HW 7

Math 5587
Elementary Partial Differential Equations I

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1 Problem 4.3.24

Use the method in Exercise 4.3.23 to solve an Euler equation whose characteristic equation has a double root $r_1 = r_2 = r$

Solution

1.1 Part (a)

Euler ODE is

$$ax^2u''(x) + bxu'(x) + cu(x) = 0$$

By assuming $u = x^r$ then $u' = rx^{r-1}, u'' = r(r-1)x^{r-2}$. Substituting back into the above ODE gives

$$\begin{aligned} ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r &= 0 \\ ar(r-1) + br + c &= 0 \\ ar^2 - ar + br + c &= 0 \\ ar^2 + r(b-a) + c &= 0 \end{aligned}$$

Solving for r gives the roots

$$r_{1,2} = -\frac{b-a}{2a} \pm \frac{1}{2a}\sqrt{(b-a)^2 - 4ac} \tag{1}$$

Double root means that $r = r_1 = r_2 = -\frac{b-a}{2a}$. Hence the first solution of the ODE is

$$u_1 = x^{r_1}$$

And now we need to find the second solution. Using reduction of order method, we assume the second solution is

$$u_2(x) = v(x)u_1(x) \tag{2}$$

And we need to determine the function $v(x)$. Therefore

$$\begin{aligned} u_2' &= v'u_1 + vu_1' \\ u_2'' &= v''u_1 + v'u_1' + v'u_1' + vu_1'' \\ &= v''u_1 + 2v'u_1' + vu_1'' \end{aligned}$$

Substituting the above into the ODE gives

$$\begin{aligned} ax^2\left(v''u_1 + 2v'u_1' + vu_1''\right) + bx\left(v'u_1 + vu_1'\right) + cvu_1 &= 0 \\ v''\left(ax^2u_1\right) + v'\left(2ax^2u_1' + bxu_1\right) + v\left(ax^2u_1'' + bxu_1' + cu_1\right) &= 0 \end{aligned}$$

But $ax^2u_1'' + bxu_1' + cu_1 = 0$ since u_1 is a solution. The above now simplifies to

$$v''\left(ax^2u_1\right) + v'\left(2ax^2u_1' + bxu_1\right) = 0$$

But $u_1 = x^r$, hence $u_1' = rx^{r-1}$ and the above becomes

$$
\begin{aligned}
v''\left(ax^2x^r\right) + v'\left(2arx^2x^{r-1} + bxx^r\right) &= 0\\
av''x^{r+2} + v'\left(2arx^{r+1} + bx^{r+1}\right) &= 0\\
av''x^{r+2} + v'(2ar + b)\,x^{r+1} &= 0\\
(av''x + v'(2ar + b))\,x^{r+1} &= 0\\
av''x + v'(2ar + b) &= 0
\end{aligned}
$$

But $r = r_1 = -\frac{b-a}{2a}$ from (1) since double root. The above simplifies to

$$
\begin{aligned}
av''x + v'\left(2a\left(-\frac{b-a}{2a}\right) + b\right) &= 0\\
av''x + v'((-b + a) + b) &= 0\\
av''x + av' &= 0\\
v''x + v' &= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}(xv') &= 0\\
xv' &= C_1\\
v' &= \frac{C_1}{x}\\
v &= C_1 \ln x + C_2
\end{aligned}
$$

Now that we found $v(x)$, then using (2) we find the second solution to the ODE as

$$
\begin{aligned}
u_2 &= vu_1\\
&= (C_1 \ln x + C_2)\,x^2
\end{aligned}
$$

Therefore the complete solution is

$$
u = C_0x^r + (C_1 \ln x + C_2)\,x^r
$$

By combining constants, the above simplifies to

$$
u(x) = Ax^r + Bx^r \ln x
$$

2 Problem 4.3.25

Solve the following boundary value problems (c) $\nabla^2 u = 0, x^2+y^2<4, u=x^4, x^2+y^2=4$ (d) $\nabla^2 u = 0, x^2+y^2<1, \frac{\partial u}{\partial n} = x, x^2+y^2=1$

Solution

2.1 Part c

In polar coordinates, where $x = r\cos\theta, y = r\sin\theta$, we need to solve for $u(r,\theta)$ inside disk of radius $r_0 = 4$. The Laplace PDE in polar coordinates is

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \qquad 0<r<r_0, -\pi<\theta<\pi \\ u(r_0,\theta) &= f(\theta) = (r_0\cos\theta)^4 \\ u(-\pi) &= u(\pi) \\ u_\theta(-\pi) &= u_\theta(\pi) \end{aligned}$$

The solution to Laplace PDE of radius r_0 can be found using separation of variables and derived in the textbook (full derivation is also given in this HW in problem 4.3.33 below). The Fourier series solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum a_n \left(\frac{r}{r_0}\right)^n \cos(n\theta) + b_n \left(\frac{r}{r_0}\right)^n \sin(n\theta)$$

Since $r_0 = 4$ the above becomes

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{r}{4}\right)^n \cos(n\theta) + b_n \left(\frac{r}{4}\right)^n \sin(n\theta) \tag{1A}$$

Where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} 256\cos^4\theta d\theta \\ &= \frac{256}{\pi}\int_{-\pi}^{\pi}\cos^4\theta d\theta \\ &= \frac{256}{\pi}\left(\frac{3\theta}{8} + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right)_{-\pi}^{\pi} \\ &= \frac{256}{\pi}\left(\frac{3\pi}{8} + \frac{3\pi}{8}\right) \\ &= \frac{256}{\pi}\left(\frac{3\pi}{4}\right) \\ &= 192 \end{aligned}$$

And

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} 256\cos^4(\theta)\cos(n\theta)\,d\theta \\ &= \frac{256}{\pi}\int_{-\pi}^{\pi} \cos^4(\theta)\cos(n\theta)\,d\theta \end{aligned}$$

To evaluate the above integral, we will start by using the identity

$$\cos^4(\theta) = \frac{3}{8} + \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta)$$

Therefore the integral now becomes

$$\begin{aligned} a_n &= \frac{256}{\pi}\int_{-\pi}^{\pi}\left(\frac{3}{8} + \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta)\right)\cos(n\theta)\,d\theta \\ &= \frac{256}{\pi}\left[\frac{3}{8}\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta + \frac{1}{8}\int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta + \frac{1}{2}\int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta\right] \end{aligned} \tag{1}$$

But $\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta = 0$ and $\int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta$ is not zero, only for $n = 4$ by orthogonality of cosine functions. Hence

$$\begin{aligned} \int_{-\pi}^{\pi}\cos(4\theta)\cos(n\theta)\,d\theta &= \int_{-\pi}^{\pi}\cos^2(4\theta)\,d\theta \\ &= \pi \end{aligned}$$

And similarly, $\int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta$ is not zero, only for $n = 2$ by orthogonality of cosine functions. Hence

$$\begin{aligned} \int_{-\pi}^{\pi}\cos(2\theta)\cos(n\theta)\,d\theta &= \int_{-\pi}^{\pi}\cos^2(2\theta)\,d\theta \\ &= \pi \end{aligned}$$

Using these results in (1) gives, for $n = 2$

$$\begin{aligned} a_2 &= \frac{256}{\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\cos^2(2\theta)\,d\theta\right] \\ &= \frac{256}{\pi}\left(\frac{\pi}{2}\right) \\ &= 128 \end{aligned}$$

And for $n = 4$

$$\begin{aligned} a_4 &= \frac{256}{\pi}\left[\frac{1}{8}\int_{-\pi}^{\pi}\cos^2(4\theta)\,d\theta\right] \\ &= \frac{256}{\pi}\left(\frac{\pi}{8}\right) \\ &= 32 \end{aligned}$$

And all other a_n are zero. Now that we found all a_n, and since $b_n = 0$ for all n (because $f(\theta)$ is even function) then the solution (1A) becomes

$$\begin{aligned} u(r,\theta) &= \frac{192}{2} + a_2 \left(\frac{r}{4}\right)^2 \cos(2\theta) + a_4 \left(\frac{r}{4}\right)^4 \cos(4\theta) \\ &= 96 + 128\left(\frac{r^2}{16}\right)\cos(2\theta) + 32\frac{r^4}{256}\cos(4\theta) \end{aligned}$$

Therefore

$$\boxed{u(r,\theta) = 96 + 8r^2\cos(2\theta) + \frac{1}{8}r^4\cos(4\theta)}$$

Here is plot of the above solution.

```
sol = 96 + 8 r^2 Cos[2 θ] + 1/8 r^4 Cos[4 θ];
ParametricPlot3D[{r Cos[θ], r Sin[θ], sol}, {r, 0, 4}, {θ, 0, 2 Pi},
   AxesLabel → {x, y, "u(x,y"}, ImageSize → 400, BoxRatios → {1,  1,  1}, BaseStyle → 14]
```

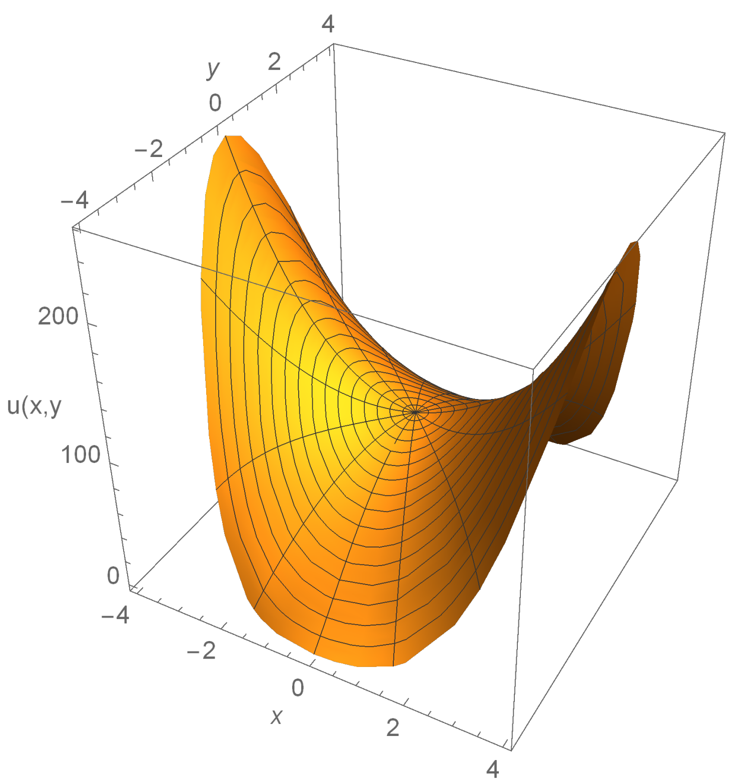

Figure 1: Solution plot to the above problem with code used

It is also possible to use, as shown in textbook, the closed form sum as given in theorem 4.6

as

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(\phi\right)\frac{1-r^2}{1+r^2-2r\cos\left(\theta-\phi\right)}d\phi$$

Notice that theorem 4.6 is for a unit disk. Since the disk here has radius 4 then r is changed to $\frac{r}{4}$ in 4.126 as given in book. Here $f(\theta)=(4\cos\theta)^4$. Hence the above becomes

$$\begin{aligned}
u(r,\theta)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}256\cos^4\left(\phi\right)\frac{1-\left(\frac{r}{4}\right)^2}{1+\left(\frac{r}{4}\right)^2-2\left(\frac{r}{4}\right)\cos\left(\theta-\phi\right)}d\phi\\
&=\frac{128}{\pi}\int_{-\pi}^{\pi}\cos^4\left(\phi\right)\frac{1-\frac{r^2}{16}}{1+\frac{r^2}{16}-\frac{r}{2}\cos\left(\theta-\phi\right)}d\phi\\
&=\frac{128}{\pi}\int_{-\pi}^{\pi}\cos^4\left(\phi\right)\frac{16-r^2}{16+r^2-8r\cos\left(\theta-\phi\right)}d\phi
\end{aligned}$$

But evaluating the above integral was hard to do by hand. It should of course give the same solution as found above using Fourier series.

2.2 Part d

In polar coordinates, where $x=r\cos\theta, y=r\sin\theta$, we need to solve for $u(r,\theta)$ inside disk of radius $r_0=1$. The Laplace PDE in polar coordinates is

$$\begin{aligned}
u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}&=0 \qquad 0<r<1,-\pi<\theta<\pi\\
u_r(1,\theta)&=f(\theta)=\cos\theta\\
u(-\pi)&=u(\pi)\\
u_\theta(-\pi)&=u_\theta(\pi)
\end{aligned}$$

Using separation of variables, let $u(r,\theta)=R(r)\Theta(\theta)$ the solution is given by

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nr^n\cos(n\theta)+b_nr^n\sin(n\theta) \tag{1}$$

At $r=r_0=1$ we have that $\frac{\partial u(r,\theta)}{\partial r}=\cos\theta$ (since $x=r\cos\theta$ but $r=1$ at boundary). The above becomes

$$\cos\theta=\sum_{n=1}^{\infty}na_nr^{n-1}\cos(n\theta)+nb_nr^{n-1}\sin(n\theta)$$

Therefore $n=1$ is only term that survives in the sum. Hence $a_1=1$ and all others are zero. The solution (1) becomes

$$u(r,\theta)=\frac{a_0}{2}+r\cos(\theta)$$

The solution is not unique as there is a_0 arbitrary constant.

3 Problem 4.3.33

Write out the series solution to the boundary value problem $u(1,\theta) = 0, u(2,\theta) = h(\theta)$ for the Laplace equation on an annulus $1 < r < 2$.

Solution

Using a for the inner radius and b for the outer radius to keep the solution more general. At the end these are replaced with $a = 1, b = 2$.

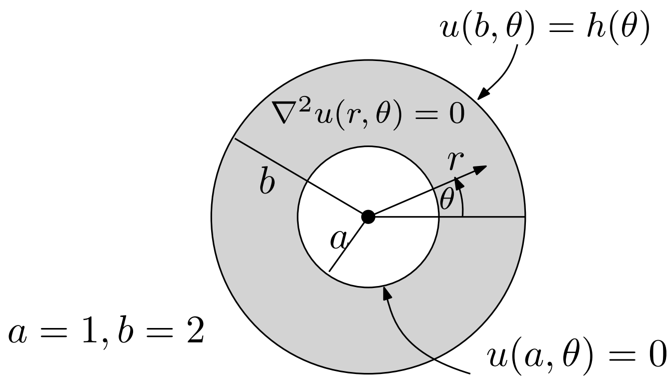

Figure 2: PDE to solve using polar coordinates

The Laplace PDE in polar coordinates is

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With

$$\begin{aligned} u(a,\theta) &= 0 \\ u(b,\theta) &= h(\theta) \end{aligned} \tag{B}$$

Let the solution be

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting this assumed solution back into the (A) gives

$$r^2 R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta$ gives

$$\begin{aligned} r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta} \end{aligned}$$

Since each side depends on different independent variable and they are equal, they must be equal to the same constant. say λ.

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. Hence

$$\begin{aligned}\Theta''+\lambda\Theta&=0 \qquad (1)\\ \Theta(-\pi)&=\Theta(\pi)\\ \Theta'(-\pi)&=\Theta'(\pi)\end{aligned}$$

And

$$\begin{aligned}r^2R''+rR'-\lambda R&=0 \qquad (2)\\ R(a)&=0\end{aligned}$$

Starting with ODE (1) with periodic boundary conditions.

Case $\lambda<0$ The solution is

$$\Theta(\theta)=A\cosh\left(\sqrt{|\lambda|}\theta\right)+B\sinh\left(\sqrt{|\lambda|}\theta\right)$$

First B.C. gives

$$\begin{aligned}\Theta(-\pi)&=\Theta(\pi)\\ A\cosh\left(-\sqrt{|\lambda|}\pi\right)+B\sinh\left(-\sqrt{|\lambda|}\pi\right)&=A\cosh\left(\sqrt{|\lambda|}\pi\right)+B\sinh\left(\sqrt{|\lambda|}\pi\right)\\ A\cosh\left(\sqrt{|\lambda|}\pi\right)-B\sinh\left(\sqrt{|\lambda|}\pi\right)&=A\cosh\left(\sqrt{|\lambda|}\pi\right)+B\sinh\left(\sqrt{|\lambda|}\pi\right)\\ 2B\sinh\left(\sqrt{|\lambda|}\pi\right)&=0\end{aligned}$$

But $\sinh=0$ only at zero and $\lambda\neq0$, hence $B=0$ and the solution becomes

$$\begin{aligned}\Theta(\theta)&=A\cosh\left(\sqrt{|\lambda|}\theta\right)\\ \Theta'(\theta)&=A\sqrt{\lambda}\cosh\left(\sqrt{|\lambda|}\theta\right)\end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned}\Theta'(-\pi)&=\Theta'(\pi)\\ A\sqrt{|\lambda|}\cosh\left(-\sqrt{|\lambda|}\pi\right)&=A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right)\\ A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right)&=A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right)\\ 2A\sqrt{|\lambda|}\cosh\left(\sqrt{|\lambda|}\pi\right)&=0\end{aligned}$$

But $\cosh$ is never zero, hence $A=0$. Therefore trivial solution and $\lambda<0$ is not an eigenvalue.

Case $\lambda=0$ The solution is $\Theta=A\theta+B$. Applying the first B.C. gives

$$\begin{aligned}\Theta(-\pi)&=\Theta(\pi)\\ -A\pi+B&=\pi A+B\\ 2\pi A&=0\\ A&=0\end{aligned}$$

And the solution becomes $\Theta=B_0$. A constant. Hence $\lambda=0$ is an eigenvalue.

<u>Case $\lambda > 0$</u>

The solution becomes

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$
$$\Theta' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)$$

Applying first B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi)\\
A\cos\left(-\sqrt{\lambda}\pi\right) + B\sin\left(-\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right)\\
A\cos\left(\sqrt{\lambda}\pi\right) - B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right)\\
2B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)
\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}
\Theta'(-\pi) &= \Theta'(\pi)\\
-A\sqrt{\lambda}\sin\left(-\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(-\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)\\
2A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (4)
\end{aligned}$$

Equations (3,4) can be both zero only if $A = B = 0$ which gives trivial solution, or when $\sin\left(\sqrt{\lambda}\pi\right) = 0$. Therefore taking $\sin\left(\sqrt{\lambda}\pi\right) = 0$ gives a non-trivial solution. Hence

$$\begin{aligned}
\sqrt{\lambda}\pi &= n\pi \qquad n = 1,2,3,\cdots\\
\lambda_n &= n^2 \qquad n = 1,2,3,\cdots
\end{aligned}$$

Hence the eigenfunctions are

$$\{1, \cos(n\theta), \sin(n\theta)\} \qquad n = 1,2,3,\cdots \qquad (5)$$

<u>Now the R equation is solved</u>

The <u>case for $\lambda = 0$</u> gives from (2)

$$\begin{aligned}
r^2R'' + rR' &= 0\\
R'' + \frac{1}{r}R' &= 0 \qquad r \neq 0
\end{aligned}$$

As was done in last problem, the solution to this is

$$R_0(r) = A\ln r + C$$

Applying the B.C. $R(a) = 0$ gives

$$0 = A\ln a + C$$
$$C = -A\ln a$$

Hence the solution becomes

$$R_0(r) = A\ln r - A\ln a$$
$$= A\ln\frac{r}{a}$$

<u>Case $\lambda > 0$</u> The ODE (2) becomes

$$r^2R'' + rR' - n^2R = 0 \qquad n = 1, 2, 3, \cdots$$

Let $R = r^p$, the above becomes

$$r^2p\left(p-1\right)r^{p-2} + rpr^{p-1} - n^2r^p = 0$$
$$p\left(p-1\right)r^p + pr^p - n^2r^p = 0$$
$$p\left(p-1\right) + p - n^2 = 0$$
$$p^2 = n^2$$
$$p = \pm n$$

Hence the solution is

$$R_n(r) = Cr^n + D\frac{1}{r^n} \qquad n = 1, 2, 3, \cdots$$

Applying the boundary condition $R(a) = 0$ gives

$$0 = Ca^n + D\frac{1}{a^n}$$
$$-Ca^n = D\frac{1}{a^n}$$
$$D = -Ca^{2n}$$

The solution becomes

$$R_n(r) = Cr^n - Ca^{2n}\frac{1}{r^n} \qquad n = 1, 2, 3, \cdots$$
$$= C_n\left(r^n - \frac{a^{2n}}{r^n}\right)$$

Hence the complete solution for $R(r)$ is

$$R(r) = A\ln\frac{r}{a} + \sum_{n=1}^{\infty} C_n\left(r^n - \frac{a^{2n}}{r^n}\right) \tag{6}$$

Using (5),(6) gives

$$u_n(r,\theta) = R_n\Theta_n$$
$$u(r,\theta) = \left(A\ln\frac{r}{a} + \sum_{n=1}^{\infty} C_n\left(r^n - \frac{a^{2n}}{r^n}\right)\right)\left(A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

Combining constants to simplify things gives

$$u(r,\theta) = A\ln\frac{r}{a} + \sum_{n=1}^{\infty}\left(r^n - \frac{a^{2n}}{r^n}\right)(A_n\cos(n\theta) + B_n\sin(n\theta))$$

But $a = 1$, then above simplifies to

$$u(r,\theta) = A\ln r + \sum_{n=1}^{\infty}\left(r^n - \frac{1}{r^n}\right)(A_n\cos(n\theta) + B_n\sin(n\theta)) \tag{7}$$

At $r = b$ we use $u(b,\theta) = h(\theta)$ to find A_0, A_n, B_n.

$$\begin{aligned} u(b,\theta) &= h(\theta) \\ h(\theta) &= A_0\ln b + \sum_{n=1}^{\infty}\left(b^n + \frac{1}{b^n}\right)(A_n\cos(n\theta) + B_n\sin(n\theta)) \end{aligned}$$

Hence

$$\begin{aligned} A_0\ln b &= \frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta \\ A_n\left(b^n + \frac{1}{b^n}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta \\ B_n\left(b^n + \frac{1}{b^n}\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin(n\theta)\,d\theta \end{aligned}$$

The solution (7) becomes

$$u(r,\theta) = \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln b} + \sum_{n=1}^{\infty}\frac{\left(r^n - \frac{1}{r^n}\right)}{b^n + \frac{1}{b^n}}\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right)$$

But $b = 2$ and the above becomes

$$\begin{aligned} u(r,\theta) &= \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln 2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left(r^n - \frac{1}{r^n}\right)}{2^n + \frac{1}{2^n}}\left(\left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right) \\ &= \left(\frac{2}{\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta\right)\frac{\ln r}{\ln 2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2^n}{r^n}\frac{\left(r^{2n} - 1\right)}{2^{2n} + 1}\left(\left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\cos(n\theta) + \left(\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta\right)\sin(n\theta)\right) \end{aligned}$$

4 Problem 4.3.38

Suppose $\int_{-\pi}^{\pi} |h(\theta)| \, d\theta < \infty$. Prove that (4.115) converges uniformly to the solution to the boundary value problem (4.101) on any smaller disk $D_{r_*} = \{r \leq r_* < 1\} \subsetneq D_1$

Solution

4.115 is solution for $u(r,\theta)$ inside unit disk $0 < r < 1$ and $u = h(\theta)$ at $r = 1$.

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \tag{4.115}$$

This problem is asking to show that the Fourier series solution 4.115 converges uniformly to solution of Laplace PDE $\nabla^2 u = 0$ inside disk with radius less than unity with above boundary conditions.

Let $f_n = r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$, then to show uniform convergence, we need to show that for any $\varepsilon > 0$, there exist integer $N(\varepsilon)$ such that for all $n > N$ the following is true

$$|u_n - u_*| < \varepsilon$$

Where

$$u_* = \left(\frac{r}{r_*}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

Hence we need to show, we can find N such that for all $n > N$

$$\left| r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) - \left(\frac{r}{r_*}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \right| < \varepsilon$$

But

$$\begin{aligned} \left| r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) - \left(\frac{r}{r_*}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \right| &= \left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \right| \\ &= \left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) \right| \left| a_n \cos(n\theta) + b_n \sin(n\theta) \right| \end{aligned} \tag{1}$$

But $|a_n \cos(n\theta) + b_n \sin(n\theta)|$ can be made as small as we want by increasing n. This is because

$$|a_n \cos(n\theta) + b_n \sin(n\theta)| \leq |a_n \cos(n\theta)| + |b_n \sin(n\theta)|$$

And since $\int_{-\pi}^{\pi} |h(\theta)| \, d\theta < \infty$ it implies the Fourier series coefficients $a_n, b_n \to 0$ as $n \to \infty$ per Lemma 3.40 on page 112. Hence (1) can be made as small as we want for large n and it will remain smaller as n increases because $\left| \left(r^n - \left(\frac{r}{r_*}\right)^n\right) \right| < 1$.

Therefore there exist such an $N(\varepsilon)$. Hence u converges uniformly to u_*.

5 Problem 4.3.42

Complete the proof of Theorem 4.9 by showing that $u(x,y) = M^*$ for all $(x,y) \in \Omega$. Hint: Join (x_0, y_0) to (x,y) by curve $C \subset \Omega$ of finite length, and use the preceding part of the proof to inductively deduce the existence of a finite sequence of points $(x_i, y_i) \in C, i = 0, \cdots, n$ with $(x_n, y_n) = (x,y)$ and such that $u(x_i, y_i) = M^*$

Solution

Theorem 4.9 : Let u be a nonconstant harmonic function defined on a bounded domain Ω and continuous on $\partial\Omega$. Then u achieves its maximum and minimum values only at boundary points of the domain. In other words, if $m = \min\{u(x,y) | (x,y) \in \partial\Omega\}$, $M = \max\{u(x,t) | (x,y) \in \partial\Omega\}$ are respectively, its maximum and minimum values on the boundary, then $m < u(x,y) < M$ at all interior points $(x,y) \in \Omega$.

The book gives the proof showing that maximum M^* occurs on the boundary $\partial\Omega$. We are asked here to show that once we determined that given a circle inside Ω and assuming the maximum is at it center meaning all points inside this disk are $u = M^*$ then this implies that all points inside Ω must also be $u = M^*$ leading to contradiction of the nonconstant requirement. Hence the starting point is this diagram

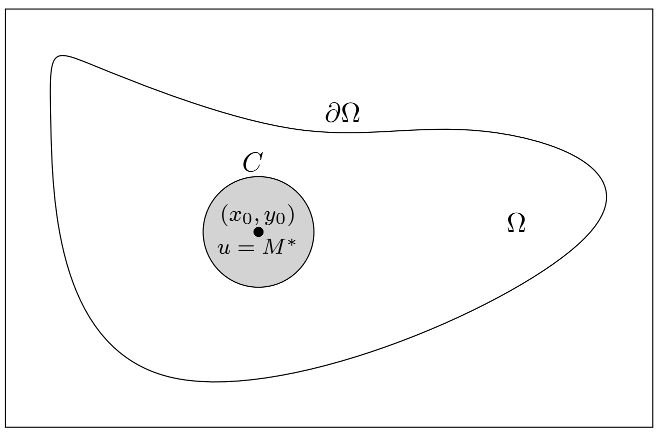

Figure 3: All points inside C have same value M^*

Now, we pick a new point from inside the disk C near the edge and apply the first part of the proof to show that all points inside the new disk C_2 also have $u = M^*$ there. So we have this new diagram.

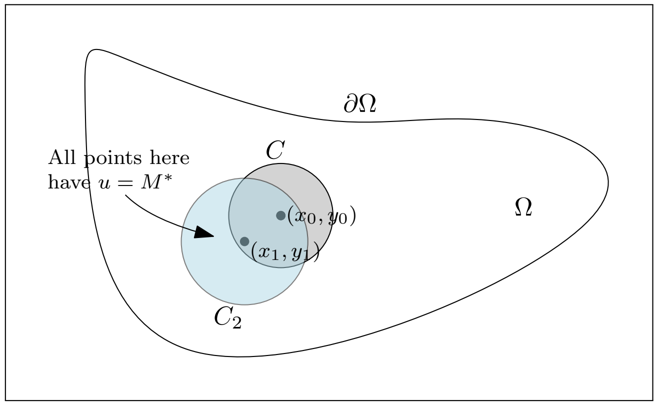

Figure 4: All points inside C_2 have same value M^*

We continue this way connecting points and adding the domain where all points have $u = M^*$ values.

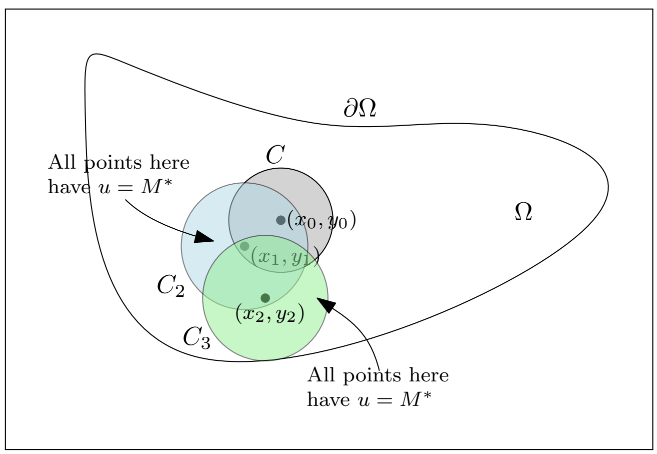

Figure 5: All points inside C_3 have same value M^*

Since Ω is connected then we can cover the whole region Ω this way all the way to the boundary $\partial\Omega$. This complete the proof given in the book.

6 Problem 4.3.46

Write down an integral formula for the solution to the Dirichlet boundary value problem on a disk of radius $R > 0$, namely, $\nabla^2 u = 0$, $x^2 + y^2 < R^2, u = h, x^2 + y^2 = R^2$

Solution

The closed form sum as given in theorem 4.6 in the book as the Poisson kernel integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi$$

Theorem 4.6 is for a unit disk. Since the disk here has radius R then r is changed to $\frac{r}{R}$ in the above giving

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-\left(\frac{r}{R}\right)^2}{1+\left(\frac{r}{R}\right)^2-2\left(\frac{r}{R}\right)\cos(\theta-\phi)}d\phi$$

Which can be simplified to

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi$$

7 Problem 4.4.4

Consider the following partial differential equations. At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

(a) $x^2 u_{xx} + x u_x + u_{yy} = 0$ (c) $u_t = \frac{\partial}{\partial x}\left((x+t)\, u_x\right)$

Solution

7.1 Part a

The general form of two variables $\left(x, y\right)$ PDE is

$$L[u] = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

The type of PDE depends on value of the discriminant

$$\Delta = B^2 - 4AC$$

Comparing the PDE $x^2 u_{xx} + x u_x + u_{yy}$ to (1) shows that $A = x^2, B = 0, C = 1$. Hence

$$\Delta = -4x^2$$

This is always negative ($x = 0$ is not possible, since this would made the PDE not a PDE any more). Therefore using definition 4.12 this means the PDE is elliptic.

7.2 Part b

$$\begin{aligned} u_t &= \frac{\partial}{\partial x}\left((x+t)\, u_x\right) \\ &= \left(\frac{\partial}{\partial x}(x+t)\right) u_x + (x+t)\frac{\partial}{\partial x} u_x \\ &= u_x + (x+t)\, u_{xx} \end{aligned}$$

Hence

$$u_x + (x+t)\, u_{xx} - u_t = 0 \tag{2}$$

The general form of two variables (t, x) PDE is

$$L[u] = Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G \tag{3}$$

Comparing (2) to (3) shows that $C = (x+t), A = 0, B = 0$. Hence

$$\begin{aligned} \Delta &= B^2 - 4AC \\ &= 0 \end{aligned}$$

Hence PDE is parabolic.

8 Problem 4.4.11

Prove that the complex change of variables $x = x, t = iy$, maps the Laplace equation $u_{xx} + u_{yy} = 0$ to the wave equation $u_{tt} = u_{xx}$. Explain why the type of a partial differential equation is not necessarily preserved under a complex change of variables.

Solution

Given $u_{xx} + u_{yy} = 0$, let $x = x, t = iy$. Hence we are to go from $u(x, y)$ to $v(t, x)$. Therefore

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x} &= \overbrace{\frac{\partial u}{\partial t}\frac{dt}{dx}}^{0} + \frac{\partial u}{\partial x}\frac{dx}{dx} \\ &= \frac{\partial u}{\partial x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\ &= \frac{\partial^2 u}{\partial x \partial t}\frac{dt}{dx} + \frac{\partial^2 u}{\partial x^2}\frac{dx}{dx} \\ &= \frac{\partial^2 u}{\partial x^2} \qquad (1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t}\frac{dt}{dy} + \overbrace{\frac{\partial u}{\partial x}\frac{dx}{dy}}^{0} \\ &= i\frac{\partial u}{\partial t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) \\ &= i\left(\frac{\partial^2 u}{\partial t^2}\frac{dt}{dy} + \overbrace{\frac{\partial^2 u}{\partial t \partial x}\frac{dx}{dy}}^{0}\right) \\ &= i\left(i\frac{\partial^2 u}{\partial t^2}\right) \\ &= -\frac{\partial^2 u}{\partial t^2} \qquad (2)\end{aligned}$$

Substituting (1,2) into $u_{xx} + u_{yy} = 0$ gives

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$u_{tt} = u_{xx}$$

Which is the wave equation.

When change of variables contains only real quantities, then no sign change will occur. Only stretching (scaling) can occur, so the type of the PDE do not change. But with complex variables, a sign change can occur as in this example due to multiplying i with i. And this is what causes the PDE type to change.

9 Problem 4.4.16

True or false: The characteristic curves of the Helmholtz equation $u_{xx} + u_{yy} - u = 0$ are circles.

Solution

Comparing the above to $L[u] = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ shows that

$$A = 1$$
$$B = 0$$
$$C = 1$$

Hence the characteristic curves are given by (4.151) as (where we choose $y \equiv y(x)$ and hence $s = x$ here)

$$A(x,y)\left(\frac{dy}{dx}\right)^2 - B(x,y)\frac{dy}{dx} + C(x,y) = 0$$
$$\left(\frac{dy}{dx}\right)^2 + 1 = 0$$
$$\left(\frac{dy}{dx}\right)^2 = -1$$
$$\frac{dy}{dx} = \pm i$$

There are no real characteristic curves. Therefore the answer is false.