4.3.24(a)

(a) $v(y) = u(e^y)$ solves a constant coefficient second-order ordinary differential equation with a double root r, and hence $v(y) = c_1 e^{ry} + c_2 y e^{ry}$. Therefore,

$$u(x) = c_1 |x|^r + c_2 |x|^r \log|x|.$$

4.3.25(c)(d)

(c) $u(x,y) = \frac{1}{8} r^4 \cos 4\theta + 2r^2 \cos 2\theta + 6 = \frac{1}{8} x^4 - \frac{3}{4} x^2 y^2 + \frac{1}{8} y^4 + 2x^2 - 2y^2 + 6$

(d) $u(x,y) = r\cos\theta = x$.

4.3.33

$$u(r,\theta) = \frac{a_0}{2} \frac{\log r}{\log 2} + \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2^n - 2^{-n}} (a_n \cos n\theta + b_n \sin n\theta),$$

where a_n, b_n are the usual Fourier coefficients of $h(\theta)$.

4.3.38

First, if $C = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} |h(\theta)|\, d\theta$, then the Fourier coefficients are bounded by

$$\left.\begin{aligned} |a_n| &\le \frac{1}{\pi}\int_{-\pi}^{\pi} |h(\theta)\cos n\theta|\, d\theta \\ |b_n| &\le \frac{1}{\pi}\int_{-\pi}^{\pi} |h(\theta)\sin n\theta|\, d\theta \end{aligned}\right\} \le \frac{1}{\pi}\int_{-\pi}^{\pi} |h(\theta)|\, d\theta = C.$$

Thus, the summands in (4.115) are bounded by

$$|a_n r^n \cos n\theta + b_n r^n \sin n\theta| \le r^n \big(|a_n| + |b_n|\big) \le 2C r_\star^n.$$

According to the Weierstrass M test, since the geometric series $\sum_{n=1}^{\infty} 2C r_\star^n < \infty$ converges, the series (4.115) converges uniformly. *Q.E.D.*

4.3.42

Given such a curve, let $\delta > 0$ be the minimum distance between C and the boundary $\partial\Omega$, which is positive since C is assumed to lie in the interior of Ω and both curves are compact (closed and bounded). Let $(x_i, y_i) \in C$, $i = 0, \ldots, n$, be a finite sequence of points on the curve with $(x_n, y_n) = (x, y)$ and such that the distance from (x_i, y_i) to (x_{i+1}, y_{i+1}) is $\le \frac{1}{2}\delta$, which implies that the disk centered at (x_i, y_i) whose boundary circle passes through (x_{i+1}, y_{i+1}) is contained in Ω. Using the preceding argument, a straightforward induction then shows that

$$M^\star = u(x_0, y_0) = u(x_1, y_1) = u(x_2, y_2) = \cdots = u(x_n, y_n) = u(x, y),$$

as desired. *Q.E.D.*

4.3.46

The rescaled function $\widehat{u}(x, y) = u(Rx, Ry)$ satisfies the boundary value problem (4.101) on the unit disk, and hence by (4.126)

$$u(r, \theta) = \widehat{u}(r/R, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) \, \frac{1 - r^2/R^2}{1 + r^2/R^2 - 2\,(r/R) \cos(\theta - \phi)} \, d\phi.$$

4.4.4(a)(c)

(a) Elliptic when $x \neq 0$; parabolic when $x = 0$.

(c) Parabolic when $x + t \neq 0$; degenerate when $t = -x$.

4.4.11

By the chain rule,

$$\frac{\partial u}{\partial y} = \mathrm{i}\, \frac{\partial u}{\partial t}, \qquad \frac{\partial^2 u}{\partial y^2} = \mathrm{i}^2\, \frac{\partial^2 u}{\partial t^2} = -\, \frac{\partial^2 u}{\partial t^2}, \qquad \text{and hence} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}.$$

Thus, this complex change of variables maps the elliptic Laplace equation to the hyperbolic wave equation, and the type is not preserved.

4.4.16

False. The equation is elliptic and so has no real characteristics.