MATH 5587 (FALL 2019): FINAL

PROFESSOR: SVITLANA MAYBORODA

Name (legibly!): ----------------------------------

Problem 1 (35 points)

Write down the solution to the following initial-boundary value problem in the form of the Fourier series.

$u=X(x)T(t)$

$$u_{tt} = u_{xx} \quad u(t,0) = u(t,\pi) = 0, \quad u(0,x) = 0, \quad u_t(0,x) = 1$$

1.1) (5 pts) Write $u(t,x) = v(t)w(x)$ and find the equations that v and w satisfies

1.2) (5 pts) Solve the equations for v and w and use boundary data $u(t,0)$, $u(t,\pi)$, as well as $u(0,x)$ to identify possible solutions v, w

1.3) (5 pts) Write the global solution $u(t,x)$ as a series and use $u_t(0,x)$ to write integral formulas for coefficients of the series.

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1.4) (5 pts) Evaluate explicitly coefficients of the series and write the final formula for solution $u(t, x)$

1.5) (15 pts) Now solve the same equation on an infinite interval (be careful to evaluate and write the solution legibly using cases depending on the values of x and t):

$$u_{tt} = u_{xx} \quad u(0, x) = 0 \text{ for all } x\,, \quad u_t(0, x) = 1 \text{ for } x < 0\,, \quad u_t(0, x) = 0 \text{ for } x > 0$$

Problem 2 (10 points) Find the Fourier transform of the Gaussian $h(x) = e^{-x^2}$ (you have to show your work, do not use a table value).

Problem 3 (20 points) (a) (10 pts) Find the Green function and write the solution using the Green function representation for the equation $-\frac{d^2u}{dx^2} + 4u = h(x)$.

(b) (10 pts) Now find the Green function and write the solution using the Green function representation for a similar equation on a finite interval:

$$-u''(x) = f(x) \quad \text{on } (0,1), \quad u(0) = 0, \quad u(1) = 2u'(1).$$

Concise Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\mathrm{i}\,\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a\,k}{k}$
$e^{-a\,x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a+\mathrm{i}\,k)}$
$e^{a\,x}\,(1-\sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a-\mathrm{i}\,k)}$
$e^{-a\,\lvert x\rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2+a^2}$
$e^{-a\,x^2}$	$\dfrac{e^{-k^2/(4a)}}{\sqrt{2a}}$
$\tan^{-1} x$	$-\mathrm{i}\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\lvert k\rvert}}{k}$
$f(cx+d)$	$\dfrac{e^{\mathrm{i}\,k\,d/c}}{\lvert c\rvert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.