Problem 1

a)

$$\lambda_n = l^2\pi^2/a^2 + m^2\pi^2/b^2 \text{ with eigenfunctions } \sin(l\pi x/a)\sin(m\pi y/b),\ l, m > 0$$

b)

$$\lambda_n = l^2\pi^2/a^2 + m^2\pi^2/b^2 \quad l, m \geq 0$$

Problem 2

We would like to solve the wave and diffusion equations

$$u_{tt} = c^2\,\Delta u \qquad \textit{and} \qquad u_t = k\,\Delta u$$

in any bounded domain D with one of the classical homogeneous conditions on bdy D and with the standard initial condition. We denote

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{or} \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in two or three dimensions, respectively. For brevity, we continue to use the vector notation $\mathbf{x} = (x, y)$ or (x, y, z). The general discussion that follows works in either dimension, but for definiteness, let's say that we're in three dimensions. Then D is a solid domain and bdy D is a surface.

The first step is to separate the time variable only,

$$u(x, y, z, t) = T(t)v(x, y, z). \tag{1}$$

Then

$$-\lambda = \frac{T''}{c^2T} = \frac{\Delta v}{v} \quad \text{or} \quad -\lambda = \frac{T'}{kT} = \frac{\Delta v}{v}, \tag{2}$$

depending on whether we are doing waves or diffusions. In *either* case we get the eigenvalue problem

$$\boxed{\begin{aligned} -\Delta v = \lambda v \quad &\text{in } D \\ v \text{ satisfies } (D), (N), (R) \quad &\text{on bdy } D. \end{aligned}} \tag{3}$$

Therefore, if this problem has eigenvalues λ_n (all positive, say) and eigenfunctions $v_n(x, y, z) = v_n(\mathbf{x})$, then the solutions of the wave equation are

$$\boxed{u(\mathbf{x}, t) = \sum_n [A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)]\, v_n(\mathbf{x})} \tag{4}$$

and the solutions of the diffusion equation are

$$\boxed{u(\mathbf{x}, t) = \sum_n A_n\, e^{-\lambda_n kt}\, v_n(\mathbf{x}).} \tag{5}$$

As usual, the coefficients will be determined by the initial conditions. However, to carry this out, we'll need to know that the eigenfunctions are orthogonal. This is our next goal. One point of notation in (4) and (5): in three dimensions the index n in the sums (4) and (5) will be a *triple index* [(l, m, n), say] and the various series will be *triple series*, one sum for each coordinate.

6.3.9

We rewrite $f(x,y) = \sigma(3x - 2y - 1)$ in terms of the step function. Thus, by the chain rule,

$$\frac{\partial f}{\partial x} = 3\,\delta(3x - 2y - 1) = \delta\big(x - \tfrac{2}{3}y - \tfrac{1}{3}\big), \qquad \frac{\partial f}{\partial y} = -2\,\delta(3x - 2y - 1) = -\,\delta\big(y - \tfrac{3}{2}y + \tfrac{1}{2}\big).$$

6.3.10

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2 \sin \dfrac{n\pi x}{a} \ \sin \dfrac{n\pi \xi}{a} \ \sinh \dfrac{n\pi(b-y)}{a}}{a \sinh \dfrac{n\pi b}{a}} .$$

Referring back to (4.98), since $|b_n| = \left| \dfrac{2}{a} \sin \dfrac{n\pi\xi}{a} \right| \leq \dfrac{2}{a}$, the uniform bound (4.99) holds, and thus the ensuing argument establishes infinite differentiability.

6.3.18

(a) Using the image point $(\xi, -\eta)$, we find $G(x,y;\xi,\eta) = \dfrac{1}{4\pi} \log \dfrac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}$.

(b) $u(x,y) = \dfrac{1}{4\pi} \displaystyle\int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{1+\eta} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \, d\xi \, d\eta.$

6.3.21

Solution:

(a) We set

$$u(x,y) = \sum_{n=1}^{\infty} b_n(x) \sin n\pi y, \qquad f(x,y) = \sum_{n=1}^{\infty} g_n(x) \sin n\pi y,$$

where

$$b_n(x) = 2\int_0^1 u(x,y) \sin n\pi y \, dy, \qquad g_n(x) = 2\int_0^1 f(x,y) \sin n\pi y \, dy.$$

(b) Substituting into the Poisson equation and the boundary conditions, the resulting boundary value problems for the individual coefficients $b_n(x)$ are

$$-b_n'' + n^2\pi^2 b_n = g_n(x), \qquad b_n(0) = b_n(1) = 0.$$

(c) Using the Green's function solution to the boundary value problem given in (6.65),

$$b_n(x) = \int_0^x \frac{\sinh n\pi(1-x) \sinh n\pi\xi}{n\pi \sinh n\pi} g_n(y) \, d\xi + \int_x^1 \frac{\sinh n\pi x \sinh n\pi(1-\xi)}{n\pi \sinh n\pi} g_n(\xi) \, d\xi.$$

(d) (*i*) $u(x,y) = \dfrac{1}{\pi^2}\left(1 - \dfrac{e^{\pi x} + e^{\pi(1-x)}}{e^{\pi}+1}\right) \sin \pi y = \dfrac{1}{\pi^2}\left(1 - \dfrac{\cosh \pi\left(\frac{1}{2} - x\right)}{\cosh \frac{1}{2}\pi}\right) \sin \pi y;$

(*ii*) $u(x,y) = \dfrac{\sin \pi x \ \sin 2\pi y}{5\pi^2};$

(*iii*) $u(x,y) = \displaystyle\sum_{k=0}^{\infty} \frac{4}{(2k+1)^3\pi^3} \left(1 - \frac{\cosh\left(k+\frac{1}{2}\right)\pi(1-2x)}{\cosh\left(k+\frac{1}{2}\right)\pi}\right) \sin(2k+1)\pi y.$

6.3.23

Set

$$\mathbf{x} = (r\cos\theta, r\sin\theta), \qquad \boldsymbol{\xi} = (\rho\cos\varphi, \rho\sin\varphi).$$

Applying the Law of Cosines to the triangle with vertices $\mathbf{0}, \mathbf{x}, \boldsymbol{\xi}$ in Figure 6.13 shows

$$\|\mathbf{x} - \boldsymbol{\xi}\|^2 = \|\mathbf{x}\|^2 + \|\boldsymbol{\xi}\|^2 - 2\|\mathbf{x}\|\,\|\boldsymbol{\xi}\|\cos(\theta - \varphi) = r^2 + \rho^2 - 2r\rho\cos(\theta - \varphi).$$

Applying the Law of Cosines to the triangle with vertices $\mathbf{0}, \mathbf{x}, \boldsymbol{\eta}$ in Figure 6.13 shows

$$\|\mathbf{x} - \boldsymbol{\eta}\|^2 = \|\mathbf{x}\|^2 + \|\boldsymbol{\eta}\|^2 - 2\|\mathbf{x}\|\,\|\boldsymbol{\eta}\|\cos(\theta - \varphi) = r^2 + \frac{1}{\rho^2} - 2\,\frac{r}{\rho}\,\cos(\theta - \varphi),$$

and so

$$\frac{\|\,\|\boldsymbol{\xi}\|^2\,\mathbf{x} - \boldsymbol{\xi}\,\|^2}{\|\boldsymbol{\xi}\|^2} = \|\boldsymbol{\xi}\|^2\,\|\mathbf{x} - \boldsymbol{\eta}\|^2 = 1 + r^2\rho^2 - 2r\rho\cos(\theta - \varphi).$$

Thus,

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi}\log\frac{\|\,\|\boldsymbol{\xi}\|^2\,\mathbf{x} - \boldsymbol{\xi}\,\|^2}{\|\boldsymbol{\xi}\|^2\,\|\mathbf{x} - \boldsymbol{\xi}\|^2} = \frac{1}{4\pi}\log\left(\frac{1 + r^2\rho^2 - 2r\rho\cos(\theta - \varphi)}{r^2 + \rho^2 - 2r\rho\cos(\theta - \varphi)}\right).$$

6.3.27

Solution:

$$u(t,x) = \tfrac{1}{2}\,\delta(x - ct - a) + \tfrac{1}{2}\,\delta(x + ct - a) \qquad (*)$$

consisting of two half-strength delta spikes traveling away from the starting position concentrated on the two characteristic lines. This solution is the limit of a sequence of classical solutions $u^{(n)}(t,x) \longrightarrow u(t,x)$ as $n \longrightarrow \infty$ which have initial conditions that converge to the delta function: $u^{(n)}(0,x) \longrightarrow \delta(x - a)$, $u_t^{(n)}(0,x) = 0$. For example, using (6.10), the initial conditions

$$u^{(n)}(0,x) = \frac{n}{\pi(1 + n^2(x - a)^2)}$$

lead to the classical solutions

$$u^{(n)}(t,x) = \frac{n}{2\pi(1 + n^2(x - ct - a)^2)} + \frac{n}{2\pi(1 + n^2(x + ct - a)^2)}$$

that converge to the delta function solution $(*)$ as $n \longrightarrow \infty$.

6.3.31

(a) $u(t,x) = \displaystyle\sum_{k=0}^{\infty} \cos\left(k + \tfrac{1}{2}\right)\pi t\ \cos\left(k + \tfrac{1}{2}\right)\pi x;$

(b) For any integer k, and $-1 \le x \le 1$,

$$u(t,x) = \begin{cases} \tfrac{1}{2}\,\delta(x - t + 4k) + \tfrac{1}{2}\,\delta(x + t - 4k), & 4k - 1 < t < 4k + 1, \\ -\tfrac{1}{2}\,\delta(x - t + 4k + 2) - \tfrac{1}{2}\,\delta(x + t - 4k - 2), & 4k + 1 < t < 4k + 3, \\ 0, & t = 2k + 1. \end{cases}$$

(c) Because the Fourier series only converges weakly, it cannot be used to approximate the solution; see Figure 6.7 for the one-dimensional version.