HW 8

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 6.1.4c

Find and sketch a graph of the derivative (in the context of generalized functions) of the following functions

(c) $h(x) = \begin{cases} \sin(\pi x) & x > 1 \\ 1 - x^2 & -1 < x < 1 \\ e^x & x < -1 \end{cases}$

Solution

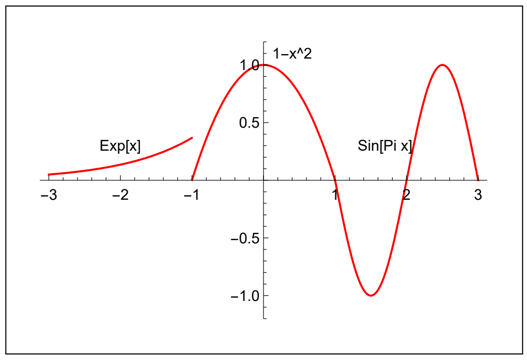

Figure 1: Sketch of the function $h(x)$

There is only one jump discontinuity at $x = -1$. The amount of jump [1] at $x = -1$ is $\frac{-1}{e}$. Hence

$$h'(x) = -e^{-1}\delta(x+1) + \begin{cases} \pi\cos(\pi x) & x > 1 \\ -2x & -1 < x < 1 \\ e^x & x < -1 \end{cases}$$

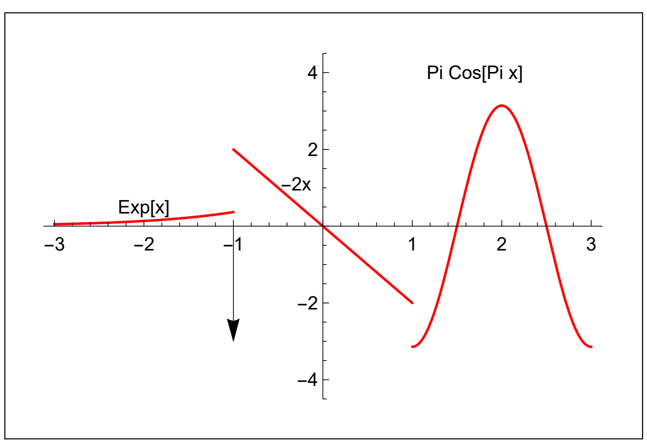

Figure 2: Sketch of the function $h'(x)$

[1]When determining the sign of the jump, we go from left to right always. Dropping down means negative sign and moving higher means positive sign.

2 Problem 6.1.5b

Find the first and second derivatives of the functions

(b) $k(x) = \begin{cases} |x| & -2 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Solution

First, the function $k(x)$ is shown below

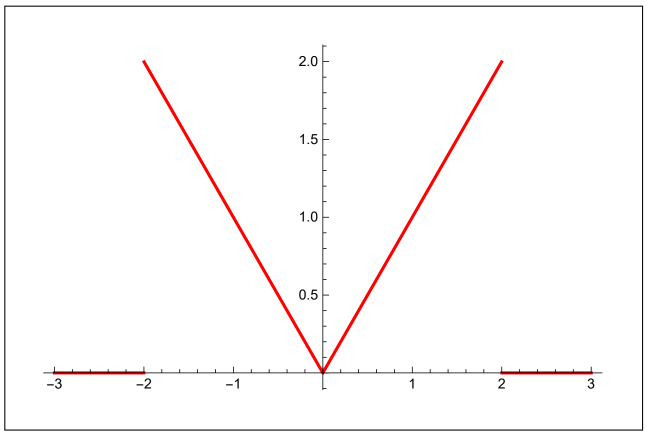

Figure 3: Sketch of the function $k(x)$

We see there is a jump discontinuity at $x = -2$ of value 2 and at $x = 2$ of value -2. Now, when $-2 < x < 0$, then $k(x) = -x$ and when $0 < x < 2$, then $k(x) = x$. Hence

$$k'(x) = 2\delta(x+2) - 2\delta(x-2) + \begin{cases} 0 & x < -2 \\ -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2 \end{cases}$$

The derivative is not defined at $x = 0$. A plot of the above gives

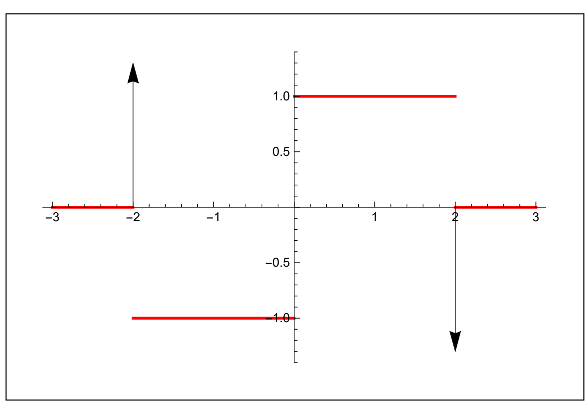

Figure 4: Sketch of the function $k'(x)$

We see that there is now a jump discontinuity at $x = -2$ of value -1 and jump discontinuity at $x = 0$ of value 2 and jump discontinuity at $x = 2$ of value -1. Hence

$$k''(x) = 2\delta'(x+2) - 2\delta'(x-2) - \delta(x+2) + 2\delta(x) - \delta(x-2)$$

Where $\delta'(x+2)$ and $\delta'(x-2)$ are called "doublets" at $x = -2$ and at $x = 2$ respectively.

3 Problem 6.1.9

For each positive integer n, let $g_n(x) = \begin{cases} \frac{1}{2}n & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ (a) Sketch a graph of $g_n(x)$. (b) Show that $\lim_{n\to\infty} g_n(x) = \delta(x)$. (c) Evaluate $f_n(x) = \int_{-\infty}^{x} g_n(y)\,dy$ and sketch a graph. Does the sequence $f_n(x)$ converge to the step function $\sigma(x)$ as $n \to \infty$? (d) Find the derivative $h_n(x) = g_n'(x)$. (e) Does the sequence $h_n(x)$ converge to $\delta'(x)$ as $n \to \infty$?

Solution

3.1 Part a

Lets try few values of n.

$\underline{n=1}$ $g_1(x) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$

$\underline{n=2}$ $g_2(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

$\underline{n=3}$ $g_2(x) = \begin{cases} \frac{3}{2} & |x| < \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$

And so on. We see that as n increases, the function value increases and the domain it is not zero on becomes smaller. As $n \to \infty$ this becomes a $\delta(x)$ function. Here is a plot of few values of increasing n.

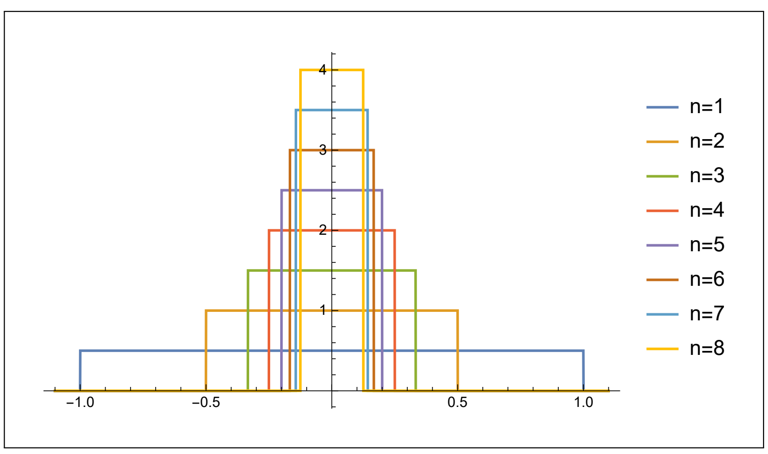

Figure 5: $g_n(x)$ for increasing n

3.2 Part b

$$\begin{aligned} \lim_{n\to\infty} g_n(x) &= \begin{cases} \lim_{n\to\infty} \frac{1}{2}n & \lim_{n\to\infty} |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \infty & |x| \to 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \delta(x) \end{aligned}$$

3.3 Part c

We want to integrate this function

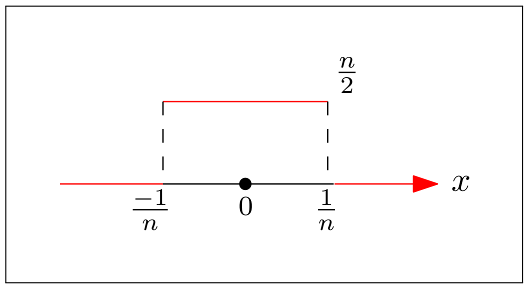

Figure 6: Integrating $g_n(x)$

Therefore

$$f_n(x) = \int_{-\infty}^{x} g_n(y)\,dy = \begin{cases} 0 & x < \frac{-1}{n} \\ \left(\frac{1}{n} + x\right)\frac{n}{2} & \frac{-1}{n} < x < 0 \\ \left(\frac{1}{n} - x\right)\frac{n}{2} & 0 < x < \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}$$

This is a sketch of the above We see that as $n \to \infty$ then $f_n(x)$ becomes

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Which is the step function $\sigma(x)$

3.4 Part d

From the plot of $g_n(x)$ above, we see there is a jump discontinuity at $x = -\frac{1}{n}$ of value $\frac{n}{2}$ and a jump discontinuity at $x = \frac{1}{n}$ of value $-\frac{n}{2}$. And since $g_n(x)$ is constant everywhere else, then

$$h_n(x) = g_n'(x) = \frac{n}{2}\delta\left(x + \frac{1}{n}\right) - \frac{n}{2}\delta\left(x - \frac{1}{n}\right)$$

3.5 Part e

Yes, $\lim_{n\to\infty} h_n(x) = \delta'(x)$. By definition, and as shown in figure 6.6 in textbook, $\delta'(x)$ is "doublets". Which is an impulse in positive direction just to the left of x and another impulse in negative direction just to the right of x and this is what happens when $\lim_{n\to\infty} h_n(x)$ as seen from the result in part d.

4 Problem 6.1.30

(a) Find the complex Fourier series for the derivative of the delta function $\delta'(x)$ by direct evaluation of the coefficient formulas (b) Verify that your series can be obtained by term-by-term differentiation of the series for $\delta(x)$. (c) Write a formula for the n^{th} partial sum of your series. (d) Use a computer graphics package to investigate the convergence of the series.

Solution

4.1 Part a

By first doing 2π periodic extension (similar to Dirac comb) we can calculate the coefficients. First we find the Fourier series for $\delta(x)$

$$\delta(x) \sim \sum_{k=-\infty}^{k=\infty} c_k e^{ikx}$$

Where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)e^{-ikx}dx = \frac{1}{2\pi}$. Hence

$$\begin{aligned}\delta(x) &\sim \frac{1}{2\pi}\sum_{k=-\infty}^{k=\infty} e^{ikx}\\ &\sim \frac{1}{2\pi}\left(\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots\right) \end{aligned} \tag{1}$$

Now

$$\delta'(x) \sim \sum_{k=-\infty}^{k=\infty} d_k e^{ikx} \tag{2}$$

Where

$$\begin{aligned} d_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta'(x)e^{-ikx}dx\\ &= \frac{1}{2\pi}\left[\left(e^{-ikx}\right)'\right]_{x=0}\\ &= \frac{1}{2\pi}\left[-ike^{-ikx}\right]_{x=0}\\ &= \frac{1}{2\pi}[-ik]\\ &= -i\frac{k}{2\pi} \end{aligned}$$

Hence from (2) we obtain the Fourier series for $\delta'(x)$ as

$$\begin{aligned}\delta'(x) &\sim \frac{-i}{2\pi}\sum_{k=-\infty}^{k=\infty} ke^{ikx}\\ &\sim \frac{-i}{2\pi}\left(\cdots - 2e^{-2ix} - e^{-ix} + e^{ix} + 2e^{2ix} + \cdots\right)\\ &\sim \frac{1}{2\pi}\left(\cdots + 2ie^{-2ix} + ie^{-ix} - ie^{ix} - 2ie^{2ix} + \cdots\right) \end{aligned} \tag{3}$$

4.2 Part b

To do term by term differentiation of $\delta(x)$, we first have to note the use of the following relation and the sign change needed to add

$$\begin{aligned}\lim_{n\to\infty} g_n(x) &\to \delta(x)\\ -\lim_{n\to\infty} g_n'(x) &\to \delta'(x)\end{aligned}$$

The above means we need to add a minus sign to the RHS when taking derivative of $\delta(x)$. Therefore, term by term differentiation of the Fourier series for $\delta(x)$ given in (1) now gives

$$
\begin{aligned}
\delta'(x) &\sim (-)\frac{1}{2\pi}\frac{d}{dx}\left(\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots\right) \\
&\sim (-)\frac{1}{2\pi}\left(\cdots - 2ie^{-2ix} - ie^{-ix} + ie^{ix} + 2ie^{2ix} + \cdots\right) \\
&\sim \frac{1}{2\pi}\left(\cdots + 2ie^{-2ix} + ie^{-ix} - ie^{ix} - 2ie^{2ix} + \cdots\right) \qquad (4)
\end{aligned}
$$

Comparing (4) and (3) shows they are the same.

4.3 Part c

It is easier to use normal Fourier series for this.

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\delta'(x)\cos(kx)\,dx \\
&= \frac{1}{\pi}\left[(\cos kx)'\right]_{x=0} \\
&= \frac{1}{\pi}\left[-k\sin kx\right]_{x=0} \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\delta'(x)\sin(kx)\,dx \\
&= \frac{1}{\pi}\left[(\sin kx)'\right]_{x=0} \\
&= \frac{1}{\pi}\left[k\cos kx\right]_{x=0} \\
&= \frac{k}{\pi}
\end{aligned}
$$

Hence

$$
\delta'(x) \sim \frac{1}{\pi}\sum_{k=1}^{\infty} k\sin(kx) \qquad (1)
$$

Therefore the n^{th} partial sum is

$$
\delta'_n(x) \sim \frac{1}{\pi}\sum_{k=1}^{n} k\sin(kx)
$$

Since $|\sin(kx)| \leq 1$, used partial sum formula for the above given by

$$
\sum_{k=1}^{n} k\sin(kx) = \frac{n\sin((n+1)x) - (n+1)\sin(nx)}{2\cos(x) - 2} \qquad (2)
$$

Hence

$$
\delta'_n(x) \sim \frac{1}{\pi}\frac{n\sin((n+1)x) - (n+1)\sin(nx)}{2\cos(x) - 2}
$$

It is possible to obtain the above formula by writing $\sin(kx) = \operatorname{Im}\left(e^{ikx}\right)$ and then using $\operatorname{Im}\sum_{k=1}^{n} ke^{ikx} = \operatorname{Im}\sum_{k=1}^{n} kz^k$ where $z = e^{ix}$. Since $|z| \leq 1$ then using the partial sum formula

$$
\begin{aligned}
\sum_{k=1}^{n} kz^k &= \frac{z(1-z^n)}{(1-z)^2} - \frac{nz^{n+1}}{1-z} \\
&= \frac{z(1-z^n) - nz^{n+1}(1-z)}{(1-z)^2} \\
&= \frac{z - z^{n+1} - nz^{n+1} + nz^{n+2}}{(1-z)^2} \\
&= \frac{z - (1+n)z^{n+1} + nz^{n+2}}{(1-z)^2}
\end{aligned}
$$

Then replacing z back by e^{ix} in the above, and using $e^{ix} = \cos x + i\sin x$ and simplifying and taking the imaginary part to obtain (2).

4.4 Part d

Using computer graphics, the following is plot of (2) for increasing values of n. This shows that as n increases $\delta_n'(x)$ approaches "doublets", which is a pulse to the left of $x = 0$ and one to the right of $x = 0$.

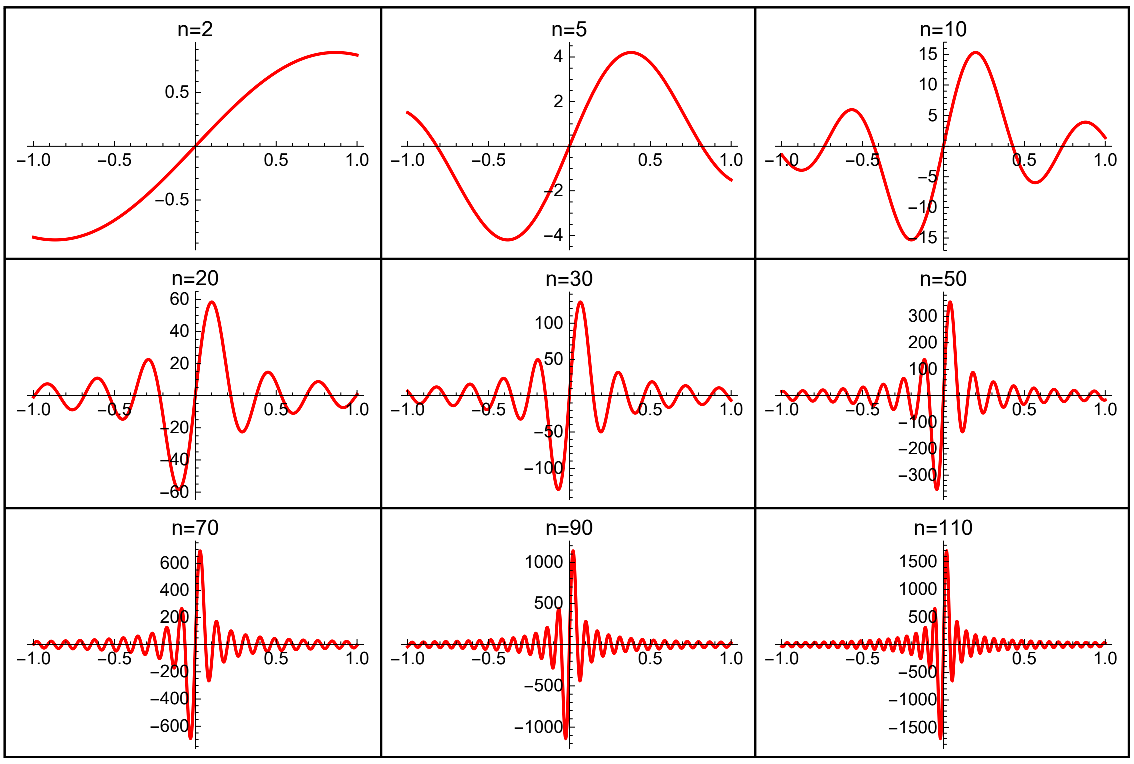

Figure 7: Convergence of Fourier series of $\delta'(x)$ as n increases

```
f[x_, n_] := 1/π (n Sin[(n + 1) x] - (n + 1) Sin[n x])/(2 Cos[x] - 2)
data = Table[Plot[f[x, n], {x, -1, 1}, PlotRange → All, PlotStyle → Red,
        PlotLabel → Row[{"n=", n}]], {n, {2, 5, 10, 20, 30, 50, 70, 90, 110}}];
p = Grid[Partition[data, 3], Frame → All];
```

Figure 8: Code used for the above plot

5 Problem 6.1.36

True or false: If you integrate the Fourier series for the delta function $\delta(x)$ term by term, you obtain the Fourier series for the step function $\sigma(x)$.

Solution

The Fourier series for delta function $\delta(x)$ is (assuming 2π periodic extension)

$$\delta(x) \sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos nx$$

Integrating RHS term by term gives

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{2\pi} dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} \cos nx dx &= 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} \overbrace{\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}}^{0} \\ &= 1 \end{aligned} \tag{1}$$

The step function $\sigma(x)$ is defined as

$$\sigma(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Its Fourier series was already found on page 83 (assuming 2π periodic extension) in Example 3.9 as

$$\begin{aligned} \sigma(x) &\sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)x) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) \end{aligned} \tag{2}$$

Comparing (1) and (2), the answer is false.

6 Problem 6.2.4

The boundary value problem $-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), u(0) = u(1) = 0$, models the displacement $u(x)$ of a nonuniform elastic bar with stiffness $c(x) = \frac{1}{1+x^2}$ for $0 \leq x \leq 1$. (a) Find the displacement when the bar is subjected to a constant external force, $f = 1$. (b) Find the Green's function for the boundary value problem (c) Use the resulting superposition formula to check your solution to part (a). (d) Which point $0 < \xi < 1$ on the bar is the "weakest", i.e., the bar experiences the largest displacement under a unit impulse concentrated at that point?

Solution

6.1 Part a

The ode to solve is

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\frac{du}{dx}\right) = -1$$

Integrating once gives

$$\begin{aligned}\frac{1}{1+x^2}\frac{du}{dx} &= -x + C_1\\ \frac{du}{dx} &= \left(1+x^2\right)(-x+C_1)\\ &= C_1 - x + C_1x^2 - x^3\end{aligned}$$

Integrating once more gives

$$\begin{aligned}u(x) &= C_1x - \frac{x^2}{2} + C_1\frac{x^3}{3} - \frac{x^4}{4} + C_2\\ &= -\frac{x^4}{4} + C_1\frac{x^3}{3} - \frac{x^2}{2} + C_1x + C_2 \qquad (1)\end{aligned}$$

Applying left B.C. $u(0) = 0$ gives

$$0 = C_2$$

Hence solution (1) becomes

$$u(x) = -\frac{x^4}{4} + C_1\frac{x^3}{3} - \frac{x^2}{2} + C_1x \qquad (2)$$

Applying left B.C. $u(1) = 0$ gives

$$\begin{aligned}0 &= -\frac{1}{4} + C_1\frac{1}{3} - \frac{1}{2} + C_1\\ C_1 &= \frac{9}{16}\end{aligned}$$

Hence the solution (2) becomes

$$\begin{aligned}u(x) &= -\frac{x^4}{4} + \frac{3}{16}x^3 - \frac{x^2}{2} + \frac{9}{16}x\\ &= \frac{1}{16}\left(-4x^4 + 3x^3 - 8x^2 + 9x\right)\end{aligned}$$

```
u[x_] := -x^4/4 + 3/16 x^3 - x^2/2 + 9/16 x;
Plot[u[x], {x, 0, 1}, PlotStyle → Red,
  GridLines → Automatic, GridLinesStyle → LightGray]
```

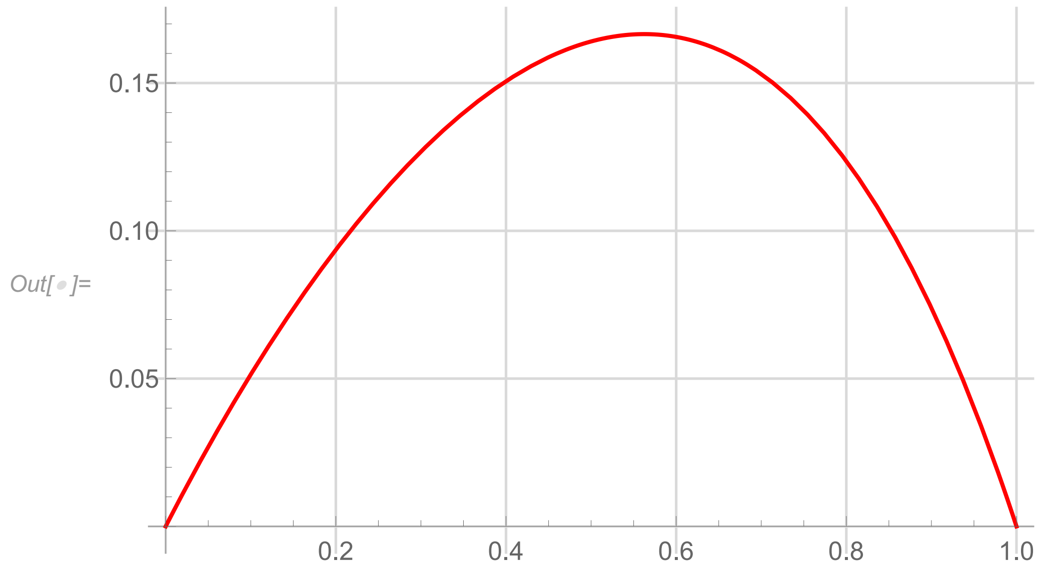

Figure 9: Plot of the above solution

6.2 Part b

When $x \neq y$, then Green function satisfies $\frac{d}{dx}\left(c(x)\frac{dG(x,y)}{dx}\right) = 0$. This means that

$$c(x)\frac{dG(x,y)}{dx} = A_1$$

But $c(x) = \frac{1}{1+x^2}$, therefore

$$\frac{dG(x,y)}{dx} = A_1\left(1+x^2\right)$$

Integrating gives

$$G(x,y) = A_1 x + A_1 \frac{x^3}{3} + A_2$$

Therefore Green function is

$$G(x,y) = \begin{cases} A_1 x + A_1 \frac{x^3}{3} + A_2 & x < y \\ B_1 x + B_1 \frac{x^3}{3} + B_2 & x > y \end{cases} \tag{1}$$

Notice we used different constants of integrations for each side of the delta location y. Now we use boundary conditions on the left and right end to find these unknowns. Since Green function satisfies same boundary conditions as the solution, then at $x = 0$ we need

$$\begin{aligned} G(0,y) &= 0 \\ &= A_2 \end{aligned}$$

And at $x = 1$

$$\begin{aligned} G(1,y) &= 0 \\ &= B_1 + B_1\frac{1}{3} + B_2 \end{aligned}$$

Which means $-\frac{4}{3}B_1 = B_2$. Using these results in (1) gives

$$G(x,y) = \begin{cases} A_1\left(x+\frac{x^3}{3}\right) & x<y \\ B_1x + B_1\frac{x^3}{3} - \frac{4}{3}B_1 & x>y \end{cases}$$

$$= \begin{cases} A_1\left(x+\frac{x^3}{3}\right) & x<y \\ B_1\left(x+\frac{x^3}{3}-\frac{4}{3}\right) & x>y \end{cases} \tag{1A}$$

We now need to determine A_1, B_1. From continuity condition of $G(x,y)$ at $x=y$ we obtain the first equation

$$A_1\left(y+\frac{y^3}{3}\right) = B_1\left(y+\frac{y^3}{3}-\frac{4}{3}\right) \tag{2}$$

And

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1\left(1+x^2\right) & x<y \\ B_1\left(1+x^2\right) & x>y \end{cases}$$

Evaluated at $x=y$

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1\left(1+y^2\right) & x<y \\ B_1\left(1+y^2\right) & x>y \end{cases}$$

There is a jump discontinuity in $\frac{dG(x,y)}{dx}$ of value $\frac{1}{p}$ where $-\left(py''\right)=0$. Comparing this with $-\frac{d}{dx}\left(c(x)\frac{dG(x,y)}{dx}\right) = f(x)$ shows that $p = \frac{1}{c(x)} = \left(1+x^2\right)$ or $\left(1+y^2\right)$ at $x=y$. Therefore this condition gives the second equation we need

$$A_1\left(1+y^2\right) - B_1\left(1+y^2\right) = \frac{1}{p} \tag{3}$$

$$= \left(1+y^2\right) \tag{1}$$

We now have the two equations we want (2,3) to solve for A_1, B_1. Solving for A_1, B_1 gives

$$A_1 = \frac{1}{4}\left(4-3y-y^3\right)$$
$$B_1 = \frac{1}{4}\left(-3y-y^3\right)$$

Substituting the above into (1A) gives the Green function

$$G(x,y) = \begin{cases} \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right) & x<y \\ \frac{1}{4}\left(-3y-y^3\right)\left(x+\frac{x^3}{3}-\frac{4}{3}\right) & x>y \end{cases}$$

$$= \begin{cases} \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right) & x<y \\ \frac{1}{4}\left(4-3x-x^3\right)\left(y+\frac{y^3}{3}\right) & x>y \end{cases}$$

we now see the symmetry above as expected.

6.3 Part (c)

Now we check the solution of part (a) for $f(x)=1$ using the superposition formula and noting that $f(y)=1$ we obtain

$$u(x) = \overbrace{\int_0^x G(x,y)f(y)\,dy}^{y<x} + \overbrace{\int_x^1 G(x,y)f(y)\,dy}^{y>x}$$
$$= \int_0^x \frac{1}{4}\left(4-3x-x^3\right)\left(y+\frac{y^3}{3}\right)dy + \int_x^1 \frac{1}{4}\left(4-3y-y^3\right)\left(x+\frac{x^3}{3}\right)dy$$

Hence

$$
\begin{aligned}
u(x) &= \frac{1}{4}\left(4-3x-x^3\right)\int_0^x\left(y+\frac{y^3}{3}\right)dy+\frac{1}{4}\left(x+\frac{x^3}{3}\right)\int_x^1\left(4-3y-y^3\right)dy\\
&= \frac{1}{4}\left(4-3x-x^3\right)\left(\frac{y^2}{2}+\frac{y^4}{12}\right)_0^x+\frac{1}{4}\left(x+\frac{x^3}{3}\right)\left(4y-\frac{3y^2}{2}-\frac{y^4}{4}\right)_x^1\\
&= \frac{1}{4}\left(4-3x-x^3\right)\left(\frac{x^2}{2}+\frac{x^4}{12}\right)+\frac{1}{4}\left(x+\frac{x^3}{3}\right)\left(4-\frac{3}{2}-\frac{1}{4}-\left(4x-\frac{3x^2}{2}-\frac{x^4}{4}\right)\right)\\
&= \frac{1}{16}x\left(-4x^3+3x^2-8x+9\right)
\end{aligned}
$$

Which agree with solution obtain in part (a)

6.4 Part (d)

From the solution above $u(x)=\frac{1}{16}\left(-4x^4+3x^3-8x^2+9x\right)$. Hence

$$
\frac{du}{dx}=\frac{1}{16}\left(-16x^3+9x^2-16x+9\right)
$$

Solving for $\frac{du}{dx}=0$ gives

$$
\begin{aligned}
\frac{1}{16}\left(-16x^3+9x^2-16x+9\right)&=0\\
-\frac{1}{16}(16x-9)\left(1+x^2\right)&=0
\end{aligned}
$$

$\left(1+x^2\right)=0$ does not give real solutions. Hence $-\frac{1}{16}(16x-9)=0$ or $16x-9=0$ or

$$
x=\frac{9}{16}
$$

At this x is the largest displacement which is found by evaluating the solution at this x

$$
\begin{aligned}
u\left(\frac{9}{16}\right) &= \frac{1}{16}\left(-4\left(\frac{9}{16}\right)^4+3\left(\frac{9}{16}\right)^3-8\left(\frac{9}{16}\right)^2+9\left(\frac{9}{16}\right)\right)\\
&= \frac{43659}{262144}\\
&= 0.167
\end{aligned}
$$

7 Problem 6.2.7

For n a positive integer, set $f_n(x) = \begin{cases} \frac{1}{2}n & |x-\xi| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ (a) Find the solution $u_n(x)$ to the boundary value problem $-u'' = f_n(x), u(0) = 0, u(1) = 0$, assuming $0 < \xi - \frac{1}{n} < \xi + \frac{1}{n} < 1$. (b) Prove that $\lim_{n\to\infty} u_n(x) = G(x;\xi)$ converges to the Green's function (6.51) given by solution to $-cu'' = f(x)$ with same BC as

$$G(x;\xi) = \frac{(1-\xi)x - \rho(x-\xi)}{c} = \begin{cases} (1-\xi)\frac{x}{c} & x \leq \xi \\ (1-x)\frac{\xi}{c} & x \geq \xi \end{cases}$$

But here $c = 1$, so the above becomes

$$G(x;\xi) = (1-\xi)x - \rho(x-\xi) = \begin{cases} (1-\xi)x & x \leq \xi \\ (1-x)\xi & x \geq \xi \end{cases}$$

Where ρ is the ramp function. Why should this be the case? (c) Reconfirm the result in part (b) by graphing $u_5(x), u_{15}(x), u_{25}(x)$, along with $G(x;\xi)$ when $\xi = 0.3$.

Solution

7.1 Part a

When $x \neq \xi$, then Green function satisfies $\frac{d^2G(x,y)}{dx^2} = 0$. This means that

$$G(x,y) = A_1x + A_2$$

Hence Green function is

$$G(x,y) = \begin{cases} A_1x + A_2 & x \leq \xi \\ B_1x + B_2 & x \geq \xi \end{cases}$$

At $x = 0$, $G(0,y) = 0 = A_2$ and at $x = 1$, $G(1,y) = 0 = B_1 + B_2$. Hence $B_2 = -B_1$. The above becomes

$$G(x,y) = \begin{cases} A_1x & x \leq \xi \\ B_1x - B_1 & x \geq \xi \end{cases}$$

$$= \begin{cases} A_1x & x \leq \xi \\ B_1(x-1) & x \geq \xi \end{cases} \tag{A}$$

Where A_1, B_1 are constants to be found. These are found from the continuity condition and the jump discontinuity condition on $\frac{dG}{dx}$ both at $x = \xi$. The continuity condition at $x = \xi$ gives the first equation as

$$A\xi = B(\xi - 1) \tag{1}$$

And $\frac{dG}{dx}$ at $x = \xi$ gives

$$\lim_{x\to\xi} \frac{dG}{dx} = \begin{cases} A_1 & x \leq \xi \\ B_1 & x \geq \xi \end{cases}$$

Hence the jump discontinuity condition gives the second equation we want which is

$$A_1 - B_1 = 1 \tag{2}$$

Where 1 is used in RHS above since $c = 1$. From (1,2) we solve for A_1, B_1. Which gives

$$B_1 = -\xi$$
$$A_1 = 1 - \xi$$

Substituting the above back into Eq (A) gives the Green function

$$G(x;\xi) = \begin{cases} (1-\xi)x & x \leq \xi \\ (1-x)\xi & x \geq \xi \end{cases} \tag{3}$$

The solution is now found using superposition formula

$$
\begin{aligned}
u_n(x) &= \overbrace{\int_0^x G(x,\xi) f_n(\xi)\, d\xi}^{\xi<x} + \overbrace{\int_x^1 G(x,\xi) f_n(\xi)\, d\xi}^{\xi>x} \\
&= \int_0^x (1-x)\,\xi f_n(\xi)\, d\xi + \int_x^1 (1-\xi)\, x f_n(\xi)\, d\xi \\
&= (1-x)\int_0^x \xi f_n(\xi)\, d\xi + x\int_x^1 (1-\xi) f_n(\xi)\, d\xi \qquad (4)
\end{aligned}
$$

But $f_n(x) = \begin{cases} \frac{1}{2}n & |x-\xi| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. We are told that $0 < \xi - \frac{1}{n} < \xi + \frac{1}{n} < 1$. Hence (4) becomes

$$
\begin{aligned}
u_n(x) &= (1-x)\int_{x-\frac{1}{n}}^{x} \xi \frac{n}{2} d\xi + x\int_x^{x+\frac{1}{n}} (1-\xi)\frac{n}{2} d\xi \\
&= \frac{(1-x)}{2} n \int_{x-\frac{1}{n}}^{x} \xi d\xi + \frac{x}{2} n \int_x^{x+\frac{1}{n}} (1-\xi)\, d\xi \\
&= \frac{(1-x)}{2} n \left(\frac{\xi^2}{2}\right)_{x-\frac{1}{n}}^{x} + \frac{x}{2} n \left(\xi - \frac{\xi^2}{2}\right)_x^{x+\frac{1}{n}} \\
&= \frac{(1-x)}{2} n \left(\frac{x^2}{2} - \frac{\left(x-\frac{1}{n}\right)^2}{2}\right) + \frac{x}{2} n \left(\left(\left(x+\frac{1}{n}\right) - \frac{\left(x+\frac{1}{n}\right)^2}{2}\right) - \left(x - \frac{x^2}{2}\right)\right) \\
&= \left(\frac{1}{2}x + \frac{1}{4n}x - \frac{1}{4n} - \frac{1}{2}x^2\right) - \left(\frac{1}{4n}x\left(2nx - 2n + 1\right)\right) \\
&= -\frac{1}{4n}\left(4nx^2 - 4nx + 1\right) \\
&= x - x^2 - \frac{1}{4n}
\end{aligned}
$$

7.2 Part b

$$
\begin{aligned}
\lim_{n\to\infty} u_n(x) &= \lim_{n\to\infty} x - x^2 - \frac{1}{4n} \\
&= x(1-x)
\end{aligned}
$$

7.3 Part c

This is plot of Green function $G(x;\xi) = \begin{cases} (1-\xi)\,x & x \le \xi \\ \xi(1-x) & x \ge \xi \end{cases}$ for $\xi = 0.3$

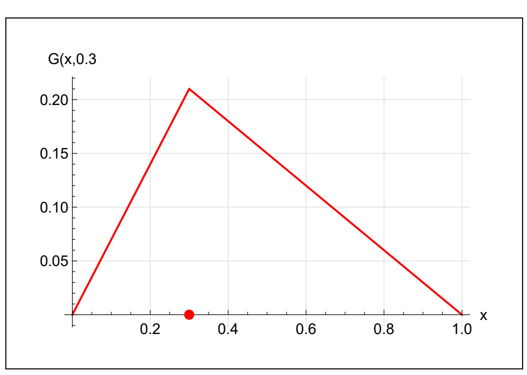

Figure 10: Green function

```
green[x_, z_] := Piecewise[{{(1 - z) x, x < z}, {(1 - x) z, x > z}}]
p = Plot[green[x, 0.3], {x, 0, 1}, PlotStyle → Red,
        GridLines → Automatic, GridLinesStyle → LightGray,
        AxesLabel → {"x", "G(x,0.3"}, BaseStyle → 12,
        Epilog → {Red, {PointSize[.025], Point[{0.3, 0}]}}];
```

Figure 11: Code for the above plot

These are plots of $u_n(x) = x - x^2 - \frac{1}{4n}$ for different n values.

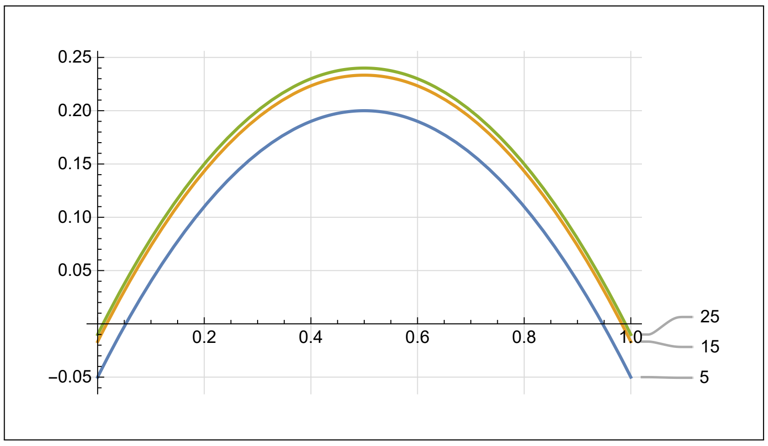

Figure 12: Plot of $u_n(x)$ for different n values

```
u[x_, n_] := x - x^2 - 1/(4 n)
p = Plot[Evaluate[Table[Callout[u[x, n], n], {n, {5, 15, 25}}]], {x, 0, 1},
        AxesOrigin → {0, 0}, GridLines → Automatic,
        GridLinesStyle → LightGray];
```

Figure 13: Code for the above plot

Please note that the plots above do not seem to converge well with what is expected which is the Green function plot earlier. I am not able to find out so far where the problem is.

8 Problem 6.2.11

Let $\omega > 0$. (a) Find the Green's function for the mixed boundary value problem

$$-u'' + \omega^2 u = f(x), \qquad u(0) = 0, u'(1) = 0$$

(b) Use your Green's function to find the solution when $f(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \leq 1 \end{cases}$

Solution

8.1 Part a

When $x \neq \xi$, then Green function satisfies $-\frac{d^2G(x,y)}{dx^2} + \omega^2 G(x,y) = 0$. This means that $\frac{d^2G(x,y)}{dx^2} - \omega^2 G(x,y) = 0$ which has solution

$$G(x,y) = A_1 \cosh(\omega x) + A_2 \sinh(\omega x)$$

Hence Green function is

$$G(x,y) = \begin{cases} A_1 \cosh(\omega x) + A_2 \sinh(\omega x) & 0 < x < y \\ B_1 \cosh(\omega x) + B_2 \sinh(\omega x) & y < x < 1 \end{cases} \tag{1A}$$

At $x = 0$, $G(0,y) = 0 = A_1$. And to find conditions at $x = 1$, then $G'(x,y) = \omega B_1 \sinh(\omega x) + \omega B_2 \cosh(\omega x)$. Hence at $x = 1$ this gives

$$\begin{aligned} G'(1,y) &= 0 \\ &= \omega B_1 \sinh \omega + \omega B_2 \cosh \omega \end{aligned}$$

Therefore $B_1 \sinh \omega + B_2 \cosh \omega = 0$. Or $B_2 = -B_1 \tanh \omega$. Hence (1A) becomes

$$\begin{aligned} G(x,y) &= \begin{cases} A_2 \sinh(\omega x) & 0 < x < y \\ B_1 \cosh(\omega x) - B_1 \tanh \omega \sinh(\omega x) & y < x < 1 \end{cases} \\ &= \begin{cases} A_2 \sinh(\omega x) & 0 < x < y \\ B_1 (\cosh(\omega x) - \tanh \omega \sinh(\omega x)) & y < x < 1 \end{cases} \end{aligned}$$

But $\cosh(\omega x) - \tanh \omega \sinh(\omega x) = \frac{\cosh(\omega - \omega x)}{\cosh \omega}$. The above becomes

$$G(x,y) = \begin{cases} A_2 \sinh(\omega x) & 0 < x < y \\ B_1 \frac{\cosh(\omega - \omega x)}{\cosh \omega} & y < x < 1 \end{cases} \tag{1}$$

We now need to determine A_2, B_1. From continuity condition of $G(x,y)$ at $x = y$ we obtain the first equation

$$A_2 \sinh(\omega y) = B_1 \frac{\cosh(\omega - \omega y)}{\cosh \omega} \tag{2}$$

And

$$\frac{dG(x,y)}{dx} = \begin{cases} A_2 \omega \cosh(\omega x) & x < y \\ B_1 \left(\frac{-\omega \sinh(\omega - \omega x)}{\cosh \omega} \right) & x > y \end{cases}$$

Evaluated at $x = y$

$$\frac{dG(x,y)}{dx} = \begin{cases} A_2 \omega \cosh(\omega y) & x < y \\ B_1 \left(\frac{-\omega \sinh(\omega - \omega y)}{\cosh \omega} \right) & x > y \end{cases}$$

There is a jump discontinuity in $\frac{dG(x,y)}{dx}$ of value 1 at $x = y$. Therefore this condition gives the second equation we need

$$A_2 \omega \cosh(\omega y) + B_1 \frac{\omega \sinh(\omega - \omega y)}{\cosh \omega} = 1 \tag{3}$$

Solving (2,3) for A_2, B_1 gives

$$A_2 = \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}$$

$$B_1 = \frac{\sinh\left(\omega y\right)}{\omega}$$

Substituting the above into (1) gives the Green function

$$G\left(x,y\right) = \begin{cases} \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh(\omega)}\sinh\left(\omega x\right) & 0<x<y \\ \frac{\cosh(\omega(1-x))}{\omega\cosh\omega}\sinh\left(\omega y\right) & y<x<1 \end{cases} \tag{4}$$

8.2 Part b

Using the superposition formula

$$\begin{aligned} u\left(x\right) &= \overbrace{\int_0^x G\left(x,y\right)f\left(y\right)dy}^{y<x} + \overbrace{\int_x^1 G\left(x,y\right)f\left(y\right)dy}^{y>x} \\ &= \int_0^x \frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega y\right)f\left(y\right)dy + \int_x^1 \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)f\left(y\right)dy \end{aligned}$$

But $f\left(x\right) = \begin{cases} 1 & 0<x<\frac{1}{2} \\ -1 & \frac{1}{2}<x\leq 1 \end{cases}$, hence the above reduces to

case $x<\frac{1}{2}$

$$\begin{aligned} u\left(x\right) &= \int_0^x \frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega y\right)dy + \int_x^{\frac{1}{2}} \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy - \int_{\frac{1}{2}}^1 \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy \\ &= \frac{1}{\omega^2} - \frac{\left(e^{\frac{\omega}{2}} - e^{-\frac{\omega}{2}} + e^{-\omega}\right)e^{\omega x} + \left(e^{\omega} - e^{\frac{\omega}{2}} + e^{-\frac{\omega}{2}}\right)e^{-\omega x}}{\omega^2\left(e^{\omega}+e^{-\omega}\right)} \end{aligned}$$

case $x>\frac{1}{2}$

$$\begin{aligned} u\left(x\right) &= \int_0^{\frac{1}{2}} \frac{\cosh\left(\omega\left(1-x\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega y\right)dy - \int_{\frac{1}{2}}^x \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy - \int_x^1 \frac{\cosh\left(\omega\left(1-y\right)\right)}{\omega\cosh\left(\omega\right)}\sinh\left(\omega x\right)dy \\ &= -\frac{1}{\omega^2} - \frac{\left(e^{\frac{-\omega}{2}} - e^{-\omega} + e^{-\frac{3}{2}\omega}\right)e^{\omega x} + \left(e^{\frac{3}{2}\omega} - e^{\omega} + e^{\frac{\omega}{2}}\right)e^{-\omega x}}{\omega^2\left(e^{\omega}+e^{-\omega}\right)} \end{aligned}$$

9 Problem 6.2.12

Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x), u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

Solution

To find out if it admits a Green function, we will see if we can solve for the constants that show up in the formulation of Green function. If not able to find a solution, then no Green function.

When $x \neq \xi$, then Green function satisfies $-\frac{d^2G(x,y)}{dx^2} + \omega^2 G(x,y) = 0$. This means that

$$G(x,y) = A_1 \cosh(\omega x) + A_2 \sinh(\omega x)$$

Hence Green function is

$$G(x,y) = \begin{cases} A_1 \cosh(\omega x) + A_2 \sinh(\omega x) & 0 < x < y \\ B_1 \cosh(\omega x) + B_2 \sinh(\omega x) & y < x < 1 \end{cases} \tag{1A}$$

On the left end, $\frac{d}{dx}G(x,y) = \omega A_1 \sinh(\omega x) + \omega A_2 \cosh(\omega x)$. Hence At $x = 0$, $G'(0,y) = 0 = \omega A_2$. Therefore $A_2 = 0$. On the right side $\frac{d}{dx}G(x,y) = \omega B_1 \sinh(\omega x) + \omega B_2 \cosh(\omega x)$. At $x = 1$, then $G'(x,y) = \omega B_1 \sinh(\omega) + \omega B_2 \cosh(\omega) = 0$. Therefore $B_1 \sinh\omega + B_2 \cosh\omega = 0$. Or $B_2 = -B_1 \tanh\omega$. Hence (1A) becomes

$$G(x,y) = \begin{cases} A_1 \cosh(\omega x) & 0 < x < y \\ B_1 \cosh(\omega x) - B_1 \tanh\omega \sinh(\omega x) & y < x < 1 \end{cases}$$
$$= \begin{cases} A_1 \cosh(\omega x) & 0 < x < y \\ B_1 (\cosh(\omega x) - \tanh\omega \sinh(\omega x)) & y < x < 1 \end{cases}$$

But $\cosh(\omega x) - \tanh\omega \sinh(\omega x) = \frac{\cosh(\omega(1-x))}{\cosh\omega}$. The above becomes

$$G(x,y) = \begin{cases} A_1 \cosh(\omega x) & 0 < x < y \\ B_1 \frac{\cosh(\omega(1-x))}{\cosh\omega} & y < x < 1 \end{cases}$$

Now we will try to see if we can determine A_1, B_1. Continuity condition at $x = y$ gives the first equation

$$A_1 \cosh(\omega y) = \frac{B_1}{\cosh\omega} \cosh(\omega(1-y)) \tag{1}$$

And

$$\frac{dG(x,y)}{dx} = \begin{cases} A_1 \omega \sinh(\omega x) & 0 < x < y \\ -\frac{B_1}{\cosh\omega} \omega \sinh(\omega(1-x)) & y < x < 1 \end{cases}$$

Hence at $x = y$ to satisfy the jump discontinuity in $\frac{dG(x,y)}{dx}$ the second equation is

$$A\omega \sinh(\omega y) + \frac{B_1}{\cosh\omega} \omega \sinh(\omega(1-y)) = 1 \tag{2}$$

Solving (1,2) for A, B gives

$$A_1 = \frac{\cosh(\omega(1-y))}{\omega \sinh(\omega)}$$
$$B_1 = \frac{\cosh(\omega y)}{\omega \sinh(\omega)} \cosh(\omega)$$

Hence Green function exist. Substituting the above in Green function above gives

$$G(x,y) = \begin{cases} \frac{\cosh(\omega(1-y))}{\omega \sinh(\omega)} \cosh(\omega x) & 0 < x < y \\ \frac{\cosh(\omega(1-x))}{\omega \sinh(\omega)} \cosh(\omega y) & y < x < 1 \end{cases}$$

Here is a plot of the above when the pulse at $y = 0.25$ with $\omega = 1$

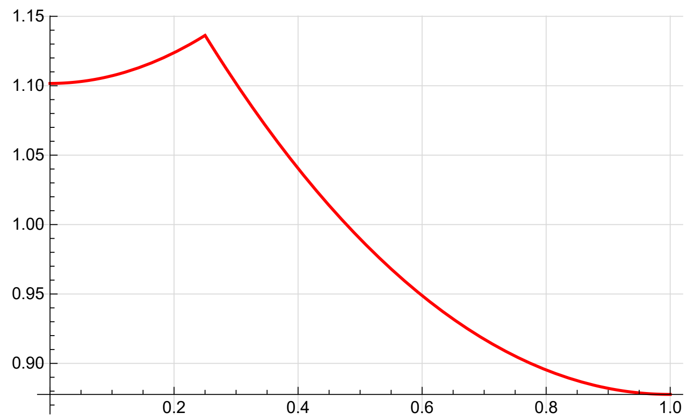

Figure 14: Plot of the Green function found

```
p = With[{y = 0.25, w = 1},
        Plot[Cosh[w (1 - y)]/(w Sinh[w]) Cosh[w x] HeavisideTheta[-x + y] +
            Cosh[w (1 - x)]/(w Sinh[w]) Cosh[w y] HeavisideTheta[x - y], {x, 0, 1},
          PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray]
    ];
```

Figure 15: Code used for the above plot

The integral formula is

$$\begin{aligned} u(x) &= \int_0^x \frac{\cosh(\omega y)}{\omega \sinh(\omega)} \cosh(\omega(1-x)) f(y)\, dy + \int_x^1 \frac{\cosh(\omega(1-y))}{\omega \sinh(\omega)} \cosh(\omega x) f(y)\, dy \\ &= \frac{\cosh(\omega(1-x))}{\omega \sinh(\omega)} \int_0^x \cosh(\omega y) f(y)\, dy + \frac{\cosh(\omega x)}{\omega \sinh(\omega)} \int_x^1 \cosh(\omega(1-y)) f(y)\, dy \end{aligned}$$