HW 9

Math 5587
Elementary Partial Differential Equations I

Fall 2019
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December 20, 2019 Compiled on December 20, 2019 at 10:38am

Contents

1 Problem 1

Find the eigenvalues and the eigenfunctions for the Dirichlet and Neumann problems for the Laplacian on a rectangle $(0,a)\times(0,b)$

Solution

1.1 Dirichlet case

$$
\begin{aligned}
\nabla^2 u &= -\lambda u\\
u(x,0) &= 0\\
u(x,b) &= 0\\
u\left(0,y\right) &= 0\\
u\left(a,y\right) &= 0
\end{aligned}
$$

Let $u\left(x,y\right)=X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-\lambda u$ gives

$$X''Y+Y''X=-\lambda XY$$

Dividing by $XY\neq 0$ gives

$$
\begin{aligned}
\frac{X''}{X}+\frac{Y''}{Y} &= -\lambda\\
\frac{X''}{X} &= -\frac{Y''}{Y}-\lambda
\end{aligned}
$$

Since the LHS depends on x only and the RHS depends on y only and they are equal, they must be both constant. Say $-\mu$. The above becomes

$$\frac{X''}{X}=-\frac{Y''}{Y}-\lambda=-\mu$$

Two ODE's are therefore obtained from the above. They are

$$
\begin{aligned}
X''+\mu X &= 0 \qquad (1)\\
X(0) &= 0\\
X(a) &= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{Y''}{Y}+\lambda &= \mu\\
\frac{Y''}{Y}+\left(\lambda-\mu\right) &= 0
\end{aligned}
$$

Let $\left(\lambda-\mu\right)=\gamma$ constant. Hence the above gives the second ODE in y as

$$
\begin{aligned}
Y''+\gamma Y &= 0 \qquad (2)\\
Y(0) &= 0\\
Y(b) &= 0
\end{aligned}
$$

Now the eigenvalues μ,γ and eigenfunctions for each ODE is found and from that result the eigenvalue λ is found using

$$\lambda=\gamma+\mu \qquad (3)$$

Starting with ODE (1) $X''+\mu X=0$

Case $\mu<0$

The solution to (1) is

$$X=A\cosh\left(\sqrt{|\mu|}x\right)+B\sinh\left(\sqrt{|\mu|}x\right)$$

At $x=0$, the above gives $0=A$. Hence $X=B\sinh\left(\sqrt{|\mu|}x\right)$. At $x=a$ this gives $0=B\sinh\left(\sqrt{|\mu|}a\right)$. But $\sinh\left(\sqrt{|\mu|}a\right)=0$ only at 0 and $\sqrt{|\mu|}a\neq 0$, therefore $B=0$ and this leads to trivial solution. Hence $\mu<0$ is not an eigenvalue.

Case $\mu = 0$

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = a$, $B = 0$. Hence the trivial solution. $\mu = 0$ is not an eigenvalue.

Case $\mu > 0$

Solution is

$$X = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\mu}x\right)$. At $x = a$

$$0 = B\sin\left(\sqrt{\mu}a\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\mu}a\right) = 0$ or $\sqrt{\mu}a = k\pi$ where $k = 1, 2, 3, \cdots$, therefore

$$\mu_k = \left(\frac{k\pi}{a}\right)^2 \qquad k = 1, 2, 3, \cdots \tag{4}$$

The corresponding Eigenfunctions are

$$X_k(x) = \sin\left(\frac{k\pi}{a}x\right) \qquad k = 1, 2, 3, \cdots \tag{5}$$

Solving ODE (2) $Y'' + \gamma Y = 0$

The same steps are repeated as above. The only difference is that now we obtain eigenvalues

$$\gamma_m = \left(\frac{m\pi}{b}\right)^2 \qquad m = 1, 2, 3, \cdots \tag{6}$$

And the corresponding eigenfunctions

$$Y_m(y) = \sin\left(\frac{m\pi}{b}y\right) \qquad m = 1, 2, 3, \cdots \tag{7}$$

From (4,6) we see that the eigenvalues for $\nabla^2 u = -\lambda u$ are, using (3)

$$\begin{aligned}\lambda_{k,m} &= \mu_k + \gamma_m \\ &= \left(\frac{k\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \qquad k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots\end{aligned}$$

And the eigenfunctions are from (5,7) are

$$\Phi_{k,m}(x, y) = \sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) \qquad k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots$$

1.2 Neumann case

$$\begin{aligned}\nabla^2 u &= -\lambda u \\ \frac{\partial}{\partial y}u(x, 0) &= 0 \\ \frac{\partial}{\partial y}u(x, b) &= 0 \\ \frac{\partial}{\partial x}u(0, y) &= 0 \\ \frac{\partial}{\partial x}u(a, y) &= 0\end{aligned}$$

Let $u(x, y) = X(x)Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u$ gives

$$X''Y + Y''X = -\lambda XY$$

Dividing by $XY \neq 0$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$
$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda$$

Since the LHS depends on x only and the RHS depends on y only and they are equal, they must be both constant. Say $-\mu$. The above becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

Two ODE's are therefore obtained from the above. They are

$$\begin{aligned} X'' + \mu X &= 0 \\ X'(0) &= 0 \\ X'(a) &= 0 \end{aligned} \tag{1}$$

And

$$\frac{Y''}{Y} + \lambda = \mu$$
$$\frac{Y''}{Y} + (\lambda - \mu) = 0$$

Let $(\lambda - \mu) = \gamma$ constant. Hence the above gives the second ODE in y as

$$\begin{aligned} Y'' + \gamma Y &= 0 \\ Y'(0) &= 0 \\ Y'(b) &= 0 \end{aligned} \tag{2}$$

Now we find the eigenvalues μ, γ and eigenfunctions for each ODE and from this result find

$$\lambda = \gamma + \mu \tag{3}$$

Starting with ODE (1) $X'' + \mu X = 0$

Case $\mu < 0$

The solution to (1) is

$$X(x) = A\cosh\left(\sqrt{|\mu|}x\right) + B\sinh\left(\sqrt{|\mu|}x\right)$$
$$X'(x) = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}x\right) + B\sqrt{|\mu|}\cosh\left(\sqrt{|\mu|}x\right)$$

At $x = 0$, the above gives $0 = B$. Hence $X(x) = A\cosh\left(\sqrt{|\mu|}x\right)$ and $X'(x) = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}x\right)$ At $x = a$ this gives $0 = A\sqrt{|\mu|}\sinh\left(\sqrt{|\mu|}a\right)$. But $\sinh\left(\sqrt{|\mu|}a\right) = 0$ only at 0 and $\sqrt{|\mu|}a \neq 0$, therefore $A = 0$ and this leads to trivial solution. Hence $\mu < 0$ is not an eigenvalue.

Case $\mu = 0$

$$X = Ax + B$$
$$X' = A$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B$, therefore $X' = 0$. At $x = a$, $0 = 0$. Hence any constant B will work. Let this constant be C_0. Therefore $\mu = 0$ is an eigenvalue with corresponding eigenfunction $X_0(x) = C_0$, a constant.

Case $\mu > 0$

Solution is

$$X(x) = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$
$$X'(x) = -A\sqrt{\mu}\sin\left(\sqrt{\mu}x\right) + B\sqrt{\mu}\cos\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives $0 = B$ and the solution becomes $X(x) = A\cos\left(\sqrt{\mu}x\right)$. Hence $X'(x) = -A\sqrt{\mu}\sin\left(\sqrt{\mu}x\right)$. At $x = a$ this gives

$$0 = -A\sqrt{\mu}\sin\left(\sqrt{\mu}a\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\mu}a\right) = 0$ or $\sqrt{\mu}a = k\pi$ where $k = 1, 2, 3, \cdots$, therefore

$$\mu_k = \left(\frac{k\pi}{a}\right)^2 \qquad k = 1, 2, 3, \cdots \tag{4}$$

The corresponding Eigenfunctions are

$$X_k(x) = \cos\left(\frac{k\pi}{a}x\right) \qquad k = 1, 2, 3, \cdots \tag{5}$$

Solving ODE (2) $Y'' + \gamma Y = 0$

The same steps are repeated as above. The only difference is that now we obtain eigenvalues $\gamma = 0$ also and corresponding eigenfunction constant, say D_0 and also obtain

$$\gamma_m = \left(\frac{m\pi}{b}\right)^2 \qquad m = 1, 2, 3, \cdots \tag{6}$$

and corresponding eigenfunctions

$$Y_m(y) = \cos\left(\frac{m\pi}{b}y\right) \qquad m = 1, 2, 3, \cdots \tag{7}$$

From (4,6) we see that the eigenvalues for $\nabla^2 u = -\lambda u$ are

$$\begin{aligned}\lambda_{k,m} &= \begin{cases} 0 & k = 0, m = 0 \\ \mu_k + \gamma_m & k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{cases} \\ &= \begin{cases} 0 & k = 0, m = 0 \\ \left(\frac{k\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 & k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{cases}\end{aligned}$$

And the eigenfunctions are from (5,7) are

$$\Phi_n(x, y) = \begin{cases} 1 & k = 0, m = 0 \\ \cos\left(\frac{k\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right) & k = 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{cases}$$

Where in the above the constant eigenfunction that corresponds to the zero eigenvalue is taken as 1.

2 Problem 2

Prove that the wave equation $u_{tt}(x,t) = c^2\nabla^2 u,\ t > 0,\ x \in \Omega \in \Re^d$ with the Dirichlet boundary conditions $u(x,t) = 0$ for $x \in \partial\Omega, t > 0$ has solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) \tag{1}$$

Where λ_n, v_n are respectively, eigenvalues and eigenfunctions of the Dirichlet problem for the Laplacian in Ω. Write in an analogous form the solution to the heat equation $u_t(x,t) = c\nabla^2 u,\ t > 0,\ x \in \Omega \in \Re^d$ with Dirichlet boundary conditions $u(x,t) = 0$ for $x \in \partial\Omega, t > 0$.

For the wave PDE

We will show the solution given solves the PDE by substituting it into the PDE and see if it gives an identity.

$$u_t(x,t) = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x)$$

Assuming continuouseigenfunctions, term by term differential is allowed, and the above becomes

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \frac{\partial}{\partial t}\left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) \\ &= \sum_{n=1}^{\infty} \left(-A_n\sqrt{\lambda_n} c \sin\left(\sqrt{\lambda_n} ct\right) + B_n\sqrt{\lambda_n} c \cos\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) \end{aligned}$$

Taking one more time derivatives gives

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} \left(-A_n\lambda_n c^2 \cos\left(\sqrt{\lambda_n} ct\right) - B_n\lambda_n c^2 \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) \tag{2}$$

Similarly for the spatial coordinate

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) \\ &= \sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n'(x) \end{aligned}$$

Taking one more space derivatives gives

$$\nabla^2 u = \sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n''(x)$$

But since $v_n(x)$ is an eigenfunction, then $-v_n''(x) = \lambda_n v_n$ and the above simplifies to

$$\nabla^2 u = -\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x) \tag{3}$$

Substituting (2,3) into $u_{tt}(x,t) = c^2\nabla^2 u$ gives

$$\begin{aligned} \sum_{n=1}^{\infty} \left(-A_n\lambda_n c^2 \cos\left(\sqrt{\lambda_n} ct\right) - B_n\lambda_n c^2 \sin\left(\sqrt{\lambda_n} ct\right)\right) v_n(x) &= c^2\left(-\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x)\right) \\ c^2\sum_{n=1}^{\infty} \left(-A_n \cos\left(\sqrt{\lambda_n} ct\right) - B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x) &= -c^2\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x) \\ -c^2\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x) &= -c^2\sum_{n=1}^{\infty} \left(A_n \cos\left(\sqrt{\lambda_n} ct\right) + B_n \sin\left(\sqrt{\lambda_n} ct\right)\right) \lambda_n v_n(x) \end{aligned}$$

The LHS is the same as the RHS. Hence the solution given satisfies the wave PDE.

For the heat PDE

For the heat PDE, we want to show that the following solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n(x) \tag{4}$$

Satisfies $u_t(x,t) = c\nabla^2 u$.

$$u_t(x,t) = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} A_n e^{-\lambda_n ct} v_n(x)$$

Assuming term by term differential is allowed the above becomes

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{\partial}{\partial t} A_n e^{-\lambda_n c t} v_n(x)$$
$$= \sum_{n=1}^{\infty} -A_n \lambda_n c e^{-\lambda_n c t} v_n(x) \tag{5}$$

Similarly for the spatial coordinate

$$u_x(x,t) = \frac{\partial}{\partial x} \sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} v_n(x)$$
$$= \sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} v_n'(x)$$

Taking one more space derivatives gives

$$\nabla^2 u = \sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} v_n''(x)$$

But since $v_n(x)$ is an eigenfunction, then $-v_n''(x) = \lambda_n v_n$. The above becomes

$$\nabla^2 u = -\sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} \lambda_n v_n(x) \tag{6}$$

Substituting (5,6) into $u_t(x,t) = c\nabla^2 u$ gives

$$\sum_{n=1}^{\infty} -A_n \lambda_n c e^{-\lambda_n c t} v_n(x) = c\left(-\sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} \lambda_n v_n(x)\right)$$
$$-c\sum_{n=1}^{\infty} A_n \lambda_n e^{-\lambda_n^2 c t} v_n(x) = -c\sum_{n=1}^{\infty} A_n e^{-\lambda_n c t} \lambda_n v_n(x)$$

The LHS is the same as the RHS. Hence the solution (4) satisfies the heat PDE.

3 Problem 6.3.9

Suppose $f\left(x,y\right)=\begin{cases} 1 & 3x-2y>1 \\ 0 & 3x-2y<1 \end{cases}$ Compute its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ the sense of generalized functions.

Solution

The following is a plot of the above function in 3D

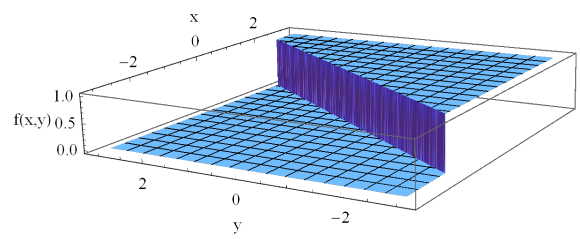

Figure 1: Plot of $f(x,y)$

```
f[x_, y_] := Piecewise[{{1, 3 x - 2 y > 1}, {0, 3 x - 2 y > 1}}]
p = ParametricPlot3D[{x, y, f[x, y]}, {x, -3, 3}, {y, -3, 3},
        AxesLabel → {"x", "y", "f(x,y)"}, ImageSize → 400,
        BaseStyle → 12, Exclusions → True,
        ExclusionsStyle → LightGray, PlotTheme -> "Classic", PlotPoints → 50];
```

Figure 2: Code used for the above plot

Similar to what we did in 1D, when taking a derivative and there is a jump discontinuity, an impulse $\delta(x)$ is generated at the location where the jump discontinuity is located. The location of the jump here is on the line $3x-2y-1=0$. This is a step function but in 3D. Hence by chain rule

$$\begin{aligned}\frac{\partial f\left(x,y\right)}{\partial x} &= \frac{\partial}{\partial x}\left(3x-2y-1\right)\delta\left(3x-2y-1\right)\\ &= 3\delta\left(3x-2y-1\right)\\ &= \delta\left(x-\frac{2}{3}y-\frac{1}{3}\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial f\left(x,y\right)}{\partial y} &= \frac{\partial}{\partial y}\left(3x-2y-1\right)\delta\left(3x-2y-1\right)\\ &= -2\delta\left(3x-2y-1\right)\\ &= \delta\left(-\frac{3}{2}x+y+\frac{1}{2}\right)\end{aligned}$$

4 Problem 6.3.10

Find a series solution to the rectangular boundary value problem 4.91-92 which is

$$\begin{aligned}
\nabla^2 u &= 0 \qquad \text{on a rectangle} \qquad R = \left\{0 < x < a, 0 < y < b\right\} \\
u(x,0) &= f(x) \\
u(x,b) &= 0 \\
u\left(0,y\right) &= 0 \\
u\left(a,y\right) &= 0
\end{aligned}$$

when the boundary data $f(x) = \delta(x-\xi)$ is a delta function at a point $0 < \xi < a$. Is your solution infinitely differentiable inside the rectangle?

Solution

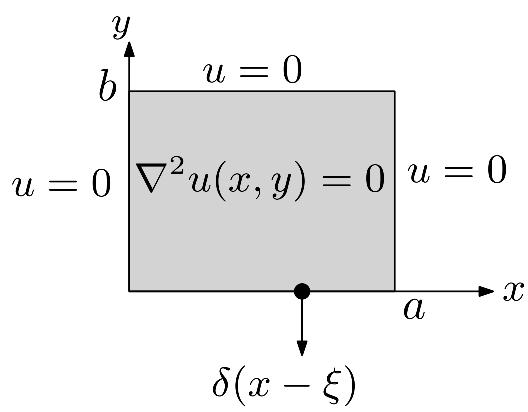

Figure 3: The problem to solve. Laplace PDE in rectangle

Let $u\left(x,y\right) = X(x)Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Each side depends on different independent variable and they are equal, therefore they must be equal to same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm\lambda$$

Since the boundary conditions along the x direction are the homogeneous ones, $-\lambda$ is selected in the above.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Two ODE's are obtained

$$X'' + \lambda X = 0 \tag{1}$$

With the boundary conditions

$$\begin{aligned} X(0) &= 0 \\ X(a) &= 0 \end{aligned}$$

And

$$Y'' - \lambda Y = 0 \tag{2}$$

With the boundary conditions

$$\begin{aligned} Y(0) &= f(x) \\ Y(b) &= 0 \end{aligned}$$

In all these cases λ will turn out to be positive. This is shown below.

Case $\lambda < 0$

The solution to (1) is

$$X = A\cosh\left(\sqrt{|\lambda|}x\right) + B\sinh\left(\sqrt{|\lambda|}x\right)$$

At $x = 0$, the above gives $0 = A$. Hence $X = B\sinh\left(\sqrt{|\lambda|}x\right)$. At $x = a$ this gives $X = B\sinh\left(\sqrt{|\lambda|}a\right)$. But $\sinh\left(\sqrt{|\lambda|}a\right) = 0$ only at 0 and $\sqrt{|\lambda|}a \neq 0$, therefore $B = 0$ and this leads to trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = a$, $B = 0$. Hence the trivial solution. $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

Solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\lambda}x\right)$. At $x = a$

$$0 = B\sin\left(\sqrt{\lambda}a\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}a\right) = 0$ or $\sqrt{\lambda}a = n\pi$ where $n = 1, 2, 3, \cdots$, therefore

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Eigenfunctions are

$$X_n(x) = B_n \sin\left(\frac{n\pi}{a}x\right) \qquad n = 1, 2, 3, \cdots \tag{3}$$

For the Y ODE, the solution is

$$Y_n = C_n \cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right) \tag{4}$$

Applying B.C. at $y = b$ gives

$$\begin{aligned}
0 &= C_n \cosh\left(\frac{n\pi}{a}b\right) + D_n \sinh\left(\frac{n\pi}{a}b\right)\\
C_n &= -D_n \frac{\sinh\left(\frac{n\pi}{a}b\right)}{\cosh\left(\frac{n\pi}{a}b\right)}\\
&= -D_n \tanh\left(\frac{n\pi}{a}b\right)
\end{aligned}$$

Hence (4) becomes

$$\begin{aligned}
Y_n &= -D_n \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right) + D_n \sinh\left(\frac{n\pi}{a}y\right)\\
&= D_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)
\end{aligned}$$

Now the complete solution is produced

$$\begin{aligned}
u_n(x, y) &= Y_n X_n\\
&= D_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right) B_n \sin\left(\frac{n\pi}{a}x\right)
\end{aligned}$$

Let $D_n B_n = B_n$ since a constant. (no need to make up a new symbol).

$$u_n(x, y) = B_n\left(\sinh\left(\frac{n\pi}{a}y\right) - \tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)$$

Sum of eigenfunctions is the solution, hence

$$u\left(x,y\right)=\sum_{n=1}^{\infty}B_n\left(\sinh\left(\frac{n\pi}{a}y\right)-\tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right) \tag{5}$$

The nonhomogeneous boundary condition is now resolved. At $y=0$

$$u\left(x,0\right)=f\left(x\right)=\delta\left(x-\xi\right)$$

Therefore (5) becomes

$$\delta\left(x-\xi\right)=\sum_{n=1}^{\infty}-B_n\tanh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{a}x\right)$ and integrating gives

$$\begin{aligned}\int_0^a\delta\left(x-\xi\right)\sin\left(\frac{m\pi}{a}x\right)dx &= -\int_0^a\sin\left(\frac{m\pi}{a}x\right)\sum_{n=1}^{\infty}B_n\tanh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right)dx\\ &= -\sum_{n=1}^{\infty}B_n\tanh\left(\frac{n\pi}{a}b\right)\int_0^a\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx\\ &= -B_n\tanh\left(\frac{m\pi}{a}b\right)\left(\frac{a}{2}\right)\end{aligned}$$

Hence

$$B_n=-\frac{2}{a}\frac{\int_0^a\delta\left(x-\xi\right)\sin\left(\frac{n\pi}{a}x\right)dx}{\tanh\left(\frac{n\pi}{a}b\right)}$$

But $\int_0^a\delta\left(x-\xi\right)\sin\left(\frac{m\pi}{L}x\right)dx=\sin\left(\frac{m\pi}{L}\xi\right)$ by the property delta function. Therefore

$$B_n=-\frac{2}{a}\frac{\sin\left(\frac{n\pi}{a}\xi\right)}{\tanh\left(\frac{n\pi}{a}b\right)}$$

This completes the solution. (4) becomes

$$\begin{aligned}u\left(x,y\right)&=-\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{a}\xi\right)}{\tanh\left(\frac{n\pi}{a}b\right)}\left(\sinh\left(\frac{n\pi}{a}y\right)-\tanh\left(\frac{n\pi}{a}b\right)\cosh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}x\right)\\ &=-\frac{2}{a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}\xi\right)\sin\left(\frac{n\pi}{a}x\right)\left(\frac{\sinh\left(\frac{n\pi}{a}y\right)}{\tanh\left(\frac{n\pi}{a}b\right)}-\cosh\left(\frac{n\pi}{a}y\right)\right)\end{aligned}$$

Looking at the solution above, it is composed of functions that are all differentiable. Hence the solution is infinitely differentiable inside the rectangle.

Here is a plot of the above solution using $a=\pi,b=\frac{1}{2},\xi=1$.

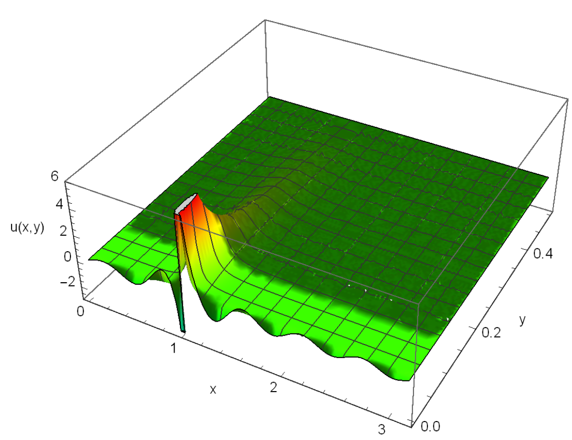

Figure 4: Plot of $u(x,y)$

```
u[x_, y_, ξ_] := (-2/a) Sum[Sin[(n π)/a ξ] Sin[(n π)/a x] (Sinh[(n π)/a y]/Tanh[(n π)/a b] - Cosh[(n π)/a y]), {n, 1, 300}];
a = Pi; b = 1/2; ξ = 1;
p = Plot3D[u[x, y, ξ], {x, 0, a}, {y, 0, b}, PlotRange → {Automatic, Automatic, {-3, 7}},
        PlotPoints → 40, AxesLabel → {"x", "y", "u(x,y)"},
        ColorFunction → Function[{x, y, z}, Hue[.45 (1 - z)]]];
```

Figure 5: Code used for the above plot

5 Problem 6.3.18

(a) Use the Method of Images to construct the Green's function for a half-plane $\{y > 0\}$ that is subject to homogeneous Dirichlet boundary conditions. Hint : The image point is obtained by reflection. (b) Use your Green's function to solve the boundary value problem With $y > 0, u(x, 0) = 0$

$$-\Delta u = \frac{1}{1+y}$$

Solution

5.1 Part (a)

The first step is to find Green function in the half-plane $G(x, y; x_0, y_0)$. To do this we will use Green function in the whole plane, called $\Gamma(x, y; x_0, y_0)$. There (x, y) is an arbitrary point in upper half plane and (x_0, y_0) is fixed point where the impulse is located. We set an impulse at the point (x_0, y_0) and a negative impulse at $(x_0, -y_0)$. This way the end effect is that at the boundary which is $x = 0$ the half plane Green function is zero which satisfies the boundary conditions of the given PDE. The following diagram helps illustrate this setup

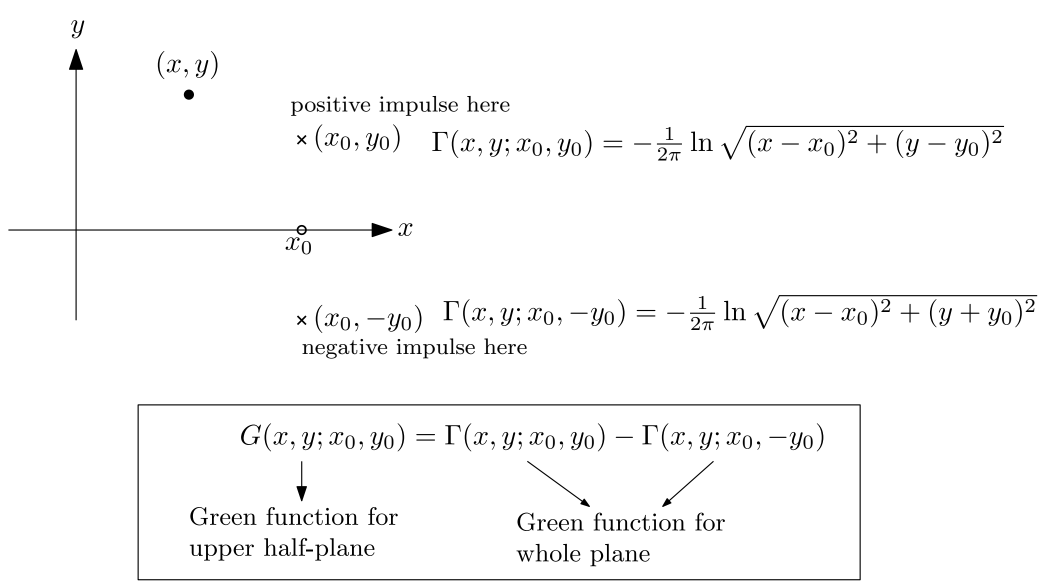

Figure 6: Using method of images

Hence

$$\begin{aligned} G(x, y; x_0, y_0) &= -\frac{1}{2\pi} \ln\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right) + \frac{1}{2\pi} \ln\left(\sqrt{(x - x_0)^2 + (y + y_0)^2}\right) \\ &= -\frac{1}{4\pi} \ln\left((x - x_0)^2 + (y - y_0)^2\right) + \frac{1}{4\pi} \ln\left((x - x_0)^2 + (y + y_0)^2\right) \\ &= \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2} \end{aligned}$$

5.2 Part b

Now that the Green function is known, the solution is

$$
\begin{aligned}
u\left(x,y\right) &= \int_{-\infty}^{x}\int_{0}^{y} G\left(x,y;x_0,y_0\right) f\left(x_0,y_0\right) dx_0 dy_0 \\
&= \int_{-\infty}^{x}\int_{0}^{y} \frac{1}{4\pi}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right)\left(\frac{1}{1+y_0}\right) dx_0 dy_0 \\
&= \frac{1}{4\pi}\int_{0}^{y}\left(\frac{1}{1+y_0}\right)\left(\int_{-\infty}^{x}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right) dx_0\right) dy_0 \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\int_{-\infty}^{x}\ln\left(\frac{(x-x_0)^2+\left(y+y_0\right)^2}{(x-x_0)^2+\left(y-y_0\right)^2}\right) dx_0 &= 2y_0\pi - x\ln\left(\left(y+y_0\right)^2\right) + x\ln\left(\frac{\left(y+y_0\right)^2}{\left(y-y_0\right)^2}\right) + x\ln\left(\left(y-y_0\right)^2\right) \\
&= 2y_0\pi - 2x\ln\left(y+y_0\right) + 2x\ln\left(\frac{y+y_0}{y-y_0}\right) + 2x\ln\left(y-y_0\right) \\
&= 2y_0\pi + 2x\ln\left(\frac{y-y_0}{y+y_0}\right) + 2x\ln\frac{y+y_0}{y-y_0} \\
&= 2y_0\pi + 2x\left(\ln\frac{y-y_0}{y+y_0} + \ln\frac{y+y_0}{y-y_0}\right) \\
&= 2y_0\pi + 2x\ln\left(\frac{y-y_0}{y+y_0}\frac{y+y_0}{y-y_0}\right) \\
&= 2y_0\pi
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
u\left(x,y\right) &= \frac{1}{2}\int_{0}^{y}\frac{y_0}{1+y_0} dy_0 \\
&= \frac{1}{2}\left(y_0 - \ln\left(y_0+1\right)\right)_0^y \\
&= \frac{1}{2}\left(y - \ln\left(y+1\right)\right)
\end{aligned}
$$

Checking: When $y=0$ then $u\left(x,y\right) = -\frac{1}{2}\ln(1) = 0$. Ok. Solution does not depend on x but only on y.

6 Problem 6.3.21

Provide the details for the following alternative method for solving the homogeneous Dirichlet boundary value problem for the Poisson equation on the unit square:

$$
\begin{aligned}
u_{xx}+u_{yy} &= -f\left(x,y\right) \qquad 0<x,y<1\\
u\left(x,0\right) &= 0\\
u\left(x,1\right) &= 0\\
u\left(0,y\right) &= 0\\
u\left(1,y\right) &= 0
\end{aligned}
$$

(a) Write both $u(x,y)$ and $f(x,y)$ as Fourier sine series in y whose coefficients depend on x. (b) Substitute these series into the differential equation, and equate Fourier coefficients to obtain an infinite system of ordinary boundary value problems for the x-dependent Fourier coefficients of u. (c) Use the Green's functions for each boundary value problem to write out the solution and hence a series for the solution to the original boundary value problem. (d) Implement this method for the following forcing functions (i) $f\left(x,y\right)=\sin\left(\pi y\right)$, (ii) $f\left(x,y\right)=\sin\left(\pi x\right)\sin\left(2\pi y\right)$, (iii) $f\left(x,y\right)=1$.

Solution

6.1 Part a

Let

$$
\begin{aligned}
u\left(x,y\right) &= \sum_{n=1}^{\infty}A_n\left(x\right)\sin\left(\sqrt{\lambda_n}y\right)\\
f\left(x,y\right) &= \sum_{n=1}^{\infty}B_n\left(x\right)\sin\left(\sqrt{\lambda_n}y\right)
\end{aligned}
$$

The eigenvalues are known to be $\lambda_n=n^2\pi^2$ for $n=1,2,\cdots$ for these boundary conditions on $x=0$ to $x=1$. Hence the above becomes

$$
u\left(x,y\right) = \sum_{n=1}^{\infty}A_n\left(x\right)\sin\left(n\pi y\right) \tag{1}
$$

$$
f\left(x,y\right) = \sum_{n=1}^{\infty}B_n\left(x\right)\sin\left(n\pi y\right) \tag{2}
$$

6.2 Part b

From (1)

$$
\begin{aligned}
u_x &= \sum_{n=1}^{\infty}A_n'\left(x\right)\sin\left(n\pi y\right)\\
u_{xx} &= \sum_{n=1}^{\infty}A_n''\left(x\right)\sin\left(n\pi y\right)\\
u_y &= \sum_{n=1}^{\infty}n\pi A_n\left(x\right)\cos\left(n\pi y\right)\\
u_{yy} &= -\sum_{n=1}^{\infty}n^2\pi^2A_n\left(x\right)\sin\left(n\pi y\right)
\end{aligned}
$$

Substituting the above back into the original $u_{xx}+u_{yy}=-f\left(x,y\right)$ gives

$$
\begin{aligned}
\sum_{n=1}^{\infty}A_n''\left(x\right)\sin\left(n\pi y\right)-\sum_{n=1}^{\infty}n^2\pi^2A_n\left(x\right)\sin\left(n\pi y\right) &= -\sum_{n=1}^{\infty}B_n\left(x\right)\sin\left(n\pi y\right)\\
\sum_{n=1}^{\infty}\left(A_n''\left(x\right)-n^2\pi^2A_n\left(x\right)\right)\sin\left(n\pi y\right) &= -\sum_{n=1}^{\infty}B_n\left(x\right)\sin\left(n\pi y\right)
\end{aligned}
$$

Equating coefficients in the above gives

$$
A_n''\left(x\right)-n^2\pi^2A_n\left(x\right)=-B_n\left(x\right)
$$

For all $n=1,2,\cdots$. This is an infinite system of ordinary boundary value problems in $A\left(x\right)$ where $B_n\left(x\right)$ acts as the external input.

6.3 Part c

We now want to find Green function for $A_n''(x) - n^2\pi^2 A_n(x) = 0$ with $A_n(0) = 0, A_n(1) = 0$. The solution is

$$A_n(x) = A\cosh(n\pi x) + B\sinh(n\pi x)$$

Hence the Green function is

$$G(x;x_0) = \begin{cases} A_1\cosh(n\pi x) + B_1\sinh(n\pi x) & x < x_0 \\ A_2\cosh(n\pi x) + B_2\sinh(n\pi x) & x > x_0 \end{cases}$$

At $x = 0$, the top branch gives $0 = A_1$ and at $x = 1$ the lower branch gives $A_2\cosh(n\pi) + B_2\sinh(n\pi) = 0$ or $A_2 = -B_2\tanh(n\pi)$. Using these in the above gives

$$\begin{aligned} G(x;x_0) &= \begin{cases} B_1\sinh(n\pi x) & x < x_0 \\ -B_2\tanh(n\pi)\cosh(n\pi x) + B_2\sinh(n\pi x) & x > x_0 \end{cases} \\ &= \begin{cases} B_1\sinh(n\pi x) & x < x_0 \\ B_2(\sinh(n\pi x) - \tanh(n\pi)\cosh(n\pi x)) & x > x_0 \end{cases} \end{aligned} \tag{1A}$$

There are two unknowns B_1, B_2 to solve for. Hence we need two equations. The first equation is found by equating the above Green function at $x = x_0$. This gives

$$B_1\sinh(n\pi x_0) = B_2(\sinh(n\pi x_0) - \tanh(n\pi)\cosh(n\pi x_0)) \tag{1}$$

Taking derivatives of $G(x;x_0)$ gives

$$\frac{d}{dx}G(x;x_0) = \begin{cases} n\pi B_1\cosh(n\pi x) & x < x_0 \\ B_2(n\pi\cosh(n\pi x) - n\pi\tanh(n\pi)\sinh(n\pi x)) & x > x_0 \end{cases}$$

The second equation is found by the condition of the jump discontinutiy on the above derivative at $x = x_0$. Hence

$$n\pi B_1\cosh(n\pi x_0) - B_2(n\pi\cosh(n\pi x_0) - n\pi\tanh(n\pi)\sinh(n\pi x_0)) = 1 \tag{2}$$

Solving (1,2) for B_1, B_2 gives

$$\begin{aligned} B_1 &= \frac{\cosh(n\pi x_0) - \coth(n\pi)\sinh(n\pi x_0)}{n\pi} = \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0 - 1)) \\ B_2 &= \frac{\coth(n\pi)\sinh(n\pi x_0)}{n\pi} = \frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)} \end{aligned}$$

Substituting these back in (1A) gives the final Green function

$$\begin{aligned} G(x;x_0) &= \begin{cases} \begin{cases} \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x < x_0 \\ \frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)}(\sinh(n\pi x) - \tanh(n\pi)\cosh(n\pi x)) & x > x_0 \end{cases} & x < x_0 \\ & x > x_0 \end{cases} \\ &= \begin{cases} \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x < x_0 \\ \sinh(n\pi x)\frac{\sinh(n\pi x_0)}{n\pi\tanh(n\pi)} - \frac{\sinh(n\pi x_0)}{n\pi}\cosh(n\pi x) & x > x_0 \end{cases} \\ &= \begin{cases} \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x) & x < x_0 \\ \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\sinh(n\pi x_0) & x > x_0 \end{cases} \end{aligned}$$

Now that the Green function is found, the solution to $A_n''(x) - n^2\pi^2 A_n(x) = B_n(x)$ is given by

$$\begin{aligned} A_n(x) = &\int_0^x \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\sinh(n\pi x_0)B_n(x_0)\,dx_0 \\ &+ \int_x^1 \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x_0-1))\sinh(n\pi x)B_n(x_0)\,dx_0 \end{aligned}$$

Or

$$\begin{aligned} A_n(x) = &\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\int_0^x \sinh(n\pi x_0)B_n(x_0)\,dx_0 \\ &+ \frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi x)\int_x^1 \sinh(n\pi(x_0-1))B_n(x_0)\,dx_0 \end{aligned} \tag{3}$$

Now that $A_n(x)$ is found, the solution to the PDE is found from

$$u(x,y)=\sum_{n=1}^{\infty}A_n(x)\sin(n\pi y)$$

Where $A_n(x)$ is given by (3). $B_n(x)$ is the Fourier series coefficient of $f(x,y)$ which needs to be found depending on $f(x,y)$. This is done below.

6.4 Part d

(i) $f(x,y)=\sin(\pi y)$

We first need to find the Fourier coefficients $B_n(x)$. Since $f(x,y)=\sum_{n=1}^{\infty}B_n(x)\sin(n\pi y)$, then multiplying both sides by $\sin(m\pi y)$ and integrating gives

$$\begin{aligned}\int_0^1\sin(\pi y)\sin(m\pi y)\,dy&=\sum_{n=1}^{\infty}B_n(x)\int_0^1\sin(m\pi y)\sin(n\pi y)\,dy\\&=\frac{1}{2}B_m(x)\end{aligned}$$

Therefore

$$B_n(x)=2\int_0^1\sin(\pi y)\sin(n\pi y)\,dy$$

For $n=1$ the above becomes

$$B_1(x)=2\int_0^1\sin^2(\pi y)\,dy=1$$

And for all other terms $B_n=0$ due to orthogonality of sin functions. Therefore now that $B_n(x)$ is found, then from (3) $A_n(x)$ can be found. Only $n=1$ term is needed.

$$\begin{aligned}A_1(x)&=\frac{1}{\pi\sinh(\pi)}\sinh(\pi(x-1))\int_0^x\sinh(\pi x_0)\,dx_0+\frac{1}{\pi\sinh(\pi)}\sinh(\pi x)\int_x^1\sinh(\pi(x_0-1))\,dx_0\\&=\frac{1}{\pi\sinh(\pi)}\sinh(\pi(x-1))\left[\frac{\cosh(\pi x_0)}{\pi}\right]_0^x+\frac{1}{\pi\sinh(\pi)}\sinh(\pi x)\left[\frac{\cosh(\pi(x_0-1))}{\pi}\right]_x^1\\&=\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi(x-1))(\cosh(\pi x)-1)+\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi x)(1-\cosh(\pi(x-1)))\end{aligned}$$

Hence the solution to the PDE is

$$\begin{aligned}u(x,y)&=\sum_{n=1}^{\infty}A_n(x)\sin(n\pi y)\\&=A_1(x)\sin(\pi y)\\&=\left(\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi(x-1))(\cosh(\pi x)-1)+\frac{1}{\pi^2\sinh(\pi)}\sinh(\pi x)(1-\cosh(\pi(x-1)))\right)\sin(\pi y)\\&=\frac{1}{\pi^2\sinh\pi}(\sinh(\pi(x-1))-\sinh(\pi x)+\sinh\pi)\sin(\pi y)\end{aligned}$$

The following is a plot of the above solution

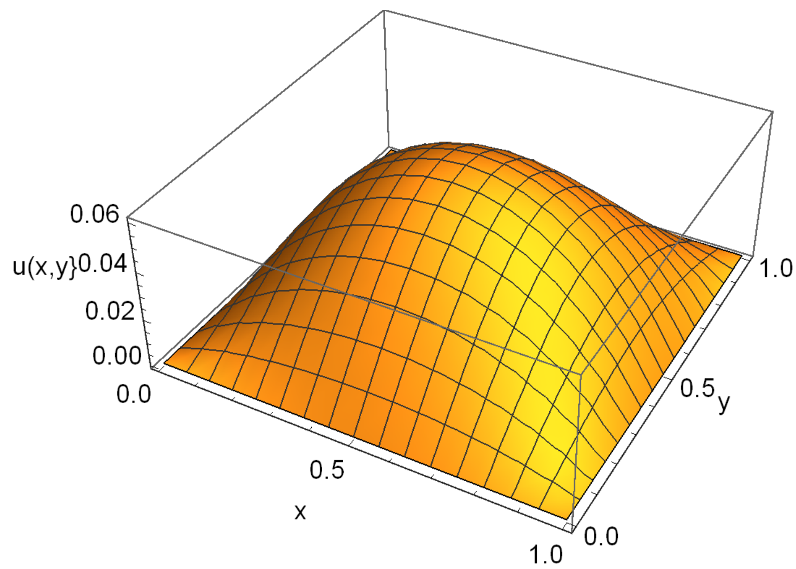

Figure 7: Plot of above solution

```
u[x_, y_] := 1/(π^2 Sinh[π]) (Sinh[π (x - 1)] - Sinh[π x] + Sinh[π]) Sin[π y]
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
        AxesLabel → {"x", "y", "u(x,y)"},
        BaseStyle → 12];
```

Figure 8: Code for the above plot

(ii) $f(x,y) = \sin(\pi x)\sin(2\pi y)$

We first need to find the Fourier coefficients $B_n(x)$. Since $f(x,y) = \sum_{n=1}^{\infty} B_n(x)\sin(n\pi y)$, then multiplying both sides by $\sin(m\pi y)$ and integrating gives

$$\int_0^1 \sin(\pi x)\sin(2\pi y)\sin(m\pi y)\,dy = \sum_{n=1}^{\infty} B_n(x)\int_0^1 \sin(m\pi y)\sin(n\pi y)\,dy$$

$$\sin(\pi x)\int_0^1 \sin(2\pi y)\sin(m\pi y)\,dy = \frac{1}{2}B_m(x)$$

Therefore

$$B_n(x) = 2\sin(\pi x)\int_0^1 \sin(2\pi y)\sin(n\pi y)\,dy$$

For $n = 2$ the above gives

$$\begin{aligned} B_2(x) &= 2\sin(\pi x)\int_0^1 \sin^2(2\pi y)\,dy \\ &= \sin(\pi x) \end{aligned}$$

And for all other terms $B_n = 0$ due to orthogonality. Hence from (3) when $n = 2$

$$\begin{aligned} A_2(x) = &\frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi(x-1))\int_0^x \sinh(2\pi x_0)\sin(\pi x_0)\,dx_0 \\ &+ \frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi x)\int_x^1 \sinh(2\pi(x_0-1))\sin(\pi x_0)\,dx_0 \end{aligned}$$

But

$$\int_0^x \sinh(2\pi x_0)\sin(\pi x_0)\,dx_0 = \frac{1}{5\pi}(2\cosh(2\pi x)\sin(\pi x) - \cos(\pi x)\sinh(2\pi x))$$

And

$$\int_x^1 \sinh(2\pi(x_0-1))\sin(\pi x_0)\,dx_0 = \frac{-1}{5\pi}(2\cosh(2\pi(x-1))\sin(\pi x) + \cos(\pi x)\sinh(2\pi(1-x)))$$

Hence

$$A_2(x) = \frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi(x-1))\left(\frac{1}{5\pi}(2\cosh(2\pi x)\sin(\pi x) - \cos(\pi x)\sinh(2\pi x))\right) + \frac{1}{2\pi\sinh(2\pi)}\sinh(2\pi x)\left(\frac{-1}{5\pi}(2\cosh(2\pi(x-1))\sin(\pi x) + \cos(\pi x)\sinh(2\pi(1-x)))\right)$$

Or

$$A_2(x) = -\frac{1}{5\pi^2}\sin(\pi x)$$

Hence the PDE solution is

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} A_n(x)\sin(n\pi y) \\ &= A_2(x)\sin(2\pi y) \\ &= \frac{-1}{5\pi^2}\sin(\pi x)\sin(2\pi y) \end{aligned}$$

The following is a plot of the above solution

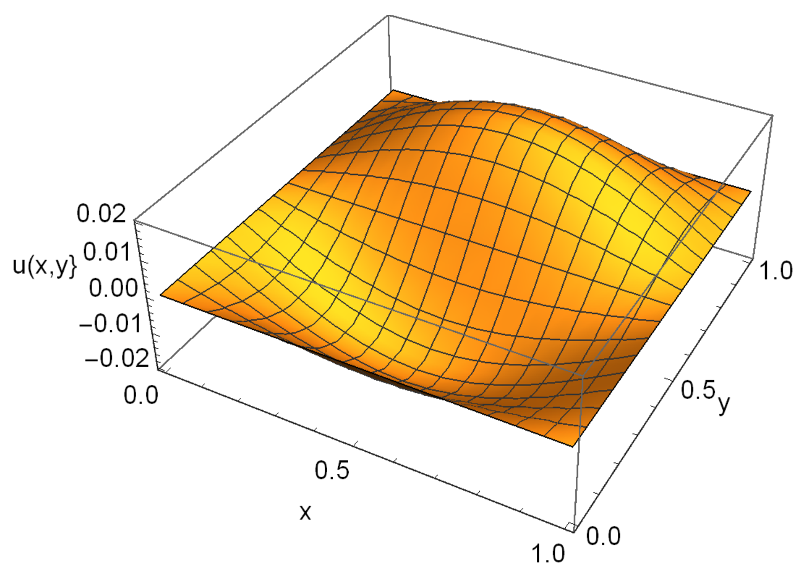

Figure 9: Plot of above solution

```
u[x_, y_] := -Sin[Pi x] Sin[2 Pi y] / (5 π^2)
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
        AxesLabel → {"x", "y", "u(x,y}"},
        BaseStyle → 12];
```

Figure 10: Code for the above plot

<u>(iii) $f(x,y) = 1$</u>

We first need to find the Fourier coefficients $B_n(x)$. Since $f(x,y) = \sum_{n=1}^{\infty} B_n(x)\sin(n\pi y)$, then multiplying both sides by $\sin\left(m\pi y\right)$ and integrating gives

$$\begin{aligned}
-\int_0^1 \sin\left(m\pi y\right) dy &= \sum_{n=1}^{\infty} B_n(x) \int_0^1 \sin\left(m\pi y\right)\sin\left(n\pi y\right) dy \\
-\int_0^1 \sin\left(n\pi y\right) dy &= B_n(x)\frac{1}{2} \\
B_n(x) &= \frac{2}{n\pi}\left(\cos\left(n\pi y\right)\right)_0^1 \\
&= \frac{2}{n\pi}\left(\cos(n\pi)-1\right) \\
&= \frac{2}{n\pi}\left((-1)^n-1\right)
\end{aligned}$$

Hence from (3)

$$\begin{aligned}
A_n(x) = {} & \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\int_0^x \sinh(n\pi x_0)\,dx_0 \\
&+ \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi x)\int_x^1 \sinh(n\pi(x_0-1))\,dx_0
\end{aligned}$$

Or

$$\begin{aligned}
A_n(x) = {} & \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi(x-1))\left[\frac{\cosh(n\pi x_0)}{n\pi}\right]_0^x \\
&+ \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n\pi\sinh(n\pi)}\sinh(n\pi x)\left[\frac{\cosh(n\pi(x_0-1))}{n\pi}\right]_x^1
\end{aligned}$$

Or

$$\begin{aligned}
A_n(x) = {} & \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n^2\pi^2\sinh(n\pi)}\sinh(n\pi(x-1))(\cosh(n\pi x)-1) \\
&+ \frac{2}{n\pi}\left((-1)^n-1\right)\frac{1}{n^2\pi^2\sinh(n\pi)}\sinh(n\pi x)(1-\cosh(n\pi(x-1)))
\end{aligned}$$

Or

$$\begin{aligned}
A_n(x) &= \frac{2\left((-1)^n-1\right)}{n^3\pi^3\sinh(n\pi)}\left(\sinh(n\pi(x-1))(\cosh(n\pi x)-1)+\sinh(n\pi x)(1-\cosh(n\pi(x-1)))\right) \\
&= \frac{2\left((-1)^n-1\right)}{n^3\pi^3\sinh(n\pi)}\left(\sinh(\pi n x)-\sinh(\pi n)-\sinh(\pi n x-\pi n)\right)
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
u\left(x,y\right) &= \sum_{n=1}^{\infty} A_n(x)\sin\left(n\pi y\right) \\
&= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)}{n^3\sinh(n\pi)}\left[\sinh(\pi n x)-\sinh(\pi n)-\sinh(\pi n(x-1))\right]\sin\left(n\pi y\right)
\end{aligned}$$

The following is a plot of the above solution

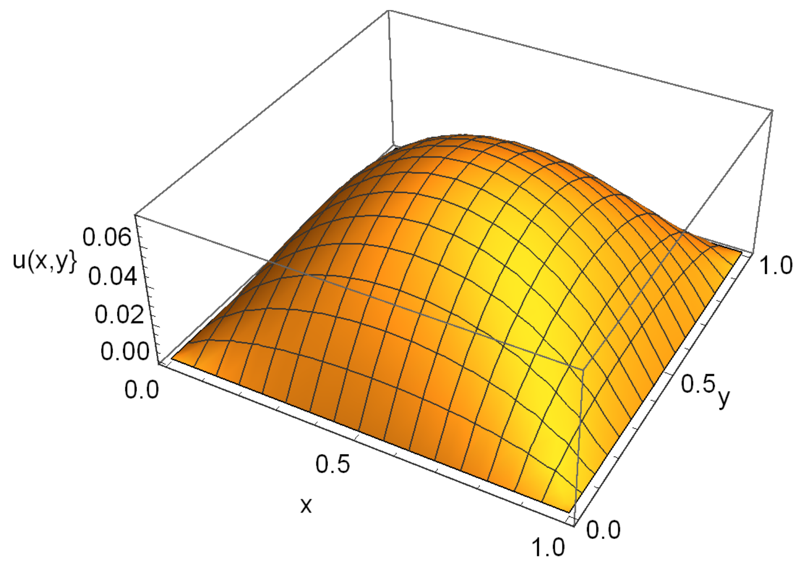

Figure 11: Plot of above solution

```
u[x_, y_] :=
      2/π^3 Sum[((-1)^n - 1)/(n^3 Sinh[n π]) (Sinh[n π x] - Sinh[n π] - Sinh[n π (x - 1)]) Sin[n π y], {n, 1, 30}];
p = Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1},
        AxesLabel → {"x", "y", "u(x,y}"},
        BaseStyle → 12];
```

Figure 12: Code for the above plot

7 Problem 6.3.23

Write out the details of how to derive (6.134) from (6.133).

$$\begin{aligned} G(\boldsymbol{x};\xi) &= -\frac{1}{2\pi}\log\|\boldsymbol{x}-\xi\| + \frac{1}{2\pi}\log\frac{\left\|\|\xi\|^2\boldsymbol{x}-\xi\right\|}{\|\xi\|} \\ &= \frac{1}{2\pi}\log\frac{\left\|\|\xi\|^2\boldsymbol{x}-\xi\right\|}{\|\xi\|\,\|\boldsymbol{x}-\xi\|} \end{aligned} \tag{6.133}$$

$$G\left(r,\theta;\rho,\phi\right) = \frac{1}{4\pi}\log\left(\frac{1+r^2\rho^2-2r\rho\cos\left(\theta-\phi\right)}{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)}\right) \tag{6.134}$$

<u>Solution</u>

Since $x=(r\cos\theta, r\sin\theta)$ and $\xi=\left(\rho\cos\phi,\rho\sin\phi\right)$, then

$$\begin{aligned} \|\xi\|^2 &= \rho^2\cos^2\phi+\rho^2\sin^2\phi \\ &= \rho^2 \end{aligned}$$

Hence

$$\begin{aligned} \|\xi\|^2\boldsymbol{x} &= \rho^2(r\cos\theta, r\sin\theta) \\ &= \left(r\rho^2\cos\theta, r\rho^2\sin\theta\right) \end{aligned}$$

And therefore

$$\begin{aligned} \|\xi\|^2\boldsymbol{x}-\xi &= \left(r\rho^2\cos\theta, r\rho^2\sin\theta\right) - \left(\rho\cos\phi,\rho\sin\phi\right) \\ &= \left(r\rho^2\cos\theta-\rho\cos\phi, r\rho^2\sin\theta-\rho\sin\phi\right) \end{aligned}$$

Hence

$$\begin{aligned} \left\|\|\xi\|^2\boldsymbol{x}-\xi\right\| &= \sqrt{\left(r\rho^2\cos\theta-\rho\cos\phi\right)^2+\left(r\rho^2\sin\theta-\rho\sin\phi\right)^2} \\ &= \sqrt{\left(r^2\rho^4\cos^2\theta+\rho^2\cos^2\phi-2r\rho^3\cos\theta\cos\phi\right)+\left(r^2\rho^4\sin^2\theta+\rho^2\sin^2\phi-2r\rho^3\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\rho^4\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\left(\cos^2\phi+\sin^2\phi\right)-2r\rho^3\left(\cos\theta\cos\phi+\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\rho^4+\rho^2-2r\rho^3\left(\cos\theta\cos\phi+\sin\theta\sin\phi\right)} \end{aligned}$$

But $\cos\theta\cos\phi+\sin\theta\sin\phi=\cos\left(\theta-\phi\right)$. The above becomes

$$\begin{aligned} \left\|\|\xi\|^2\boldsymbol{x}-\xi\right\| &= \sqrt{r^2\rho^4+\rho^2-2r\rho^3\cos\left(\theta-\phi\right)} \\ &= \rho\sqrt{r^2\rho^2+1-2r\rho\cos\left(\theta-\phi\right)} \end{aligned} \tag{1}$$

The above is the numerator of 6.133. Now we find the denominator $\|\xi\|\,\|\boldsymbol{x}-\xi\|$.

$$\begin{aligned} \|\xi\| &= \sqrt{\rho^2\cos^2\phi+\rho^2\sin^2\phi} \\ &= \rho \end{aligned}$$

And

$$\begin{aligned} \|\boldsymbol{x}-\xi\| &= \left\|(r\cos\theta, r\sin\theta)-\left(\rho\cos\phi,\rho\sin\phi\right)\right\| \\ &= \sqrt{\left(r\cos\theta-\rho\cos\phi\right)^2+\left(r\sin\theta-\rho\sin\phi\right)^2} \\ &= \sqrt{\left(r^2\cos^2\theta+\rho^2\cos^2\phi-2r\rho\cos\theta\cos\phi\right)+\left(r^2\sin^2\theta+\rho^2\sin^2\phi-2r\rho\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\left(\cos^2\phi+\sin^2\phi\right)-2r\rho\left(\cos\theta\cos\phi+\sin\theta\sin\phi\right)} \\ &= \sqrt{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)} \end{aligned}$$

Hence

$$\|\xi\|\,\|\boldsymbol{x}-\xi\| = \rho\sqrt{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)} \tag{2}$$

From (1,2)

$$\frac{1}{2\pi}\log\frac{\left\| \|\xi\|^2 x - \xi \right\|}{\|\xi\|\,\|x-\xi\|} = \frac{1}{2\pi}\log\frac{\rho\sqrt{r^2\rho^2+1-2r\rho\cos\left(\theta-\phi\right)}}{\rho\sqrt{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)}}$$

$$= \frac{1}{4\pi}\frac{1+r^2\rho^2-2r\rho\cos\left(\theta-\phi\right)}{r^2+\rho^2-2r\rho\cos\left(\theta-\phi\right)}$$

Which is what required to show.

8 Problem 6.3.27

Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the line $-\infty < x < \infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(x,0) = \delta(x-a), u_t(x,0) = 0$. Can you realize your solution as the limit of classical solutions?

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \tag{2.82}$$

Solution

In (2.82), the function f is the initial conditions and the function g is the initial velocity. Hence the above becomes

$$u(x,t) = \frac{1}{2}\left(\delta((x-a)-ct) + \delta((x-a)+ct)\right)$$

But $\delta((x-a)-ct) = \delta(x-a-ct) = \delta(x-(a+ct))$ and $\delta((x-a)+ct) = \delta(x-a+ct) = \delta(x-(a-ct))$. Hence the above becomes

$$u(x,t) = \frac{1}{2}\delta(x-(a+ct)) + \frac{1}{2}\delta(x-(a-ct)) \tag{1}$$

The above is two half strength delta pulses, one traveling to the left and one traveling to the right from the starting position. Using the limiting definition of delta function, the solution is the limit of sequence of classical solutions $\lim_{n\to\infty} u_n(x,t) \to u(x,t)$ which has initial position that converges to the delta function and initial velocity which converges to zero as given in this problem. Hence

$$\lim_{n\to\infty} u_n(x,0) = \delta(x-a)$$
$$\lim_{n\to\infty} \frac{\partial}{\partial t} u_n(x,0) = 0$$

Using one such definition of limiting function given in 6.10, page 218

$$u_n(x) = \frac{n}{\pi\left(1+n^2x^2\right)}$$

Then

$$u_n(x-a) = \frac{n}{\pi\left(1+n^2(x-a)^2\right)}$$

Hence

$$u_n(x-(a+ct)) = \frac{n}{\pi\left(1+n^2(x-(a+ct))^2\right)}$$

$$u_n(x-(a-ct)) = \frac{n}{\pi\left(1+n^2(x-(a-ct))^2\right)}$$

Using the classical solution $u(x,t) = \frac{1}{2}\left(u_n(x-(a+ct)) + u_n(x-(a-ct))\right)$ becomes

$$u(x,t) = \frac{1}{2}\frac{n}{\pi\left(1+n^2(x-(a+ct))^2\right)} + \frac{1}{2}\frac{n}{\pi\left(1+n^2(x-(a-ct))^2\right)}$$

Which converges to (1) $u(x,t) = \frac{1}{2}\delta(x-(a+ct)) + \frac{1}{2}\delta(x-(a-ct))$ as $n\to\infty$.

9 Problem 6.3.31

(a) Write down a Fourier series for the solution to the initial-boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(-1,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \delta(x) \\ \frac{\partial u(x,0)}{\partial t} &= 0 \end{aligned}$$

(b) Write down an analytic formula for the solution, i.e., sum your series. (c) In what sense does the series solution in part (a) converge to the true solution? Do the partial sums provide a good approximation to the actual solution?

Solution

9.1 Part (a)

Since the boundary conditions are at $x = -1$ and at $x = 1$, it is a little easier to solve this by first shifting the boundaries to $x = 0$ and $x = 2$. This is done by transformation. Let

$$z = x + 1$$

When $x = -1$ then $z = 0$ and when $x = 1$ then $z = 2$. The PDE in terms of z remains the same but the B.C. are shifted. Hence we want to solve for $v(z,x)$ in

$$\begin{aligned} v_{tt} &= v_{zz} \\ v(0,t) &= 0 \\ v(2,t) &= 0 \end{aligned}$$

No need to worry about initial conditions now, since we will transform back to x before applying initial conditions and therefore will use the original initial conditions. This PDE is now solved by separation. Let $v = Z(z)T(t)$. Substituting into the PDE gives

$$\begin{aligned} T''Z &= Z''T \\ \frac{T''}{T} &= \frac{Z''}{Z} = -\lambda \end{aligned}$$

This gives the boundary value ODE

$$\begin{aligned} Z'' + \lambda Z &= 0 \qquad (1) \\ Z(0) &= 0 \\ Z(2) &= 0 \end{aligned}$$

And the time ODE

$$T'' + \lambda T = 0 \qquad (2)$$

Solving (1). From the boundary conditions we know only $\lambda > 0$ is an eigenvalue. Hence for $\lambda > 0$ the solution is

$$Z(z) = A\cos\left(\sqrt{\lambda}z\right) + B\sin\left(\sqrt{\lambda}z\right)$$

At $z = 0$ this gives $A = 0$. Hence the solution now becomes $Z(z) = B\sin\left(\sqrt{\lambda}z\right)$. At $z = 2$ the above gives $0 = B\sin\left(2\sqrt{\lambda}\right)$. For non-trivial solution we want $\sin\left(2\sqrt{\lambda}\right) = 0$ which implies $2\sqrt{\lambda} = n\pi$ or

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 \qquad n = 1,2,3,\cdots$$

And the corresponding eigenfunctions

$$Z_n(z) = \sin\left(\frac{n\pi}{2}z\right) \qquad n = 1,2,3,\cdots$$

The time ODE (2) now becomes

$$T'' + \left(\frac{n\pi}{2}\right)^2 T = 0$$

Which has solution

$$T_n(t) = A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right)$$

Hence the complete solution is

$$v(z,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}z\right)$$

We are now ready to switch back from z to x. Since $z = x+1$ then the above becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right) \tag{3}$$

Now we apply initial conditions to find A_n, B_n. At $t=0, u(x,0) = \delta(x)$. Hence the above gives

$$\delta(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}(x+1)\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{2}(x+1)\right)$ and Integrating gives

$$\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx = \sum_{n=1}^{\infty} A_n \int_{-1}^{1}\sin\left(\frac{n\pi}{2}(x+1)\right)\sin\left(\frac{m\pi}{2}(x+1)\right)dx$$

By orthogonality of $\sin$ functions only term survives and the above simplifies to

$$\begin{aligned}\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx &= A_m \overbrace{\int_{-1}^{1}\sin^2\left(\frac{m\pi}{2}(x+1)\right)dx}^{1}\\ &= A_m\end{aligned}$$

But $\int_{-1}^{1}\delta(x)\sin\left(\frac{m\pi}{2}(x+1)\right)dx = \sin\left(\frac{m\pi}{2}\right)$ since that is where $x=0$. The above reduces to

$$A_n = \sin\left(\frac{n\pi}{2}\right) \qquad n = 1,2,3,\cdots$$

The solution (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right) \tag{4}$$

Taking time derivatives

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}t\right) + \frac{n\pi}{2}B_n\cos\left(\frac{n\pi}{2}t\right)\right)\sin\left(\frac{n\pi}{2}(x+1)\right)$$

At $t=0$ the above becomes

$$0 = \sum_{n=1}^{\infty}\frac{n\pi}{2}B_n \sin\left(\frac{n\pi}{2}(x+1)\right)$$

Therefore $B_n = 0$. Hence the solution (4) becomes

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right)\sin\left(\frac{n\pi}{2}(x+1)\right) \tag{5}$$

Notice that $\sin\left(\frac{n\pi}{2}\right)$ is zero when n is even.

9.2 Part b

$$\sin\left(\frac{n\pi}{2}(x+1)\right) = \sin\left(\frac{n\pi}{2}x + \frac{n\pi}{2}\right)$$

Using $\sin(A+B) = \cos A \sin B + \sin A \cos B$, the above becomes, where $A = \frac{n\pi}{2}x$ and $B = \frac{n\pi}{2}$

$$\sin\left(\frac{n\pi}{2}(x+1)\right) = \cos\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}\right)$$

Hence (5) becomes

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}t\right)\left(\cos\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}\right)\right)\\ &= \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{2}t\right)\left(\cos\left(\frac{n\pi}{2}x\right)\sin^2\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)\right) \end{aligned} \tag{6}$$

But $\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) = 0$, since using $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ gives

$$\begin{aligned}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) &= \frac{1}{2}(\sin(n\pi) + \sin(0)) \\ &= 0\end{aligned}$$

Therefore (6) simplifies to

$$u(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}t\right)\cos\left(\frac{n\pi}{2}x\right)\sin^2\left(\frac{n\pi}{2}\right)$$

But $\sin^2\left(\frac{n\pi}{2}\right) = 0$ when n is even and 1 when n is odd. Hence the above becomes

$$\begin{aligned}u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} \cos\left(\frac{n\pi}{2}t\right)\cos\left(\frac{n\pi}{2}x\right) \\ &= \sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{2}t\right)\cos\left(\frac{(2n+1)\pi}{2}x\right)\end{aligned}$$

Using $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$, then using $A = \frac{(2n+1)\pi}{2}t, B = \frac{(2n+1)\pi}{2}x$ the above becomes

$$\begin{aligned}u(x,t) &= \sum_{n=0}^{\infty} \frac{1}{2}\left(\cos\left(\frac{(2n+1)\pi}{2}t + \frac{(2n+1)\pi}{2}x\right) + \cos\left(\frac{(2n+1)\pi}{2}t - \frac{(2n+1)\pi}{2}x\right)\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{2}(t+x)\right) + \frac{1}{2}\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{2}(t-x)\right) \qquad (7)\end{aligned}$$

But with help of the computer, found that the sums give

$$\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{2}(t+x)\right) = 0$$

$$\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{2}(t-x)\right) = 0$$

Hence (7) becomes

$$u(x,t) = 0$$

9.3 Part c

The solution given by the part b converges to the true solution in the mean sense. Since with wave PDE, there will two pulses, each of half strength moving back and forth on the string each wave with very small width but large amplitude. Solution in part b is giving an averaging value for the solution as zero.