t) + 4s - 5t &= 17 \\ x_1 &= 19t - 8s + 3 \end{aligned}$$

Hence the solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 19t - 8s + 3 \\ 7 + 2s - 7t \\ s \\ t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -8 \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 19 \\ -7 \\ 0 \\ 1 \end{pmatrix} t \end{aligned}$$

There are infinite number of solutions.

5 Problem 9 section 3.2

Problem

The linear systems in Problems 1–10 are in echelon form. Solve each by back substitution

$$
\begin{aligned}
2x_1 + x_2 + x_3 + x_4 &= 6 \\
3x_2 - x_3 - 2x_4 &= 2 \\
3x_3 + 4x_4 &= 9 \\
x_4 &= 6
\end{aligned}
$$

Solution

The augmented matrix is

$$
\begin{pmatrix}
2 & 1 & 1 & 1 & 6 \\
0 & 3 & -1 & -2 & 2 \\
0 & 0 & 3 & 4 & 9 \\
0 & 0 & 0 & 1 & 6
\end{pmatrix}
$$

The leading variables are x_1, x_2, x_3, x_4. There are no free variables. Backsubstitution gives

$$x_4 = 6$$

And

$$
\begin{aligned}
3x_3 + 4x_4 &= 9 \\
3x_3 &= 9 - 4(6) \\
3x_3 &= -15 \\
x_3 &= -5
\end{aligned}
$$

And

$$
\begin{aligned}
3x_2 - x_3 - 2x_4 &= 2 \\
3x_2 + 5 - 12 &= 2 \\
x_2 &= 3
\end{aligned}
$$

And

$$
\begin{aligned}
2x_1 + x_2 + x_3 + x_4 &= 6 \\
2x_1 + 3 - 5 + 6 &= 6 \\
x_1 &= 1
\end{aligned}
$$

Hence the solution is

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \\ 6 \end{pmatrix}
$$

6 Problem 15 section 3.2

Problem

In Problems 11–22, use elementary row operations to transform each augmented coefficient matrix to echelon form. Then solve the system by back substitution.

$$\begin{aligned} 3x_1 + x_2 - 3x_3 &= -4 \\ x_1 + x_2 + x_3 &= 1 \\ 5x_1 + 6x_2 + 8x_3 &= 8 \end{aligned}$$

Solution

The augmented matrix is

$$\begin{pmatrix} 3 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

Exchanging row 1 with row 2 gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & -4 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

$R_2 \rightarrow (-3)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & -7 \\ 5 & 6 & 8 & 8 \end{pmatrix}$$

$R_3 \rightarrow (-5)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & -7 \\ 0 & 1 & 3 & 3 \end{pmatrix}$$

Exchanging row 2 with row 3 gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & -2 & -6 & -7 \end{pmatrix}$$

$R_3 \rightarrow (2)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence the equations are

$$
\begin{aligned}
x_1 + x_2 + x_1 &= 1\\
x_2 + 3x_3 &= 3\\
0 &= -1
\end{aligned}
$$

Therefore the system is inconsistent due to the last row result above which is not valid. No solution exist.

7 Problem 8 section 3.3

Problem

Find the reduced echelon form

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

Solution

First we convert the matrix to echelon form by elimination.

$R_2 \rightarrow (-3)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 1 & -2 & 3 \end{pmatrix}$$

$R_3 \rightarrow (-1)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{pmatrix}$$

Scale second row by 3 and scale third row by 2 gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 6 & 36 \\ 0 & 6 & 24 \end{pmatrix}$$

$R_3 \rightarrow (-1)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 6 & 36 \\ 0 & 0 & -12 \end{pmatrix}$$

The above is now in echelon form. We now convert it to reduced echelon form. Scaling the second row by $\frac{1}{6}$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & -12 \end{pmatrix}$$

Scaling the third row by $\frac{-1}{12}$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Starting from right to left, we now zero out all entries above the diagonal elements.

$R_2 \to (-6)R_3 + R_2$ gives

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \to (5)R_3 + R_1$ gives

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \to (4)R_2 + R_1$ gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above is the reduced echelon form. It is the identity matrix.

8 Problem 11 section 3.3

Problem

Find the reduced echelon form

$$\begin{pmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{pmatrix}$$

Solution

First we convert the matrix to echelon form by elimination. Exchanging row 3 and first row gives

$$\begin{pmatrix} 1 & 3 & -6 \\ 2 & 6 & 7 \\ 3 & 9 & 1 \end{pmatrix}$$

$R_2 \rightarrow (-2)R_1 + R_2$ gives

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 3 & 9 & 1 \end{pmatrix}$$

$R_2 \rightarrow (-3)R_1 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 19 \end{pmatrix}$$

$R_3 \rightarrow (-1)R_2 + R_3$ gives

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 0 \end{pmatrix}$$

The above is now in echelon form. We now convert it to reduced echelon form. Scaling the second row by $\frac{1}{19}$ gives

$$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Starting from right to left, we now zero out all entries above the leading elements starting from second row.

$R_1 \rightarrow (6)R_2 + R_1$ gives

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The above is the reduced echelon form.

9 Problem 1 extra

Problem

Draw pictures to illustrate the three possibilities for the solution set of a linear system of two equations in two variables.

Solution

For homogeneous system

$$a_{11}x + a_{12}y = 0$$
$$a_{21}x + a_{22}y = 0$$

There can be either one solution, which is the trivial solution $x = 0, y = 0$ where the two lines meet at the origin, or infinite number of solutions, which is when the two lines are on top of each others. The reason for this is that there is no intercept in the equation of the lines above. Only the slope of each line can change. Hence all lines must pass though the origin. This diagram illustrates this.

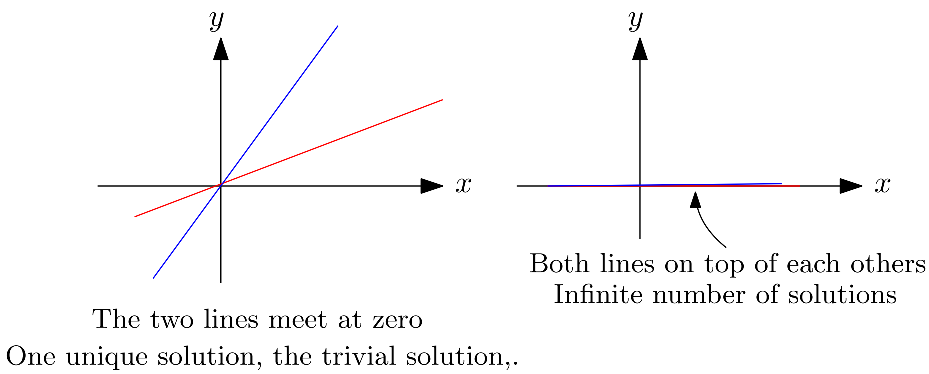

Figure 1: Possibilites for homogeneous system

For nonhomogeneous system

$$a_{11}x + a_{12}y = c_1$$
$$a_{21}x + a_{22}y = c_2$$

Now there can be three possibilities. Either one solution where the two lines meet or infinite number of solutions, which is when the two lines are on top of each others or no solutions, which is when the two lines are parallel but not on top of each others.

This diagram illustrates this.

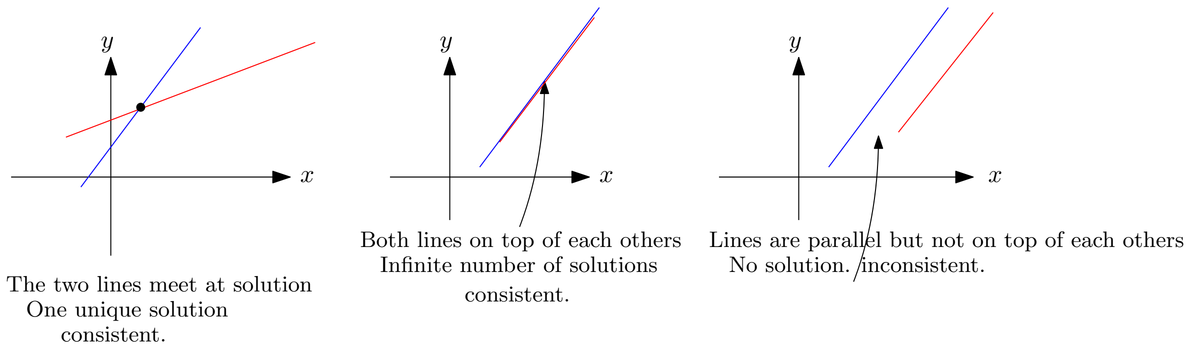

Figure 2: Possibilites for homogeneous system

10 Problem 2 extra

Problem

Give an example of a 3×3 matrix in echelon form with exactly 2 nonzero entries.

Solution

$$\begin{pmatrix} \bigstar & 0 & 0 \\ 0 & \bigstar & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Where $\bigstar$ is a nonzero entry.

11 Problem 3 extra

Problem

Give an example of a 3×3 matrix in reduced echelon form with exactly 4 nonzero entries.

Solution

$$\begin{pmatrix} 1 & 0 & \bigstar \\ 0 & 1 & \bigstar \\ 0 & 0 & 0 \end{pmatrix}$$

Where $\bigstar$ is a nonzero entry. In reduced echelon form only the leading entries (which must be 1) has to be zero.