MATH 5587 (FALL2019): MIDTERM 2

PROFESSOR: SVITLANA MAYBORODA

Name (legibly!): ------------------------------------

Problem 1 (40 points)

A metal bar, of length $l = 1$ and thermal diffusivity $\gamma = 1$, is taken out of a 100° oven and fully insulated except for a left end which is fixed to a large ice cube, and hence, kept at the constant temperature 0° and the right end which is kept at temperature 50°.

(a) (5 pts) Write down an initial boundary value problem that describes the the temperature of the bar $u(t, x)$.

Equation: ------------------- for $x \in (0, 1)$, $t > 0$

Boundary data: ---------------------

Initial data: -------------------

(b) Use separation of variables to write a series formula for solution $u(t, x)$.

b1) (5 pts) Write $u(t, x) = v(t)w(x)$ and find the equations that v and w satisfies

b2) (10 pts) Solve the equations for v and w and use boundary data to identify possible solutions v, w

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b3) (5 pts) Write the global solution $u(t, x)$ as a series and use initial data to write **integral formulas** for coefficients of the series.

b4) (5 pts) Evaluate explicitly coefficients of the series and write the final formula for solution $u(t, x)$

c) (5 pts) What is the equilibrium temperature (that is, the limit as $t \to \infty$) and how fast does the solution go to equilibrium?

d) (5pts) Is the solution smooth? For which t? Why?

Problem 2 (10 points) The solution to the Laplace's equation on a unit disc subject to Dirichlet boundary conditions $u(1,\theta) = h(\theta)$ is given by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi.$$

Show that if u achieves its maximum at the center of the disc then u is constant on the entire disc.

Problem 3 (15 points) The solution to the Laplace's equation $\Delta u = 0$ on a square $0 < x < a$, $0 < y < b$, with boundary data

$$u(x,0) = f(x),\ u(x,b) = 0,\ u(0,y) = 0,\ u(a,y) = 0$$

is given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}}$$

where $b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$.

Show that whenever $\int_0^a |f(x)|\,dx$ is finite, the coefficients of the series in the formula for u,

$$b_n\frac{\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}}$$

go to zero exponentially fast as $n \to \infty$.

Now assume that $a = b = 1$ and $f(x) = x$ when $x < 1/2$ and $f(x) = 1 - x$ when $x > 1/2$. What can you say about the smoothness of f? What can you say about the smoothness of the solution and why?