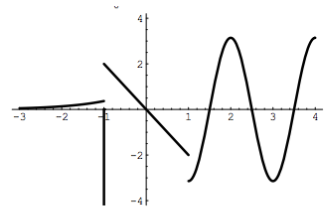
6.1.4c

$$\bigstar \text{ (c) } h'(x) = -e^{-1}\,\delta(x+1) + \begin{cases} \pi\cos\pi x, & x > 1, \\ -2x, & -1 < x < 1, \\ e^x, & x < -1. \end{cases}$$



6.1.5b

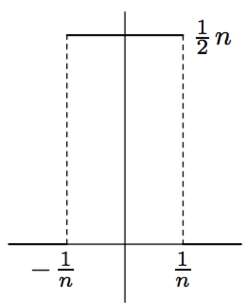
$$\text{(b)} \quad k'(x) = 2\,\delta(x+2) - 2\,\delta(x-2) + \begin{cases} -1, & -2 < x < 0, \\ 1, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$= 2\,\delta(x+2) - 2\,\delta(x-2) - \sigma(x+2) + 2\,\sigma(x) - \sigma(x-2),$$

$$k''(x) = 2\,\delta'(x+2) - 2\,\delta'(x-2) - \delta(x+2) + 2\,\delta(x) - \delta(x-2).$$

6.1.9

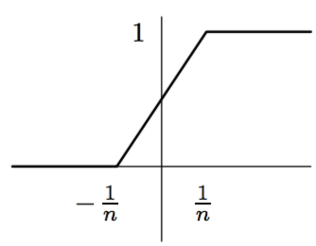
(a)



(b) First, $\lim_{n\to\infty} g_n(x) = 0$ for any $x \neq 0$ since $g_n(x) = 0$ whenever $n > 1/|\,x\,|$. Moreover, $\int_{-\infty}^{\infty} g_n(x)\,dx = 1$, and hence the sequence satisfies (6.11–12), proving

$$\lim_{n\to\infty} g_n(x) = \delta(x).$$

(c) $f_n(x) = \int_{-\infty}^{x} g_n(y)\,dy = \begin{cases} 0, & x < -\frac{1}{n}, \\ \frac{1}{2}nx + \frac{1}{2}, & |\,x\,| < \frac{1}{n}, \\ 1, & x > n. \end{cases}$



Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, the limiting function is $\lim_{n\to\infty} f_n(x) = \sigma(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{2} & x = 0, \\ 1 & x > 0. \end{cases}$

(d) $h_n(x) = \frac{1}{2}n\,\delta\big(x + \frac{1}{n}\big) - \frac{1}{2}n\,\delta\big(x - \frac{1}{n}\big)$.

6.1.30

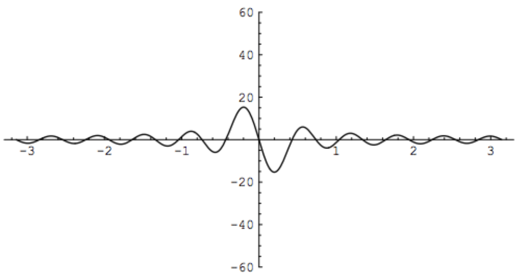
(a) $\delta'(x) \sim \dfrac{\mathrm{i}}{2\pi} \displaystyle\sum_{k=-\infty}^{\infty} k\, e^{\mathrm{i}\,k\,x}.$

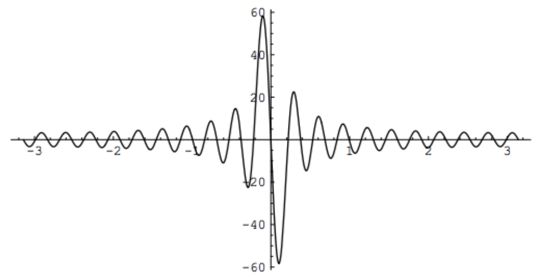
(b) This is evident since the derivative of $e^{\mathrm{i}\,k\,x}$ is $\mathrm{i}\,k\,e^{\mathrm{i}\,k\,x}$.

(c) Differentiate formula (6.39) to obtain

$$\frac{\mathrm{i}}{2\pi}\sum_{k=-n}^{n} k\, e^{\mathrm{i}\,k\,x} = \frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x\ \sin\frac{1}{2}x - \frac{1}{2}\sin\left(n+\frac{1}{2}\right)x\ \cos\frac{1}{2}x}{2\pi\sin^2\frac{1}{2}x}.$$

(d) The graphs of the partial sums $s_{10}(x)$ and $s_{20}(x)$ are:





They indicate weak convergence of the Fourier series, with increasingly rapid oscillations between an envelope, namely $\left(n+\frac{1}{2}\right)/\left(2\pi\sin\frac{1}{2}x\right)$, that has ever-increasing height.

6.1.36

False. Integrating both sides of

$$\delta(x) - \frac{1}{2\pi} \sim \frac{1}{\pi}\left(\cos x + \cos 2x + \cos 3x + \cdots\right),$$

and using (3.72) to find the constant term, yields

$$\sigma(x) - \frac{x}{2\pi} \sim \frac{1}{2} + \frac{1}{\pi}\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right),$$

which agrees with the appropriate combination of (3.49) and (3.73).

6.2.4

(a) $u(x) = \frac{9}{16}x - \frac{1}{2}x^2 + \frac{3}{16}x^3 - \frac{1}{4}x^4$;

(b) $G(x;\xi) = \begin{cases} \left(1 - \frac{3}{4}\xi - \frac{1}{4}\xi^3\right)\left(x + \frac{1}{3}x^3\right), & x \le \xi, \\ \left(1 - \frac{3}{4}x - \frac{1}{4}x^3\right)\left(\xi + \frac{1}{3}\xi^3\right), & x \ge \xi. \end{cases}$

(c) $$\begin{aligned} u(x) &= \int_0^1 G(x;\xi)\,d\xi \\ &= \int_0^x \left(1 - \tfrac{3}{4}x - \tfrac{1}{4}x^3\right)\left(\xi + \tfrac{1}{3}\xi^3\right)d\xi + \int_x^1 \left(1 - \tfrac{3}{4}\xi - \tfrac{1}{4}\xi^3\right)\left(x + \tfrac{1}{3}x^3\right)d\xi \\ &= \tfrac{9}{16}x - \tfrac{1}{2}x^2 + \tfrac{3}{16}x^3 - \tfrac{1}{4}x^4. \end{aligned}$$

(d) Under an impulse force at $x = \xi$, the maximal displacement is at the forcing point, namely $g(x) = G(x,x) = x - \frac{3}{4}x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 - \frac{1}{12}x^6$. The maximum value of $g(x^\star) = \frac{1}{3}$ occurs at the solution $x^\star = (1+\sqrt{2})^{1/3} - (1+\sqrt{2})^{-1/3} \approx .596072$ to the equation $g'(x) = 1 - \frac{3}{2}x + x^2 - 2x^3 - \frac{1}{2}x^5 = 0$.

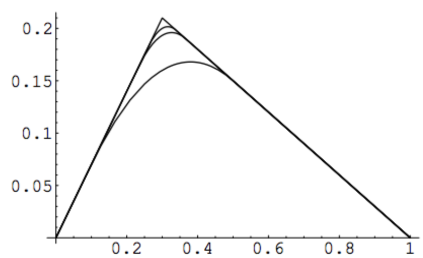
6.2.7

equation $g'(x) = 1 - \frac{3}{2}x + x^2 - 2x^3 - \frac{1}{2}x^5 = 0$.

6.2.7. (a) $u_n(x) = \begin{cases} x(1-\xi), & 0 \le x \le \xi - \frac{1}{n}, \\ -\frac{1}{4}nx^2 + \left(\frac{1}{2}n - 1\right)x\xi - \frac{1}{4}n\xi^2 + \frac{1}{2}x + \frac{1}{2}\xi - \frac{1}{4n}, & |x-\xi| \le \frac{1}{n}, \\ \xi(1-x), & \xi + \frac{1}{n} \le x \le 1. \end{cases}$

(b) Since $u_n(x) = G(x;\xi)$ for all $|x-\xi| \ge \frac{1}{n}$, we have $\lim_{n\to\infty} u_n(x) = G(x;\xi)$ for all $x \ne \xi$, while $\lim_{n\to\infty} u_n(\xi) = \lim_{n\to\infty}\left(\xi - \xi^2 - \frac{1}{4n}\right) = \xi - \xi^2 = G(\xi,\xi)$. (Or one can appeal to continuity to infer this.) This limit reflects the fact that the external forces converge to the delta function: $\lim_{n\to\infty} f_n(x) = \delta(x-\xi)$.

(c)



6.2.11

. (a) $G(x;\xi) = \begin{cases} \dfrac{\sinh \omega x \, \cosh \omega\,(1-\xi)}{\omega \cosh \omega}, & x \le \xi, \\[2ex] \dfrac{\cosh \omega\,(1-x)\, \sinh \omega \xi}{\omega \cosh \omega}, & x \ge \xi. \end{cases}$ (b) If $x \le \frac{1}{2}$, then

$$u(x) = \int_0^x \frac{\cosh \omega\,(1-x)\, \sinh \omega \xi}{\omega \cosh \omega}\, d\xi \; + \; \int_x^{1/2} \frac{\sinh \omega x\, \cosh \omega\,(1-\xi)}{\omega \cosh \omega}\, d\xi$$

$$- \int_{1/2}^{1} \frac{\sinh \omega x\, \cosh \omega\,(1-\xi)}{\omega \cosh \omega}\, d\xi$$

$$= \frac{1}{\omega^2} - \frac{\left(e^{\omega/2} - e^{-\omega/2} + e^{-\omega}\right) e^{\omega x} + \left(e^{\omega} - e^{\omega/2} + e^{-\omega/2}\right) e^{-\omega x}}{\omega^2 (e^{\omega} + e^{-\omega})},$$

while if $x \ge \frac{1}{2}$, then

$$u(x) = \int_0^{1/2} \frac{\cosh \omega\,(1-x)\, \sinh \omega \xi}{\omega \cosh \omega}\, d\xi \; - \; \int_{1/2}^{x} \frac{\cosh \omega\,(1-x)\, \sinh \omega \xi}{\omega \cosh \omega}\, d\xi$$

$$- \int_{x}^{1} \frac{\sinh \omega x\, \cosh \omega\,(1-\xi)}{\omega \cosh \omega}\, d\xi$$

$$= -\frac{1}{\omega^2} + \frac{\left(e^{-\omega/2} - e^{-\omega} + e^{-3\omega/2}\right) e^{\omega x} + \left(e^{3\omega/2} - e^{\omega} + e^{\omega/2}\right) e^{-\omega x}}{\omega^2 (e^{\omega} + e^{-\omega})}.$$

6.2.12

6.2.12. Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x)$, $u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

Solution: Yes, it does:

$$G(x;\xi) = \begin{cases} \dfrac{\cosh \omega x\, \cosh \omega\,(1-\xi)}{\omega \sinh \omega}, & x \le \xi, \\[2ex] \dfrac{\cosh \omega\,(1-x)\, \cosh \omega \xi}{\omega \sinh \omega}, & x \ge \xi. \end{cases}$$

The solution to the boundary value problem is

$$u(x) = \int_0^1 G(x;\xi) f(\xi)\, d\xi$$

$$= \int_0^x \frac{\cosh \omega\,(1-x)\, \cosh \omega \xi}{\omega \sinh \omega} f(\xi)\, d\xi \; + \; \int_x^1 \frac{\cosh \omega x\, \cosh \omega\,(1-\xi)}{\omega \sinh \omega} f(\xi)\, d\xi.$$