By Section 11.3, the eigenfunctions for all three problems (in D, in D_1, and in D_2) are complete. Among the eigenfunctions of $-\Delta$ in the rectangle D are the products $v_n w_m$. Suppose now that there were an eigenfunction $u(x, y)$ in the rectangle, other than these products. Then, for some λ, $-\Delta u = \lambda u$ in D and u would satisfy the boundary conditions. If λ were different from every one of the sums $\alpha_n + \beta_m$, then we would know (from Section 10.1) that u is orthogonal to all the products $v_n w_m$. Hence

$$0 = (u, v_n w_m) = \int\left[\int u(x, y) v_n(x)\, dx\right] w_m(y)\, dy. \tag{13}$$

So, by the completeness of the w_m,

$$\int u(x, y) v_n(x)\, dx = 0 \qquad \text{for all } y. \tag{14}$$

By the completeness of the v_n, (14) would imply that $u(x, y) = 0$ for all x, y. So $u(x, y)$ wasn't an eigenfunction after all.

One possibility remains, namely, that $\lambda = \alpha_n + \beta_m$ for certain n and m. This could be true for one pair m, n or several such pairs. If λ were such a sum, we would consider the difference

$$\psi(x, y) = u(x, y) - \sum c_{nm} v_n(x) w_m(y), \tag{15}$$

where the sum is over all the n, m pairs for which $\lambda = \alpha_n + \beta_m$ and where $c_{nm} = (u, v_n w_m)/\|v_n w_m\|^2$. The function ψ defined by (15) is constructed so as to be orthogonal to *all* the products $v_n w_m$, for both $\alpha_n + \beta_m = \lambda$ and $\alpha_n + \beta_m \neq \lambda$. It follows by the same reasoning as above that $\psi(x, y) \equiv 0$. Hence $u(x, y) = \Sigma\, c_{nm} v_n(x) w_m(y)$, summed over $\alpha_n + \beta_m = \lambda$. That is, u was *not* a new eigenfunction at all, but was just a linear combination of those old products $v_n w_m$ which have the same eigenvalue λ. This completes the proof of Theorem 2. □

EXERCISES

1. Verify that all the functions (7) are solutions of (1) if a is an eigenvalue λ_N and if $\int f v_N\, dx = 0$. Why does the series in (7) converge?
2. Use the completeness to show that the solutions of the wave equation in any domain with a standard set of BC satisfy the usual expansion $u(\mathbf{x}, t) = \Sigma_{n=1}^{\infty} [A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)] v_n(\mathbf{x})$. In particular, show that the series converges in the L^2 sense.
3. Provide the details of the proof that $\psi(x, y)$, defined by (15), is identically zero.

11.6 ASYMPTOTICS OF THE EIGENVALUES

The main purpose of this section is to show that $\lambda_n \to +\infty$. In fact, we'll show exactly *how fast* the eigenvalues go to infinity. For the case of the Dirichlet boundary condition, the precise result is as follows.

Theorem 1. For a *two*-dimensional problem $-\Delta u = \lambda u$ in any plane domain D with $u = 0$ on bdy D, the eigenvalues satisfy the limit relation

$$\boxed{\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{A},} \tag{1}$$

where A is the *area* of D.

For a *three*-dimensional problem in any solid domain, the relation is

$$\boxed{\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{V},} \tag{2}$$

where V is the *volume* of D.

Example 1. The Interval

Let's compare Theorem 1 with the *one*-dimensional case where $\lambda_n = n^2\pi^2/l^2$. In that case,

$$\lim_{n\to\infty} \frac{\lambda_n^{1/2}}{n} = \frac{\pi}{l}, \tag{3}$$

where l is the length of the interval! The same result (3) was also derived for the one-dimensional Neumann condition in Section 4.2 and the Robin conditions in Section 4.3. □

Example 2. The Rectangle

Here the domain is $D = \{0 < x < a,\ 0 < y < b\}$ in the plane. We showed explicitly in Section 10.1 that

$$\lambda_n = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \tag{4}$$

with the eigenfunction $\sin(l\pi x/a) \cdot \sin(m\pi y/b)$. Since the eigenvalues are naturally numbered using a pair of integer indices, it is difficult to see the relationship between (4) and (1). For this purpose it is convenient to introduce the *enumeration function*

$$N(\lambda) \equiv \textit{the number of eigenvalues that do not exceed } \lambda. \tag{5}$$

If the eigenvalues are written in increasing order as in (11.1.2), then $N(\lambda_n) = n$. Now we can express $N(\lambda)$ another way using (4). Namely, $N(\lambda)$ is the number of integer lattice points (l, m) which are contained within the quarter-ellipse

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} \le \frac{\lambda}{\pi^2} \qquad (l > 0,\ m > 0) \tag{6}$$

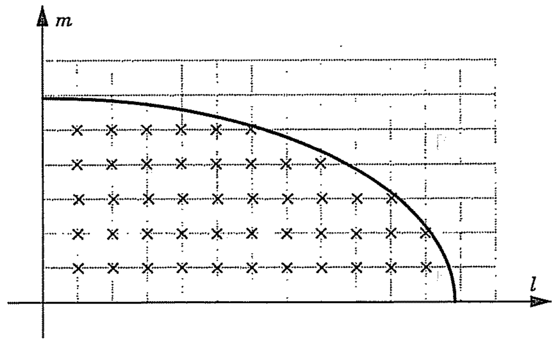

Figure 1

in the (l, m) plane (see Figure 1). Each such lattice point is the upper right corner of a square lying within the quarter ellipse. Therefore, $N(\lambda)$ is at most the area of this quarter ellipse:

$$N(\lambda) \leq \frac{\lambda ab}{4\pi}. \tag{7}$$

For large λ, $N(\lambda)$ and this area may differ by approximately the length of the perimeter, which is of the order $\sqrt{\lambda}$. Precisely,

$$\frac{\lambda ab}{4\pi} - C\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda ab}{4\pi} \tag{8}$$

for some constant C. Substituting $\lambda = \lambda_n$ and $N(\lambda) = n$, (8) takes the form

$$\frac{\lambda_n ab}{4\pi} - C\sqrt{\lambda_n} \leq n \leq \frac{\lambda_n ab}{4\pi}, \tag{9}$$

where the constant C does not depend on n. Therefore, upon dividing by n, we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab}, \tag{10}$$

which is Theorem 1 for a rectangle. □

For the Neumann condition, the only difference is that l and m are allowed to be zero, but the result is exactly the same:

$$\lim_{n\to\infty} \frac{\tilde{\lambda}_n}{n} = \frac{4\pi}{ab}. \tag{11}$$

To prove Theorem 1, we will need the maximin principle. It is like the minimum principle of Section 11.1 but with more general constraints. The idea is that *any* orthogonality constraints *other* than those in Section 11.1 will lead to smaller minimum values of the Rayleigh quotient.

Theorem 2. **Maximin Principle** Fix a positive integer $n \geq 2$. Fix $n-1$ arbitrary trial functions $y_1(\mathbf{x}), \ldots, y_{n-1}(\mathbf{x})$. Let

$$\boxed{\lambda_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2}} \tag{12}$$

among all trial functions w that are orthogonal to $y_1, \ldots, y_{n-1}$. Then

$$\boxed{\lambda_n = \max \lambda_{n*}} \tag{13}$$

over *all* choices of the $n-1$ trial functions $y_1, \ldots, y_{n-1}$.

Proof. Fix an arbitrary choice of $y_1, \ldots, y_{n-1}$. Let $w(\mathbf{x}) = \Sigma_{j=1}^n c_j v_j(\mathbf{x})$ be a linear combination of the first n eigenfunctions which is chosen to be orthogonal to $y_1, \ldots, y_{n-1}$. That is, the constants $c_1, \ldots, c_n$ are chosen to satisfy the linear system

$$0 = \left(\sum_{j=1}^{n} c_j v_j, y_k\right) = \sum_{j=1}^{n} (v_j, y_k) c_j \qquad (\text{for } k = 1, \ldots, n-1).$$

Being a system of only $n-1$ equations in n unknowns, it has a solution $c_1, \ldots, c_n$, not all of which constants are zero. Then, by definition (12) of λ_{n*},

$$\begin{aligned}\lambda_{n*} &\leq \frac{\|\nabla w\|^2}{\|w\|^2} = \frac{\Sigma_{j,k}\, c_j c_k(-\Delta v_j, v_k)}{\Sigma_{j,k}\, c_j c_k(v_j, v_k)} \\ &= \frac{\Sigma_{j=1}^n \lambda_j c_j^2}{\Sigma_{j=1}^n c_j^2} \leq \frac{\Sigma_{j=1}^n \lambda_n c_j^2}{\Sigma_{j=1}^n c_j^2} = \lambda_n,\end{aligned} \tag{14}$$

where we've again taken $\|v_j\| = 1$. This inequality (14) is true for every choice of $y_1, \ldots, y_{n-1}$. Hence, $\max \lambda_{n*} \leq \lambda_n$. This proves half of (13).

To demonstrate the equality in (13), we need only exhibit a special choice of $y_1, \ldots, y_{n-1}$ for which $\lambda_{n*} = \lambda_n$. Our special choice is the first $n-1$ eigenfunctions: $y_1 = v_1, \ldots, y_{n-1} = v_{n-1}$. By the minimum principle for the nth eigenvalue in Section 11.1, we know that

$$\lambda_{n*} = \lambda_n \qquad \text{for this choice.} \tag{15}$$

The maximin principle (13) follows directly from (14) and (15). □

The same maximin principle is also valid for the *Neumann boundary condition* if we use the "free" trial functions that don't satisfy any boundary condition. Let's denote the Neumann eigenvalues by $\tilde{\lambda}_j$. Now we shall simultaneously consider the Neumann and Dirichlet cases.

Theorem 3. $\tilde{\lambda}_j \leq \lambda_j$ for all $j = 1, 2, \ldots$.

Proof. Let's begin with the first eigenvalues. By Theorems 11.1.1 and 11.3.1, both $\tilde{\lambda}_1$ and λ_1 are expressed as the same minimum of the Rayleigh

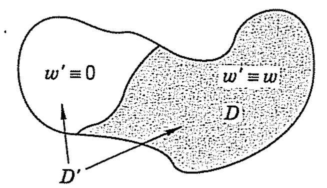

Figure 2

quotient except that the test functions for λ_1 satisfy one extra constraint (namely, that $w = 0$ on bdy D). Having one less constraint, $\tilde{\lambda}_1$ has a greater chance of being small. Thus $\tilde{\lambda}_1 \le \lambda_1$.

Now let $n \ge 2$. For the same reason of having one extra constraint, we have

$$\tilde{\lambda}_{n*} \le \lambda_{n*}. \tag{16}$$

We take the maximum of both sides of (16) over all choices of trial functions $y_1, \ldots, y_{n-1}$. By the maximin principle of this section (Theorem 2 and its Neumann analog), we have

$$\tilde{\lambda}_n = \max \tilde{\lambda}_{n*} \le \max \lambda_{n*} = \lambda_n. \qquad \square$$

Example 3.

For the interval $(0, l)$ in one dimension, the eigenvalues are $\lambda_n = n^2\pi^2/l^2$ and $\tilde{\lambda}_n = (n-1)^2\pi^2/l^2$ (using our present notation with n running from 1 to ∞). It is obvious that $\tilde{\lambda}_n < \lambda_n$. □

The general principle which is illustrated by Theorem 3 is that

any additional constraint will increase the value of the maximin. (17)

In particular, we can use this principle as follows to prove the monotonicity of the eigenvalues with respect to the domain.

Theorem 4. If the domain is enlarged, each eigenvalue is decreased.

That is, if one domain D is contained in another domain D', then $\lambda_n \ge \lambda_n'$ and $\tilde{\lambda}_n \ge \tilde{\lambda}_n'$, where we use primes on eigenvalues to refer to the larger domain D' (see Figure 2).

Proof. In the Dirichlet case, consider the maximin expression (13) for D. If $w(\mathbf{x})$ is any trial function in D, we define $w(\mathbf{x})$ in all of D' by setting it equal to zero outside D; that is,

$$w'(\mathbf{x}) = \begin{cases} w(\mathbf{x}) & \text{for } \mathbf{x} \text{ in } D \\ 0 & \text{for } \mathbf{x} \text{ in } D' \text{ but } \mathbf{x} \text{ not in } D. \end{cases} \tag{18}$$

Thus every trial function in D corresponds to a trial function in D' (but not conversely). So, compared to the trial functions for D', the trial functions for D have the extra constraint of vanishing in the rest of D'. By the general principle (17), the maximin for D is larger than the maximin for D'. It follows that $\lambda_n \ge \lambda_n'$, as we wanted to prove. But we should beware that we are avoiding the difficulty that by extending the function to be zero, it is most likely no longer a C^2 function and therefore not a trial function. The good thing about the extended function $w'(\mathbf{x})$ is that it still is continuous. For a rigorous justification of this point, see [CH] or [Ga].

The same kind of reasoning is valid in the Neumann case. Indeed, the maximin principle for the Neumann boundary condition states that

$$\tilde{\lambda}_n = \max \tilde{\lambda}_{n*} \qquad \text{where } \tilde{\lambda}_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2} \tag{19}$$

and the competing trial functions $w(\mathbf{x})$ do not satisfy any boundary condition at all. As above, these test functions on D may be extended to the larger domain D' by setting them equal to zero outside D. In this case, the new trial functions $w'(\mathbf{x})$ may be *discontinuous* at the part of the boundary of D which is internal to D' (see Figure 1.) But in any case there are again more trial functions for D' than for D. That is, the maximin for D has more constraints, so that $\tilde{\lambda}_n \ge \tilde{\lambda}_n'$. Again see [CH] for a complete proof. □

SUBDOMAINS

Our next step in establishing Theorem 1 is to *divide the general domain D into a finite number of subdomains* $D_1, \ldots, D_m$ by introducing inside D a system of smooth surfaces $S_1, S_2, \ldots$ (see Figure 3). Let D have Dirichlet eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ and Neumann eigenvalues $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots$. Each of the subdomains $D_1, \ldots, D_m$ has its own collection of eigenvalues. We combine *all* of the Dirichlet eigenvalues of *all* of the subdomains $D_1, \ldots, D_m$ into a single increasing sequence $\mu_1 \le \mu_2 \le \cdots$. We combine *all* of their Neumann eigenvalues into another single increasing sequence $\tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots$.

By the maximin principle, each of these numbers can be obtained as the maximum over trial functions $y_1, \ldots, y_{n-1}$ of the minimum over trial functions w orthogonal to $y_1, \ldots, y_{n-1}$. As discussed above, although each μ_n is a Dirichlet eigenvalue of a single one of the subdomains, the trial functions can be defined in all of D simply by making them vanish in the other subdomains. Thus each of the competing trial functions for μ_n has the extra restriction, compared with the trial functions for λ_n for D, of vanishing on the internal boundaries. It follows from the general principle (17) that

$$\lambda_n \le \mu_n \qquad \text{for each } n = 1, 2, \ldots. \tag{20}$$

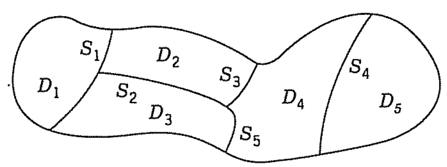

Figure 3

On the other hand, the trial functions defining $\tilde{\lambda}_n$ for the Neumann problem in D are arbitrary C^2 functions. As above, we can characterize $\tilde{\mu}_n$ as

$$\tilde{\mu}_n = \max \tilde{\mu}_{n*} \qquad \tilde{\mu}_{n*} = \min \frac{\|\nabla w\|^2}{\|w\|^2}, \tag{21}$$

where the competing trial functions are arbitrary on each subdomain and orthogonal to $y_1, \ldots, y_{n-1}$. But these trial functions *are allowed to be discontinuous on the internal boundaries*, so they comprise a significantly more extensive class than the trial functions for $\tilde{\lambda}_n$, which required to be continuous in D. Therefore, by (17) we have $\tilde{\mu}_n \le \tilde{\lambda}_n$ for each n. Combining this with Theorem 3 and (20), we have proved the following inequalities.

Theorem 5.

$$\boxed{\tilde{\mu}_n \le \tilde{\lambda}_n \le \lambda_n \le \mu_n.}$$

Example 4.

Let D be the union of a finite number of rectangles $D = D_1 \cup D_2 \cup \cdots$ in the plane as in Figure 4. Each particular μ_n corresponds to one of these rectangles, say D_p (where p depends on n). Let $A(D_p)$ denote the area of D_p. Let $M(\lambda)$ be the enumeration function for the sequence $\mu_1, \mu_2, \ldots$ defined above:

$$M(\lambda) \equiv \text{the number of } \mu_1, \mu_2, \ldots \text{ that do not exceed } \lambda. \tag{22}$$

Then, adding up the integer lattice points which are located within D, we get

$$\lim_{\lambda\to\infty} \frac{M(\lambda)}{\lambda} = \sum_p \frac{A(D_p)}{4\pi} = \frac{A(D)}{4\pi}, \tag{23}$$

as for the case of a single rectangle. Since $M(\mu_n) = n$, the reciprocal of (23) takes the form

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{4\pi}{A(D)}. \tag{24}$$

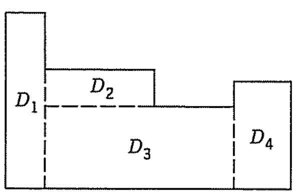

Figure 4

Similarly,

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n}{n} = \frac{4\pi}{A(D)}. \tag{25}$$

By Theorem 5 it follows that all the limits are equal: $\lim \lambda_n/n = \lim \tilde{\lambda}_n/n = 4\pi/A(D)$. This proves Theorem 1 for unions of rectangles. □

Now an arbitrary plane domain D can be approximated by unions of rectangles just as in the construction of a double integral (and as in Section 8.4). With the help of Theorem 5, it is possible to prove Theorem 1. The details are omitted but the proof may be found in [CH].

THREE DIMENSIONS

The *three-dimensional case* works the same way. We limit ourselves, however, to the basic example.

Example 5. The Rectangular Box

Let $D = \{0 < x < a, 0 < y < b, 0 < z < c\}$. As in Example 2, the enumeration function $N(\lambda)$ is approximately the volume of the ellipsoid

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{k^2}{c^2} \le \frac{\lambda}{\pi^2}$$

in the first octant. Thus for large λ

$$\begin{aligned} N(\lambda) &\sim \frac{1}{8}\,\frac{4\pi}{3}\,\frac{a\lambda^{1/2}}{\pi}\,\frac{b\lambda^{1/2}}{\pi}\,\frac{c\lambda^{1/2}}{\pi} \\ &= \lambda^{3/2}\frac{abc}{6\pi^2} \end{aligned} \tag{26}$$

and the same for the Neumann case. Substituting $\lambda = \lambda_n$ and $N(\lambda) = n$, we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{abc} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n}. \tag{27}$$

For the union of a finite number of boxes of volume V(D), we deduce that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{V(D)} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n}.$$

Then a general domain is approximated by unions of boxes. □

For the very general case of a *symmetric differential operator* as (11.4.1), the statement of the theorem is modified (in three dimensions, say) to read

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \lim_{n\to\infty} \frac{\tilde{\lambda}_n^{3/2}}{n} = \frac{6\pi^2}{\iiint_D [m(\mathbf{x})/p(\mathbf{x})]^{3/2}\,d\mathbf{x}}. \tag{28}$$

EXERCISES

1. Prove that (9) implies (10).
2. (a) For a circular drumhead (D = disk), verify Theorem 1 directly from Section 10.2 and the properties of Bessel functions.
   (b) Do the same in the Neumann case.
3. (a) For a spherical ball, verify Theorem 1 directly from Section 10.3 and the properties of Bessel functions.
   (b) Do the same in the Neumann case.
4. Explain how it is possible that λ_2 is *both* a maximin *and* a minimax.
5. For $-\Delta$ in the *ellipsoid* $D = \{x^2 + y^2/4 < 1\}$ with Dirichlet BCs use the monotonicity of the eigenvalues with respect to the domain to find estimates for the first two eigenvalues. Inscribe or circumscribe rectangles or circles, for which we already know the exact values.
   (a) Find upper bounds.
   (b) Find lower bounds.
6. In the proof of Theorem 1 for an arbitrary domain D, one must approximate D by unions of rectangles. This is a delicate limiting procedure. Outline the main steps required to carry out the proof.
7. Use the surface area of an ellipsoid to write the inequalities that make (26) a more precise statement.
8. For a symmetric differential operator in three dimensions as in (11.4.1), explain why Theorem 1 should be modified to be (28).
9. Consider the Dirichlet BCs in a domain D. Show that the first eigenfunction $v_1(\mathbf{x})$ *vanishes at no point* of D by the following method.
   (a) Suppose on the contrary that $v_1(\mathbf{x}) = 0$ at some point in D. Show that both $D^+ = \{\mathbf{x} \in D : v_1(\mathbf{x}) > 0\}$ and $D^- = \{\mathbf{x} \in D : v_1(\mathbf{x}) < 0\}$ are nonempty. (*Hint:* Use the maximum principle in Exercise 7.4.26.)
   (b) Let $v^+(\mathbf{x}) = v_1(\mathbf{x})$ for $\mathbf{x} \in D^+$ and $v^+(\mathbf{x}) = 0$ for $\mathbf{x} \in D^-$. Let $v^- = v_1 - v^+$. Notice that $|v_1| = v^+ - v^-$. Noting that $v_1 = 0$ on bdy D, we may deduce that $\nabla v^+ = \nabla v_1$ in D, and $\nabla v^+ = 0$ outside D. Similarly for ∇v^-. Show that the Rayleigh quotient Q for the function $|v_1|$ is equal to λ_1. Therefore, both v_1 and $|v_1|$ are eigenfunctions with the eigenvalue λ_1.
   (c) Use the maximum principle on $|v_1|$ to show that $v_1 > 0$ in all of D or $v_1 < 0$ in all of D.
   (d) Deduce that λ_1 is a simple eigenvalue (*Hint:* If $u(x)$ were another eigenfunction with eigenvalue λ_1, let w be the component of u orthogonal to v_1. Applying part (c) to w, we know that $w > 0$ or $w < 0$ or $w \equiv 0$ in D. Conclude that $w \equiv 0$ in D.)
10. Show that the nodes of the nth eigenfunction $v_n(\mathbf{x})$ divide the domain D into *at most n* pieces, assuming (for simplicity) that the eigenvalues are distinct, by the following method. Assume Dirichlet BCs.
   (a) Suppose on the contrary that $\{\mathbf{x} \in D : v_n(\mathbf{x}) \neq 0\}$ has at least $n+1$ disconnected components $D_1 \cup D_2 \cup \cdots \cup D_{n+1}$. Let $w_j(\mathbf{x}) = v_n(\mathbf{x})$ for $\mathbf{x} \in D_j$, and $w_j(\mathbf{x}) = 0$ elsewhere. You may assume that $\nabla w_j(\mathbf{x}) = \nabla v_n(\mathbf{x})$ for $\mathbf{x} \in D_j$, and $\nabla w_j(\mathbf{x}) = 0$ elsewhere. Show that the Rayleigh quotient for w_j equals λ_n.
   (b) Show that the Rayleigh quotient for any linear combination $w = c_1 w_1 + \cdots + c_{n+1} w_{n+1}$ also equals λ_n.
   (c) Let $y_1, \ldots, y_n$ be any trial functions. Choose the $n+1$ coefficients c_j so that w is orthogonal to each of $y_1, \ldots, y_n$. Use the maximin principle to deduce that $\lambda_{n+1} \le \|\nabla w\|^2/\|w\|^2 = \lambda_n$. Hence deduce that $\lambda_{n+1} = \lambda_n$, which contradicts our assumption.