## MATH 5587 (FALL 2019): FINAL

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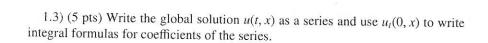
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Problem 1 (35 points)

Write down the solution to the following initial-boundary value problem in the form of the Fourier series.

$$u_{tt} = u_{xx} \quad u(t,0) = u(t,\pi) = 0, \quad u(0,x) = 0, \quad u_t(0,x) = 1$$
1.1) (5 pts) Write  $u(t,x) = v(t)w(x)$  and find the equations that  $v$  and  $w$  satisfies

1.2) (5 pts) Solve the equations for v and w and use boundary data u(t, 0),  $u(t, \pi)$ , as well as u(0, x) to identify possible solutions v, w



1.4) (5 pts) Evaluate explicitly coefficients of the series and write the final formula for solution u(t, x)

1.5) (15 pts) Now solve the same equation on an infinite interval (be careful to evaluate and write the solution legibly using cases depending on the values of x and t):

 $u_{tt} = u_{xx} \quad u(0,x) = 0 \text{ for all } x \; , \quad u_t(0,x) = 1 \text{ for } x < 0 \; , \quad u_t(0,x) = 0 \text{ for } x > 0$ 

**Problem 2 (10 points)** Find the Fourier transform of the Gaussian  $h(x) = e^{-x^2}$  (you have to show your work, do not use a table value).

**Problem 3 (20 points)** (a) (10 pts) Find the Green function and write the solution using the Green function representation for the equation  $-\frac{d^2u}{dx^2} + 4u = h(x)$ .

(b) (10 pts) Now find the Green function and write the solution using the Green function representation for a similar equation on a finite interval:

$$-u''(x) = f(x)$$
 on  $(0, 1)$ ,  $u(0) = 0$ ,  $u(1) = 2u'(1)$ .

Concise Table of Fourier Transforms

f(x)	$\widehat{f}(k)$
1	$\sqrt{2\pi}  \delta(k)$
$\delta(x)$	
$\sigma(x)$	$\sqrt{\frac{\pi}{2}}  \delta(k) - \frac{\mathrm{i}}{\sqrt{2\pi}  k}$
$\operatorname{sign} x$	$-i\sqrt{\frac{2}{\pi}}\frac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\frac{2}{\pi}}  \frac{\sin a  k}{k}$
$e^{-ax}\sigma(x)$	$\frac{1}{\sqrt{2\pi}\left(a+\mathrm{i}k\right)}$
$e^{ax}\left(1-\sigma(x)\right)$	$\frac{1}{\sqrt{2\pi}\left(a-\mathrm{i}k\right)}$
$e^{-a x }$	$\sqrt{\frac{2}{\pi}}  \frac{a}{k^2 + a^2}$
$e^{-ax^2}$	$\frac{e^{-k^2/(4a)}}{\sqrt{2a}}$
$\tan^{-1} x$	$-i\sqrt{\frac{\pi}{2}}\frac{e^{- k }}{k}$
f(cx+d)	$\frac{e^{\mathrm{i}kd/c}}{ c }\widehat{f}\left(\frac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	f(-k)
f'(x)	i $k\widehat{f}(k)$
x f(x)	i $\widehat{f}'(k)$
f * g(x)	$\sqrt{2\pi}\widehat{f}(k)\widehat{g}(k)$

Note: The parameters a, c, d are real, with a > 0 and  $c \neq 0$ .