HW 4

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 3.2.34

If $f(x)$ is odd, is $f'(x)$ (i) even? (ii) odd? (iii) neither? (iv) could be either?

solution

Answer is (i), even.

Proof: Since $f(x)$ is odd, then by definition

$$f(x) = -f(-x)$$

For all x in the domain of $f(x)$. Taking derivatives w.r.t. gives

$$\left[f(x)\right]' = \left[-f(-x)\right]'$$

Applying the chain rule to RHS gives $-f'(-x)(-1) = f'(-x)$ and the LHS gives $f'(x)$. Hence the above becomes

$$f'(x) = f'(-x)$$

But by definition $g(-x) = g(x)$ implies an even function. Hence the says that $f'(x)$ is an even function.

2 Problem 3.2.37

True or False. (a) If $f(x)$ is odd, its 2π periodic extension is odd. (b) if the 2π periodic extension of $f(x)$ is odd, then $f(x)$ is odd.

solution

2.1 Part a

True.

To show this, will use an illustration. In this illustration, and to reduce confusion, let $f(x)$ represents the original odd function defined over $-\pi \leq x \leq \pi$ and let $g(x)$ represents the 2π periodic extension of $f(x)$. For illustration, used the odd function $f(x) = x$.

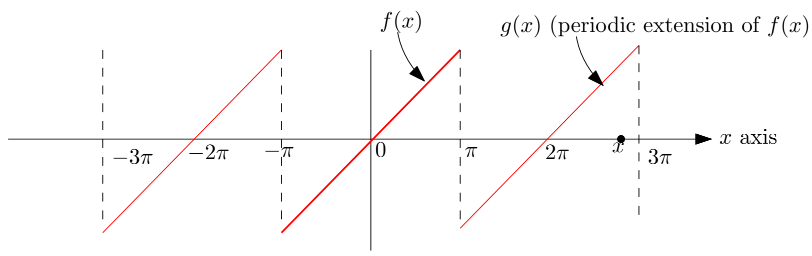

Figure 1: Showing $f(x)$ and its 2π extension

To show that $g(x)$ is odd, we pick any point x and now we need to show that $g(-x) = -g(x)$ or $g(x) = -g(-x)$.

On the right side of the x axis, $g(x) = f(x - n(2\pi))$ where n is positive integer due to the 2π extension. In the above illustration $n = 1$ but it can be any positive n. Let the point $x - n(2\pi) = z$. Hence now we have the following diagram

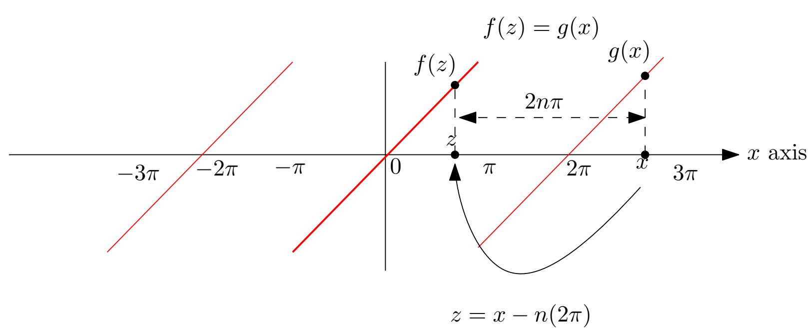

Figure 2: showing $g(x) = f(x - 2n\pi)$

Where $f(z) = g(x)$. But we are given that $f(x)$ is odd. Hence $f(z) = -f(-z)$. On the negative side of the x axis, we do the same we did on the positive side. Since the left side of $f(x)$ was also 2π extended, then $g(-x) = f(-x + n(2\pi)) = f(-z)$

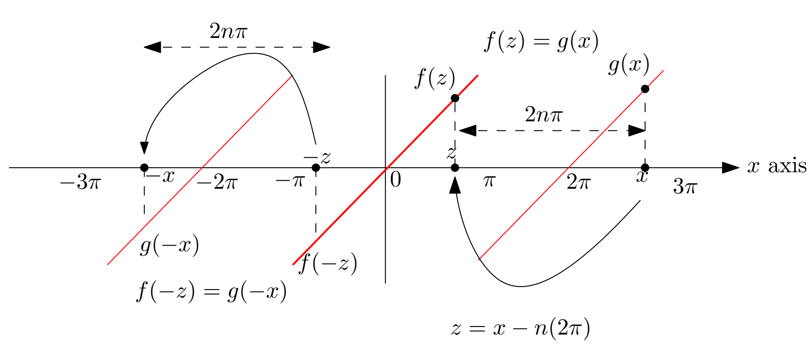

Figure 3: showing $g(-x) = f(-z)$

In conclusion, from the above we see that

$$g(-x) = f(-z)$$

But $f(-z) = -f(z)$ since f is odd. Hence the above becomes

$$g(-x) = -f(z)$$

But $f(z) = g(x)$ as shown in the first illustration, hence the above becomes

$$g(-x) = -g(x)$$

Which shows that $\underline{g(x) \text{ is odd}}$.

Since $g(x)$ is the 2π periodic extension of $f(x)$. This is what we asked to show.

2.2 Part b

(b) True. Proof by contradiction. Since $g(x)$ is odd, then we know that

$$g(-x) = -g(x)$$

We also know that by the 2π extension of $f(x)$ that

$$f(z) = g(x)$$

Where we are using the same diagrams from part (a). Where $z = x - 2n\pi$. Now, let us $\underline{\text{assume that } f(x) \text{ is even}}$. Then this means that

$$f(z) = f(-z)$$

But the 2π extension on the left side of the x axis, then we conclude that

$$g(-x) = f(-z)$$

Which means that

$$\begin{aligned} g(-x) &= f(z) \\ &= g(x) \end{aligned}$$

But this means $g(x)$ is even, which is a contradiction, since $g(x)$ is odd. Hence $f(x)$ can now now be even.

Only other choice is that $f(x)$ is neither odd or even, or an odd function. Let us now assume is neither. For example, take $f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$. Then following the above argument, we see that

$$g(x) = f(z)$$

But now $f(-z) = 0$, then by 2π extension of the left, then $g(-x) = f(-z) = 0$. But this means that $g(-x) \neq -g(x)$ which is not possible since $g(x)$ is odd. The only other choice left is that $\underline{f(x) \text{ is odd}}$. Which is what we are asked to show.

3 Problem 3.2.40a

Find the Fourier series and discuss convergence for (a) the box function $b(x) = \begin{cases} 1 & |x| < \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi < |x| < \pi \end{cases}$

solution

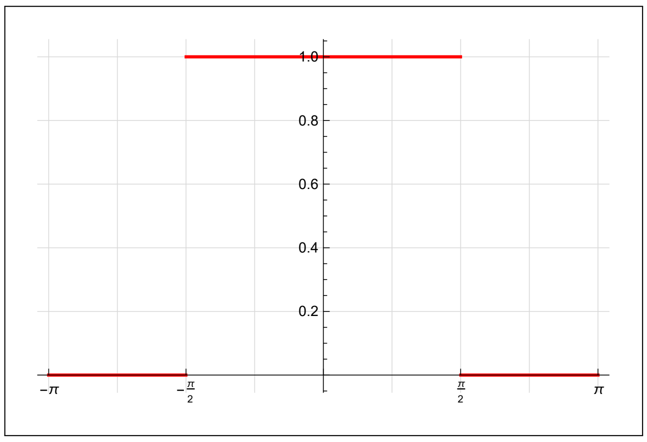

Figure 4: plot of $b(x)$

$$b(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

But $\frac{a_0}{2}$ is the average of the function over its 2π domain. Hence $\frac{a_0}{2} = \frac{\text{area}}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$, and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\frac{1}{2}\pi} \cos(nx)\,dx \\ &= \frac{2}{\pi n}\left[\sin(nx)\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

And since the function is even, then all $b_n = 0$. Hence

$$b(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)\cos nx$$

To verify the above solution, it is plotted against $b(x)$ for increasing number of terms.

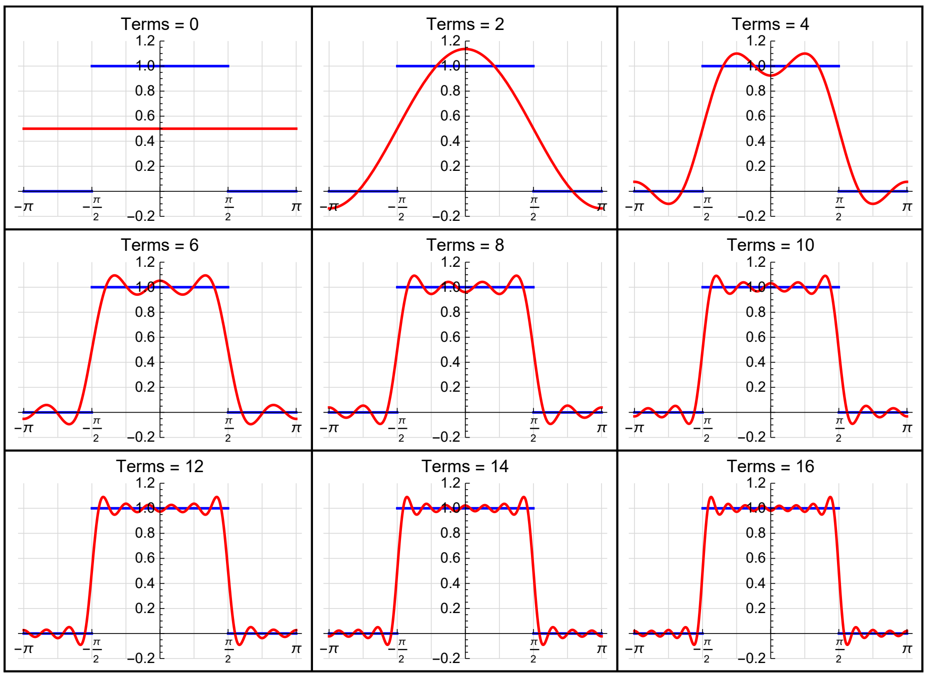

Figure 5: plot of Fourier series approximation to $b(x)$

```
In[•]:= p = Table[
            Plot[{f[x], fs[x, max]}, {x, -Pi, Pi}, PlotStyle → {Blue, Red},
              Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
              GridLines → {Range[-Pi, Pi, Pi / 4], Automatic},
              GridLinesStyle → LightGray,
              PlotRange → {Automatic, {-.2, 1.2}},
              PlotLabel → Row[{"Terms = ", max}]
            ]
            ,
            {max, 0, 16, 2}
          ];
        p = Grid[Partition[p, 3], Frame → All]
```

Figure 6: Code used for the above plot

Since there are jump discontinuities in the function $b(x)$, this will cause Gibbs effect at those points. This also implies that the convergence is not uniform. Fourier series will converge to each x where the function is continuous, but it will converge to the average of $b(x)$ at those points where there is a jump discontinuity.

In this case at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ in the fundamental domain given as shown in the plots above. At those points, Fourier series converges to $\frac{1}{2}$.

4 Problem 3.2.54

Prove that $\coth \pi = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \cdots\right)$, where $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$

solution

The complex Fourier series of e^x is

$$e^x = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx} \tag{1}$$

Where

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)} dx \\
&= \frac{1}{2\pi}\left[\frac{e^{x(1-in)}}{1-in}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left[e^x e^{-inx}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left[e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi}\right]
\end{aligned}$$

But $e^{in\pi} = \cos(n\pi)$ and also $e^{-in\pi} = \cos(n\pi)$ since n is integer. The above simplifies to

$$c_n = \frac{1}{2\pi}\frac{\cos(n\pi)}{1-in}\left[e^{\pi} - e^{-\pi}\right]$$

But $e^{\pi} - e^{-\pi} = 2\sinh(\pi)$. Therefore

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\frac{\cos(n\pi)}{1-in}\left[2\sinh(\pi)\right] \\
&= \frac{1}{\pi}\frac{\cos(n\pi)\sinh(\pi)}{1-in} \\
&= \frac{1}{\pi}\frac{\cos(n\pi)\sinh(\pi)}{1-in}\frac{(1+in)}{1+in} \\
&= \frac{1}{\pi}\cos(n\pi)\sinh(\pi)\frac{(1+in)}{1+n^2} \\
&= \frac{(-1)^n}{\pi}\sinh(\pi)\frac{(1+in)}{1+n^2}
\end{aligned}$$

Substituting this back into (1) gives

$$\begin{aligned}
e^x &= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-1)^n}{\pi}\sinh(\pi)\frac{(1+in)}{1+n^2}e^{inx} \\
&= \frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{inx}
\end{aligned}$$

At $x = \pi$

$$\begin{aligned}
\frac{1}{2}\left(e^{\pi} + e^{-\pi}\right) &= \cosh(\pi) \\
\frac{1}{2}\left(\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{in\pi} + \frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{-in\pi}\right) &= \cosh(\pi) \\
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{in\pi} + \lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^n\frac{(1+in)}{1+n^2}e^{-in\pi}\right) &= \cosh(\pi)
\end{aligned}$$

But $e^{in\pi} = \cos n\pi = (-1)^n$ and $e^{-in\pi} = \cos \pi = (-1)^n$. The above becomes

$$
\begin{aligned}
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^{2n}\frac{(1+in)}{1+n^2}+\lim_{N\to\infty}\sum_{n=-N}^{N}(-1)^{2n}\frac{(1+in)}{1+n^2}\right) &= \cosh(\pi)\\
\frac{1}{2}\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2}+\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2}\right) &= \cosh(\pi)\\
\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(1+in)}{1+n^2} &= \cosh(\pi)\\
\frac{\sinh(\pi)}{\pi}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{1+n^2}+i\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{n}{1+n^2}\right) &= \cosh(\pi)
\end{aligned}
$$

But $\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{n}{1+n^2}=0$ by symmetry. The above simplifies to

$$
\begin{aligned}
\frac{\sinh(\pi)}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{1+n^2} &= \cosh(\pi)\\
\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} &= \frac{\cosh(\pi)}{\sinh(\pi)}\\
&= \coth(\pi)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\coth(\pi) &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}\\
&= \frac{1}{\pi}\left(1+\sum_{n=-\infty}^{-1}\frac{1}{1+n^2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right)\\
&= \frac{1}{\pi}\left(1+2\sum_{n=1}^{-\infty}\frac{1}{1+n^2}\right)\\
&= \frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{-\infty}\frac{1}{1+n^2}\\
&= \frac{1}{\pi}+\frac{2}{\pi}\left(\frac{1}{1+1^2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\cdots\right)
\end{aligned}
$$

Which is what the problem asked to show.

5 Problem 3.2.60

Can you recognize whether a function is real by looking at its complex Fourier coefficients?

solution

Yes. If complex Fourier coefficients come in conjugate pairs such that $c_{-n} = \overline{c_n}$ and c_0 is real. (c_0 should always be real, since this represents the average energy at the zero (D.C.) frequency, hence must be real quantity).

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=-\infty}^{-1} c_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=1}^{\infty} c_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{inx} \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_{-n} e^{-inx} + c_n e^{inx}\right)
\end{aligned}$$

Now, If $c_{-n} = \overline{c_n}$ then the above becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} \overline{c_n} e^{-inx} + c_n e^{inx}$$

But $\left(\overline{c_n} e^{-inx} + c_n e^{inx}\right)$ is real. (This could also be written as $\overline{c_n e^{-inx}} + c_n e^{inx}$ which now looks like standard $\overline{z} + z$ in complex numbers). Hence $f(x)$ is real.

To show this, here is an example. Let $c_n = a + ib$, then $c_{-n} = a - ib$. Therefore

$$\begin{aligned}
\overline{c_n} e^{-inx} + c_n e^{inx} &= \overline{(a+ib)} e^{-inx} + (a+ib)\, e^{inx} \\
&= (a-ib)\, e^{-inx} + (a+ib)\, e^{inx} \\
&= \left(ae^{-inx} - ibe^{-inx}\right) + \left(ae^{inx} + ibe^{inx}\right) \\
&= a\left(e^{inx} + e^{-inx}\right) + bi\left(e^{inx} - e^{-inx}\right) \\
&= a(\cos nx + i\sin nx + \cos nx - i\sin nx) + bi(\cos nx + i\sin nx - \cos nx + i\sin nx) \\
&= a(2\cos nx) + bi(2i\sin nx) \\
&= 2a\cos nx - 2b\sin nx
\end{aligned}$$

Which is real value. Therefore if each c_n is a complex conjugate of c_{-n} (with c_0 real) then $f(x)$ will be a real function.

6 Problem 3.3.2

Find the Fourier series for the function $f(x) = x^3$. If you differentiate your series, do you recover the Fourier series for $f'(x) = 3x^2$? If not, explain why not.

solution

The function $f(x)$ over $-\pi \leq x \leq \pi$ is

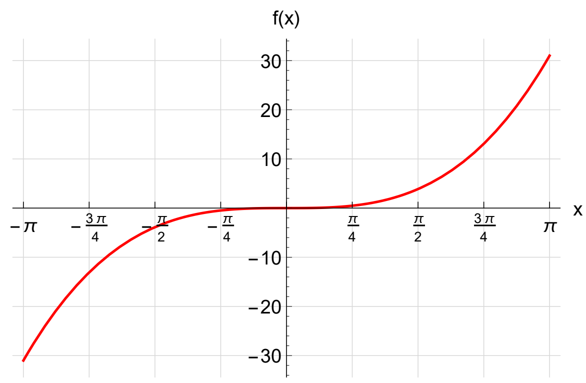

Figure 7: Plot of x^3

We see right away that differentiating term by term the Fourier series for the above function could not be justified. Even though the function x^3 has no jump discontinuity inside $-\pi \leq x \leq \pi$, which is good, it still fails the other test which requires that $f(-\pi) = f(\pi)$ for the term by term differentiation to be justified. This is because the 2π extension will now have jump discontinuities in it. The conditions under which the Fourier series for a function can be term by term differentiated are

1. $f(x)$ is piecewise continuous between $-\pi \leq x \leq \pi$ with no jump discontinuities.
2. $f(-\pi) = f(\pi)$

The function given fails condition (2) above. This explains why differentiating the Fourier series of x^3 will not give the Fourier series of $3x^2$. Now we will show this as required by the problem.

To find the Fourier series of x^3, since it is an odd function, then we only need to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(nx)\,dx$$

Since x^3 is odd, and sin is odd, then the product is even, and the above simplifies to

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin(nx)\,dx$$

Integration by parts., Let $u = x^3, \sin(nx) = dv$. Then $du = 3x^2, v = -\frac{1}{n}\cos(nx)$. Then $\int udv = uv - \int vdu$ gives

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left(-\frac{1}{n}\left[x^3\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} 3x^2\cos(nx)\,dx\right)\\
&= \frac{2}{\pi}\left(-\frac{1}{n}\left[\pi^3\cos(n\pi)\right] + \frac{3}{n}\int_0^{\pi} x^2\cos(nx)\,dx\right)\\
&= \frac{2}{\pi}\left(-\frac{1}{n}\left[\pi^3(-1)^n\right] + \frac{3}{n}\int_0^{\pi} x^2\cos(nx)\,dx\right)\\
&= -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n\pi}\int_0^{\pi} x^2\cos(nx)\,dx
\end{aligned}$$

Integration by parts again. Let $u = x^2, \cos(nx) = dv$. Then $du = 2x, v = \frac{1}{n}\sin(nx)$. Then using $\int udv = uv - \int vdu$ the above becomes

$$
\begin{aligned}
b_n &= -\frac{2(-1)^n \pi^2}{n} + \frac{6}{n\pi}\left(\frac{1}{n}\left[x^2 \sin(nx)\right]_0^\pi - \frac{2}{n}\int_0^\pi x\sin(nx)\,dx\right) \\
&= -\frac{2(-1)^n \pi^2}{n} + \frac{6}{n\pi}\left(-\frac{2}{n}\int_0^\pi x\sin(nx)\,dx\right) \\
&= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\int_0^\pi x\sin(nx)\,dx
\end{aligned}
$$

Integration by parts again. Let $u = x, \sin(nx) = dv$. Then $du = 1, v = \frac{-1}{n}\cos(nx)$. Then using $\int udv = uv - \int vdu$ the above becomes

$$
\begin{aligned}
b_n &= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\right) \\
&= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[\pi\cos(n\pi)\right] + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^\pi\right) \\
&= -\frac{2(-1)^n \pi^2}{n} - \frac{12}{n^2\pi}\left(\frac{-1}{n}\left[\pi(-1)^n\right]\right) \\
&= -\frac{2(-1)^n \pi^2}{n} + \frac{12}{n^3\pi}\pi(-1)^n \\
&= \frac{-2(-1)^n n^2\pi^2 + 12(-1)^n}{n^3} \\
&= \frac{-2(-1)^n\left(-6 + n^2\pi^2\right)}{n^3}
\end{aligned}
$$

Hence

$$
x^3 \sim \sum_{n=1}^\infty -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^3}\sin(nx) \tag{1}
$$

Now we apply Term by term differentiation to the RHS above and obtain

$$
\begin{aligned}
\left(\sum_{n=1}^\infty -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^3}\sin(nx)\right)' &= \sum_{n=1}^\infty -\frac{2(-1)^n\left(-6+n^2\pi^2\right)}{n^2}\cos(nx) \\
&= \sum_{n=1}^\infty \frac{12(-1)^n - 2(-1)^n n^2\pi^2}{n^2}\cos(nx) \\
&= \sum_{n=1}^\infty \left(\frac{12(-1)^n}{n^2} - 2(-1)^n\pi^2\right)\cos(nx) \qquad (2)
\end{aligned}
$$

And differentiation of LHS of (1) gives

$$
\left(x^3\right)' = 3x^2
$$

Let us now find the Fourier series for $3x^2$ and see if it matches (2). Since x^2 is even, it will only have a_n terms

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^\pi 3x^2 dx \\
&= \frac{3}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^\pi \\
&= \frac{1}{\pi}\left[x^3\right]_{-\pi}^\pi \\
&= \frac{1}{\pi}\left[\pi^3 - (-\pi)^3\right] \\
&= \frac{1}{\pi}\left[\pi^3 + \pi^3\right] \\
&= 2\pi^2
\end{aligned}
$$

And

$$
\begin{aligned}
a_n &= \frac{3}{\pi}\int_{-\pi}^\pi x^2\cos(nx)\,dx \\
&= \frac{12(-1)^n}{n^2}
\end{aligned}
$$

Therefore the Fourier series for $3x^2$ is

$$3x^2 \sim \pi^2 + \sum_{n=1}^{\infty} \frac{12\,(-1)^n}{n^2} \cos(nx) \tag{3}$$

Comparing (2,3) shows they are not the same. (2) has an extra term $-2\,(-1)^n\,\pi^2$ inside the sum and it also do not have the added π^2 term outside the sum. The explanation of why that is, is given earlier in the solution.

7 Problem 3.4.3 (b,d)

Find the Fourier series for the following functions on the indicated intervals, and graph the functions that it converges to. (b) x^2-4 over $-2 \le x \le 2$. (d) $\sin x$ over $-1 \le x \le 1$.

solution

7.1 Part (b)

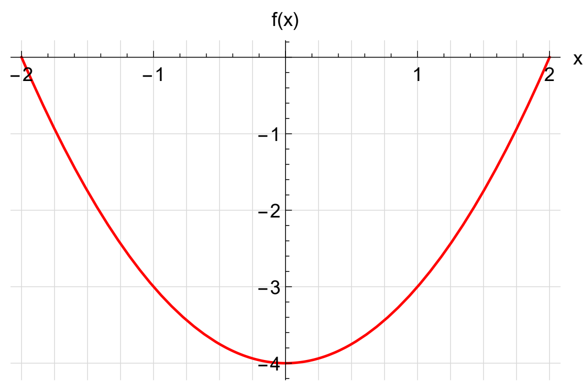

Figure 8: Plot of x^2-4

The function x^2-4 is even. Hence all b_n terms are zero. The period now is $T=4$.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)dx\\
&= \int_0^2\left(x^2-4\right)dx\\
&= \left[\frac{x^3}{3}-4x\right]_0^2\\
&= \frac{8}{3}-8\\
&= -\frac{16}{3}
\end{aligned}$$

And

$$a_n = \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)\cos\left(\frac{2\pi}{T}nx\right)dx$$

But $T=4$, hence the above becomes

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2}\left(x^2-4\right)\cos\left(\frac{\pi}{2}nx\right)dx\\
&= \int_0^2\left(x^2-4\right)\cos\left(\frac{\pi}{2}nx\right)dx\\
&= \int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx-\int_0^2 4\cos\left(\frac{\pi}{2}nx\right) \qquad \text{(1A)}
\end{aligned}$$

Looking at the term $\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx$, applying integration by parts. Let $u=x^2, dv=\cos\left(\frac{\pi}{2}nx\right)$. Then $du=2x, v=\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)$. Then using $\int udv = uv-\int vdu$ gives

$$\begin{aligned}
\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx &= \left[x^2\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)\right]_0^2-\int_0^2 2x\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)dx\\
&= \frac{2}{n\pi}\left[\overbrace{4\sin\left(\pi n\right)}^{0}-0\right]-\frac{4}{\pi n}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\\
&= -\frac{4}{\pi n}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx
\end{aligned}$$

Applying integration by parts again. Let $u=x, dv=\sin\left(\frac{\pi}{2}nx\right)$. Then $du=1, v=\frac{-2}{\pi n}\cos\left(\frac{\pi}{2}nx\right)$ then the above becomes

$$\begin{aligned}
\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx &= -\frac{4}{\pi n}\left[\frac{-2}{\pi n}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2 - \int_0^2 \frac{-2}{\pi n}\cos\left(\frac{\pi}{2}nx\right)dx\right]\\
&= -\frac{4}{\pi n}\left[\frac{-2}{\pi n}\left[2\cos(\pi n)-0\right]+\frac{2}{\pi n}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right]\\
&= -\frac{4}{\pi n}\frac{-4}{\pi n}(-1)^n+\frac{2}{\pi n}\left(\frac{2}{\pi n}\right)\left[\overbrace{\sin\left(\frac{\pi}{2}nx\right)}^{0}\right]_0^2\\
&= -\frac{4}{\pi n}\left[\frac{-4}{\pi n}(-1)^n\right]\\
&= \frac{16}{\pi^2 n^2}(-1)^n \qquad (1B)
\end{aligned}$$

The above takes care of the first term in (1A). The second term in (1A) is

$$\begin{aligned}
4\int_0^2\cos\left(\frac{\pi}{2}nx\right) &= 4\left[\frac{2}{n\pi}\sin\left(\frac{\pi}{2}nx\right)\right]_0^2\\
&= \frac{8}{n\pi}\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2\\
&= 0 \qquad (1C)
\end{aligned}$$

Using (1B,1C) results back in (1A) gives a_n as

$$a_n=\frac{16}{\pi^2n^2}(-1)^n$$

Therefore the Fourier series is

$$\begin{aligned}
x^2-4 &\sim -\frac{8}{3}+\sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n\cos\left(\frac{\pi}{2}nx\right)\\
&\sim -\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right)
\end{aligned}$$

The following shows how the above Fourier series converges for increasing number of terms. The convergence is uniform convergence.

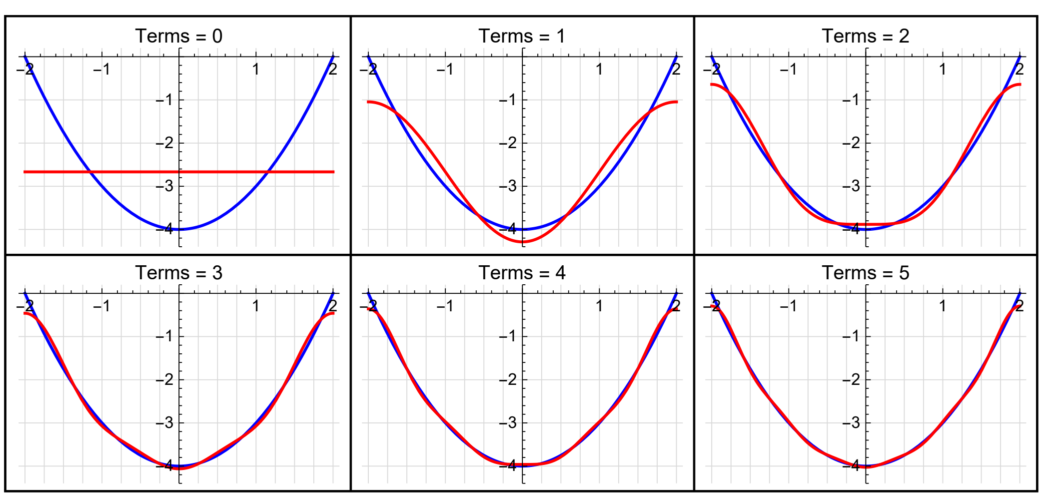

Figure 9: Plot of Fourier series for x^2-4

```
In[ ]:= ClearAll[f, x, n, max];
        f[x_] := x^2 - 4
        fs[x_, max_] := -8 / 3 + 16/π^2 Sum[(-1)^n/n^2 Cos[π/2 n x], {n, 1, max}]
        p = Table[
                Plot[{f[x], fs[x, max]}, {x, -2, 2},
                   PlotStyle → {Blue, Red},
                   GridLines → {Range[-2, 2, 1 / 4], Automatic},
                   GridLinesStyle → LightGray,
                   PlotRange → {Automatic, {-4.4, .2}},
                   PlotLabel → Row[{"Terms = ", max}]
                ]
                ,
                {max, 0, 6, 1}
              ];
        p = Grid[Partition[p, 3], Frame → All]
```

Figure 10: Code used for the above Plot

7.2 Part d

The function $\sin x$ is odd. Hence all a_n terms are zero. The period now is $T = 2$.

$$\begin{aligned} b_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(x) \sin\left(\frac{2\pi}{T} nx\right) dx \\ &= \int_{-1}^{1} \sin(x) \sin(\pi n x)\, dx \end{aligned}$$

But the integrand is even, then the above becomes

$$b_n = 2 \int_0^1 \sin(x) \sin(\pi n x)\, dx$$

Integration by parts. Let $u = \sin x, dv = \sin(\pi n x)$, then $du = \cos x, v = -\frac{1}{\pi n} \cos(\pi n x)$ and the above becomes

$$\begin{aligned} b_n &= 2\left(-\frac{1}{\pi n} [\sin x \cos(\pi n x)]_0^1 + \frac{1}{\pi n} \int_0^1 \cos x \cos(\pi n x)\, dx\right) \\ &= 2\left(-\frac{1}{\pi n} [\sin(1) \cos(\pi n)] + \frac{1}{\pi n} \int_0^1 \cos x \cos(\pi n x)\, dx\right) \\ &= \frac{2}{\pi n}\left(-\sin(1)(-1)^n + \int_0^1 \cos x \cos(\pi n x)\, dx\right) \\ &= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \int_0^1 \cos x \cos(\pi n x)\, dx\right) \end{aligned}$$

Integration by parts again. Let $u = \cos x, dv = \cos(\pi n x)$, then $du = -\sin x, v = \frac{1}{\pi n} \sin(\pi n x)$ and the above becomes

$$\begin{aligned} b_n &= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \left(\frac{1}{\pi n}\left[\overbrace{\cos x \sin(\pi n x)}^{0}\right]_0^1 + \frac{1}{\pi n} \int_0^1 \sin x \sin(\pi n x)\right)\right) \\ &= \frac{2}{\pi n}\left(\sin(1)(-1)^{n+1} + \frac{1}{\pi n} \int_0^1 \sin x \sin(\pi n x)\right) \\ &= \frac{2}{\pi n} \sin(1)(-1)^{n+1} + \frac{2}{\pi^2 n^2} \int_0^1 \sin x \sin(\pi n x) \end{aligned}$$

But $2\int_0^1 \sin x \sin(\pi n x) = b_n$. Hence the above simplifies to

$$
\begin{aligned}
b_n - \frac{b_n}{\pi^2 n^2} &= \frac{2}{\pi n}\sin(1)(-1)^{n+1} \\
b_n\left(1 - \frac{1}{\pi^2 n^2}\right) &= \frac{2}{\pi n}\sin(1)(-1)^{n+1} \\
b_n &= \frac{\frac{2}{\pi n}\sin(1)(-1)^{n+1}}{1 - \frac{1}{\pi^2 n^2}} \\
&= \frac{\left(\pi^2 n^2\right)\frac{2}{\pi n}\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1} \\
&= \frac{2n\pi\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}
\end{aligned}
$$

Hence the Fourier series is

$$
\begin{aligned}
\sin x &\sim \sum_{n=1}^{\infty} b_n \sin(\pi n x) \\
&\sim \sum_{n=1}^{\infty} \frac{2n\pi\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}\sin(\pi n x)
\end{aligned}
$$

The following shows how the above Fourier series converges for increasing number of terms. The convergence is not uniform since the function is odd. Hence there will be a jump discontinuity when periodic extended leading to Gibbs effect at the edges.

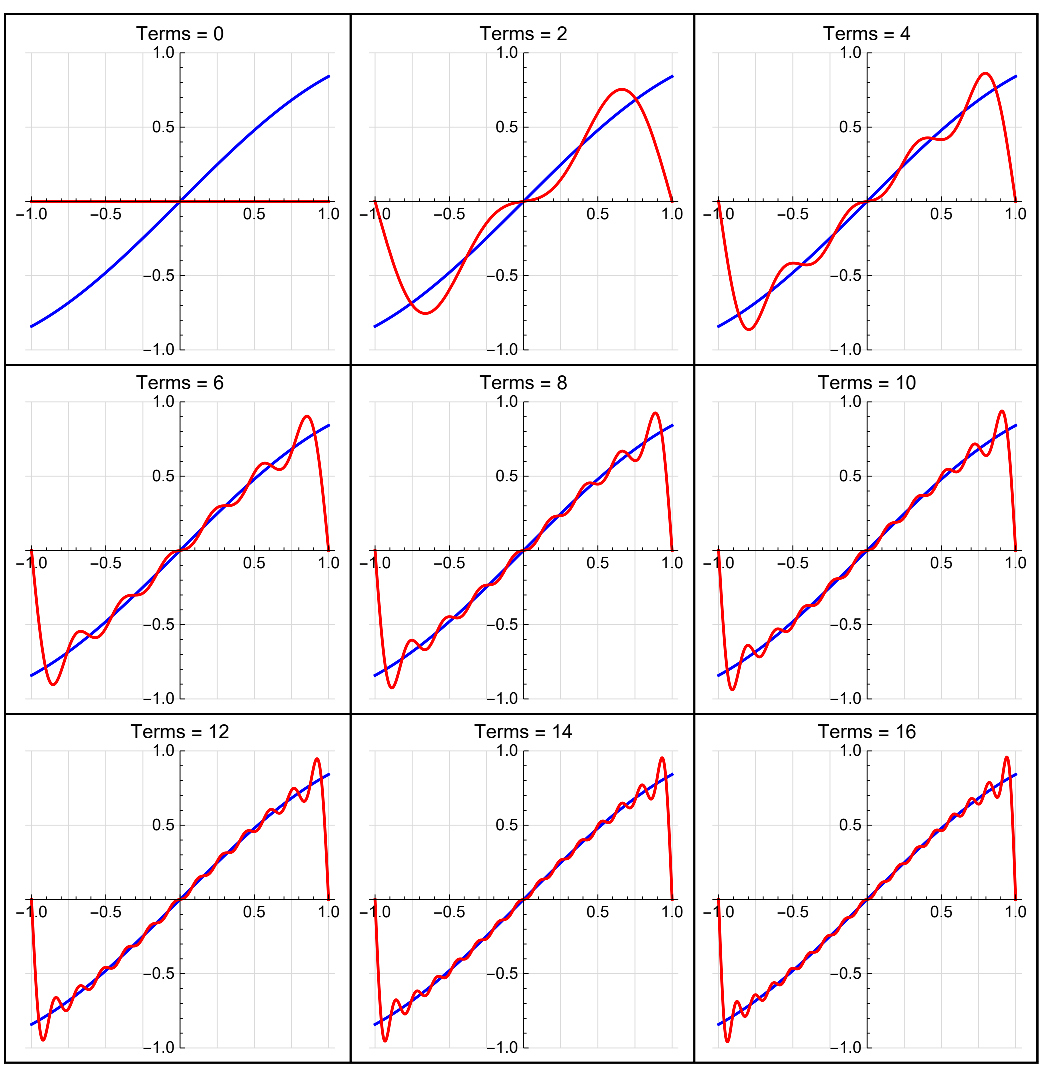

Figure 11: Plot of Fourier series for $\sin(x)$

```
In[•]:= ClearAll[f, x, n, max];
        f[x_] := Sin[x]
        fs[x_, max_] := Sum[-(2 (-1)^n n π Sin[1])/(n^2 π^2 - 1) Sin[π n x], {n, 1, max}]
        p = Table[
                  Plot[{f[x], fs[x, max]}, {x, -1, 1},
                     PlotStyle → {Blue, Red},
                     GridLines → {Range[-1, 1, 1/4], Automatic},
                     GridLinesStyle → LightGray,
                     PlotRange → {Automatic, {-1, 1}},
                     PlotLabel → Row[{"Terms = ", max}], AspectRatio -> Automatic
                  ]
                  ,
                  {max, 0, 18, 2}
               ];
        p = Grid[Partition[p, 3], Frame → All]
```

Figure 12: Code used for the above Plot

8 Problem 3.4.4

For (b) x^2-4 over $-2\leq x\leq 2$. (d) $\sin x$ over $-1\leq x\leq 1$ write out the differentiated Fourier series and determine whether it converges to the derivative of the original function.

solution

8.1 Part b

From Problem 3.4.3

$$x^2-4\sim -\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right) \tag{1}$$

Since the function x^2-4 is uniform convergent, then we expect that the differentiated Fourier series will converge to the derivative of the original function. The following calculations confirms this.

Taking derivative of the RHS of (1) gives

$$\begin{aligned}\left(-\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right)\right)' &= \frac{16}{\pi^2}\sum_{n=1}^{\infty}-\left(\frac{\pi}{2}n\right)\frac{(-1)^n}{n^2}\sin\left(\frac{\pi}{2}nx\right)\\ &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi}{2}nx\right) \end{aligned}\tag{2}$$

And taking derivative of the LHS of (1) gives

$$\left(x^2-4\right)'=2x \tag{3}$$

We now need to show if the Fourier series of $2x$ gives the RHS of (2). Let us now find the Fourier series for x, over $-2\leq x\leq 2$ (period $T=4$). Since x is odd, then all $a_n=0$.

$$\begin{aligned}b_n &= \frac{1}{2}\int_{-2}^{2}x\sin\left(\frac{2\pi}{T}nx\right)dx\\ &= \frac{1}{2}\int_{-2}^{2}x\sin\left(\frac{\pi}{2}nx\right)dx\end{aligned}$$

But $x\sin\left(\frac{\pi}{2}nx\right)$ is even. The above becomes

$$b_n=\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx$$

Integration by parts. Let $u=x, dv=\sin\left(\frac{\pi}{2}nx\right)$, then $du=1, v=\frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$\begin{aligned}b_n &= \frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2+\frac{2}{n\pi}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\\ &= \frac{-2}{n\pi}\left[2\cos(n\pi)\right]+\frac{2}{n\pi}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\\ &= \frac{2}{n\pi}\left(-2\cos(n\pi)+\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\ &= \frac{2}{n\pi}\left(-2\cos(n\pi)+\frac{2}{n\pi}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right)\\ &= \frac{2}{n\pi}\left(-2\cos(n\pi)\right)\\ &= \frac{-4}{n\pi}(-1)^n\end{aligned}$$

Hence the Fourier series for x over $-2\leq x\leq 2$ is

$$x\sim\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi}{2}nx\right)$$

Therefore the Fourier series for $2x$ is

$$2x\sim\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi}{2}nx\right) \tag{4}$$

Comparing (4) and (2) shows they are the same. Hence term by term differentiation is valid in this case.

8.2 Part d

From Problem 3.4.3, the Fourier series for $\sin x$ over $-1 \le x \le 1$ is

$$\sin x \sim \sum_{n=1}^{\infty} -\frac{2n\pi\,(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi n x) \tag{1}$$

Since the Fourier series for $\sin x$ over $-1 \le x \le 1$ is not uniform convergent, then we expect that the differentiated Fourier series will not converge to the derivative of the original function. The following calculations confirms this.

Taking derivative of the RHS of (1) gives

$$\begin{aligned}\left(\sum_{n=1}^{\infty} -\frac{2n\pi\,(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi n x)\right)' &= \sum_{n=1}^{\infty} -\pi n\frac{2n\pi\,(-1)^n}{n^2\pi^2-1}\sin(1)\cos(\pi n x)\\ &= \sum_{n=1}^{\infty} \frac{2n^2\pi^2\,(-1)^{n+1}}{n^2\pi^2-1}\sin(1)\cos(\pi n x) \qquad (2)\end{aligned}$$

And Taking derivative of the LHS of (1) gives

$$(\sin x)' = \cos x \tag{3}$$

So now we need to show that the Fourier series for $\cos x$, over $-1 \le x \le 1$ (period $T = 2$) agrees with (2).

Since $\cos x$ is even, then all $b_n = 0$.

$$\begin{aligned}a_0 &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(x)\,dx\\ &= \int_{-1}^{1} \cos(x)\,dx\\ &= 2\int_{0}^{1} \cos x dx\\ &= 2\left[\sin(x)\right]_0^1\\ &= 2\sin(1)\end{aligned}$$

And

$$\begin{aligned}a_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos x\cos\left(\frac{2\pi}{T}nx\right)dx\\ &= \int_{-1}^{1} \cos x\cos(\pi n x)\,dx\\ &= 2\int_{0}^{1} \cos x\cos(\pi n x)\,dx\end{aligned}$$

Integration by parts. Let $u = \cos x, dv = \cos(n\pi x)$, then $du = -\sin x, v = \frac{1}{\pi n}\sin(n\pi x)$ and the above becomes

$$\begin{aligned}a_n &= 2\left(\frac{1}{\pi n}\left[\cos x\sin(n\pi x)\right]_0^1 + \frac{1}{\pi n}\int_0^1 \sin x\sin(n\pi x)\,dx\right)\\ &= \frac{2}{\pi n}\left(\left[\cos x\sin(n\pi x)\right]_0^1 + \int_0^1 \sin x\sin(n\pi x)\,dx\right)\\ &= \frac{2}{\pi n}\left(\left[\cos(1)\sin(n\pi)\right] + \int_0^1 \sin x\sin(n\pi x)\,dx\right)\\ &= \frac{2}{\pi n}\int_0^1 \sin x\sin(n\pi x)\,dx\end{aligned}$$

Integration by parts again. Let $u = \sin x, dv = \sin(n\pi x)$, then $du = \cos x, v = \frac{-1}{\pi n}\cos(n\pi x)$

and the above becomes

$$
\begin{aligned}
a_n &= \frac{2}{\pi n}\left(\frac{-1}{\pi n}\left[\sin x \cos(n\pi x)\right]_0^1 + \frac{1}{\pi n}\int_0^1 \cos x \cos(n\pi x)\,dx\right)\\
&= \frac{2}{\pi^2 n^2}\left(-\left[\sin(1)\cos(n\pi)\right] + \int_0^1 \cos x \cos(n\pi x)\,dx\right)\\
&= -\frac{2}{\pi^2 n^2}\left(\sin(1)(-1)^n\right) + \frac{2}{\pi^2 n^2}\int_0^1 \cos x \cos(n\pi x)\,dx
\end{aligned}
$$

But $2\int_0^1 \cos x \cos(n\pi x)\,dx = a_n$. Hence the above becomes

$$
\begin{aligned}
a_n - \frac{a_n}{\pi^2 n^2} &= -\frac{2}{\pi^2 n^2}\left(\sin(1)(-1)^n\right)\\
a_n\left(1 - \frac{1}{\pi^2 n^2}\right) &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2}\\
a_n\left(\frac{\pi^2 n^2 - 1}{\pi^2 n^2}\right) &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2}\\
a_n &= \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}
\end{aligned}
$$

Hence the Fourier series for $\cos(x)$ over $-1 \le x \le 1$ is

$$
\begin{aligned}
\cos x &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x)\\
&\sim \sin(1) + \sum_{n=1}^{\infty} \frac{2\sin(1)(-1)^{n+1}}{\pi^2 n^2 - 1}\cos(\pi n x) \qquad (4)
\end{aligned}
$$

Comparing (4) and (2) shows they are not the same. Hence taking derivatives term by term of the Fourier series was not justified as expected.

9 Problem 3.4.5

For (b) x^2-4 over $-2\leq x\leq 2$. (d) $\sin x$ over $-1\leq x\leq 1$ find the Fourier series for the integral of the function.

solution

9.1 Part b

From Problem 3.4.3

$$x^2-4\sim-\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}nx\right)\tag{1}$$

Integrating the RHS of (1) gives

$$\begin{aligned}\int_0^x\left(-\frac{8}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{\pi}{2}ns\right)\right)ds&=-\frac{8}{3}\int_0^x ds+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^x\cos\left(\frac{\pi}{2}ns\right)ds\\&=-\frac{8}{3}x+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[\frac{\sin\left(\frac{\pi}{2}ns\right)}{\frac{\pi}{2}n}\right]_0^x\\&=-\frac{8}{3}x+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{(-1)^n}{n^2}\left[\sin\left(\frac{\pi}{2}ns\right)\right]_0^x\\&=-\frac{8}{3}x+\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{\pi}{2}nx\right)\end{aligned}\tag{2}$$

Integrating the LHS of (1) gives

$$\begin{aligned}\int_0^x\left(s^2-4\right)ds&=\left(\frac{s^3}{2}-4s\right)_0^x\\&=\frac{x^3}{2}-4x\end{aligned}\tag{3}$$

Now we find Fourier series for $\frac{x^3}{2}-4x$ and compare it with the (2) to see they match in order to see if term by term integration was justified or not above.

Let $f(x)=\frac{x^3}{2}-4x$ for $-2\leq x\leq 2$. This is an odd function. Hence only b_n exist.

$$\begin{aligned}b_n&=\frac{1}{2}\int_{-2}^{2}\left(\frac{x^3}{2}-4x\right)\sin\left(\frac{\pi}{2}nx\right)dx\\&=\frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx-2\int_{-2}^{2}x\sin\left(\frac{\pi}{2}nx\right)dx\end{aligned}\tag{4}$$

Looking at the first integral above, $\frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx$. Since the integrand is even, then $\frac{1}{4}\int_{-2}^{2}x^3\sin\left(\frac{\pi}{2}nx\right)dx=\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx$. Integration by parts. $u=x^3,dv=\sin\left(\frac{\pi}{2}nx\right)$ then $du=3x^2,v=-\frac{2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$. Therefore

$$\begin{aligned}\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx&=\frac{1}{2}\left(-\frac{2}{n\pi}\left[x^3\cos\left(\frac{\pi}{2}nx\right)\right]_0^2+\frac{2}{n\pi}\int_0^2 3x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\&=\frac{1}{n\pi}\left(-\left[8\cos(\pi n)\right]+3\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\&=\frac{1}{n\pi}\left(8(-1)^{n+1}+3\int_0^2 x^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\end{aligned}$$

Integration by parts again. $u=x^2,dv=\cos\left(\frac{\pi}{2}nx\right)$ then $du=2x,v=\frac{2}{n\pi}\sin\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$
\begin{aligned}
\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx &= \frac{1}{n\pi}\left(8(-1)^{n+1}+3\left(\frac{2}{n\pi}\left[x^2\sin\left(\frac{\pi}{2}nx\right)\right]_0^2-\frac{2}{n\pi}\int_0^2 2x\sin\left(\frac{\pi}{2}nx\right)dx\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}+\frac{6}{n\pi}\left(\overbrace{[4\sin(\pi n)]_0^2}^{0}-2\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}-\frac{12}{n\pi}\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx\right)
\end{aligned}
$$

Integration by parts again. $u=x, dv=\sin\left(\frac{\pi}{2}nx\right)$ then $du=1, v=\frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$ and the above becomes

$$
\begin{aligned}
\frac{1}{2}\int_0^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx &= \frac{1}{n\pi}\left(8(-1)^{n+1}-\frac{12}{n\pi}\left(\frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2-\int_0^2\frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)dx\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}-\frac{12}{n\pi}\left(\frac{-2}{n\pi}[2\cos(\pi n)]+\frac{2}{n\pi}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}-\frac{24}{n^2\pi^2}\left(-2(-1)^n+\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}-\frac{24}{n^2\pi^2}\left(-2(-1)^n+\frac{2}{n\pi}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right)\right)\\
&= \frac{1}{n\pi}\left(8(-1)^{n+1}+\frac{48}{n^2\pi^2}(-1)^n\right)\\
&= \frac{1}{n\pi}\left(-8(-1)^n+\frac{48}{n^2\pi^2}(-1)^n\right)\\
&= \frac{-8}{n\pi}(-1)^n+\frac{48}{n^3\pi^3}(-1)^n \qquad (5)
\end{aligned}
$$

The above takes care of the first term in (4). The second integral $2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx$ in (4) is now found. Since integrand is even then

$$
2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx = 4\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx
$$

Integration by parts. Let $u=x, dv=\sin\left(\frac{\pi}{2}nx\right)$, then $du=1, v=\frac{-2}{n\pi}\cos\left(\frac{\pi}{2}nx\right)$, therefore

$$
\begin{aligned}
4\int_0^2 x\sin\left(\frac{\pi}{2}nx\right)dx &= 4\left(\frac{-2}{n\pi}\left[x\cos\left(\frac{\pi}{2}nx\right)\right]_0^2+\frac{2}{n\pi}\int_0^2\cos\left(\frac{\pi}{2}nx\right)dx\right)\\
&= 4\left(\frac{-2}{n\pi}[2\cos(\pi n)]+\frac{4}{n^2\pi^2}\overbrace{\left[\sin\left(\frac{\pi}{2}nx\right)\right]_0^2}^{0}\right)\\
&= 4\left(\frac{-2}{n\pi}\left[2(-1)^n\right]\right)\\
&= \frac{-16}{n\pi}(-1)^n\\
&= \frac{16}{n\pi}(-1)^{n+1} \qquad (6)
\end{aligned}
$$

Substituting (5,6) back into (4) gives

$$
\begin{aligned}
b_n &= \frac{1}{4}\int_{-2}^2 x^3\sin\left(\frac{\pi}{2}nx\right)dx-2\int_{-2}^2 x\sin\left(\frac{\pi}{2}nx\right)dx\\
&= \left(\frac{-8}{n\pi}(-1)^n+\frac{48}{n^3\pi^3}(-1)^n\right)-\frac{16}{n\pi}(-1)^{n+1}\\
&= \left(\frac{-8}{n\pi}(-1)^n+\frac{48}{n^3\pi^3}(-1)^n\right)+\frac{16}{n\pi}(-1)^n\\
&= \frac{48}{n^3\pi^3}(-1)^n+\frac{8}{n\pi}(-1)^n
\end{aligned}
$$

Hence the Fourier series for $\frac{x^3}{2} - 4x$ is

$$
\begin{aligned}
\frac{x^3}{2} - 4x &\sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{2} nx\right) \\
&\sim \sum_{n=1}^{\infty} \left(\frac{48}{n^3\pi^3}(-1)^n + \frac{8}{n\pi}(-1)^n\right) \sin\left(\frac{\pi}{2} nx\right) \\
&\sim \sum_{n=1}^{\infty} \left(\frac{48 + 8\left(n^2\pi^2\right)}{n^3\pi^3}\right)(-1)^n \sin\left(\frac{\pi}{2} nx\right) \\
&\sim \sum_{n=1}^{\infty} \left(\frac{8\left(6 + n^2\pi^2\right)}{n^3\pi^3}\right)(-1)^n \sin\left(\frac{\pi}{2} nx\right) \qquad (7)
\end{aligned}
$$

Comparing (7,2) shows they are not the same. Hence integration term by term was not justified. This is because the function $x^2 - 4$ is not odd, hence its mean is not zero.

9.2 Part d

From Problem 3.4.3, the Fourier series for $\sin x$ over $-1 \leq x \leq 1$ is

$$
\sin x \sim \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2 - 1} \sin(1) \sin(\pi n x) \qquad (1)
$$

Integrating the LHS of (1) gives

$$
\begin{aligned}
\int_0^x \sin(s)\, ds &= -[\cos(s)]_0^x \\
&= -[\cos(x) - 1] \\
&= 1 - \cos x \qquad (2)
\end{aligned}
$$

Integrating the RHS of (1) gives

$$
\begin{aligned}
\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2 - 1} \sin(1) \sin(\pi n s)\, ds &= \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n}{n^2\pi^2 - 1} \sin(1) \left[\frac{\cos(n\pi s)}{n\pi}\right]_0^x \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2\pi^2 - 1} \sin(1) [\cos(n\pi s)]_0^x \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(1)}{n^2\pi^2 - 1} (\cos(n\pi x) - 1) \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(1)}{n^2\pi^2 - 1} \cos(n\pi x) - \sin(1) \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2\pi^2 - 1} \qquad (3)
\end{aligned}
$$

Let $-\sin(1) \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2\pi^2 - 1} = m$, which is a constant. The above becomes

$$
\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2 - 1} \sin(1) \sin(\pi n s)\, ds = \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(1)}{n^2\pi^2 - 1} \cos(n\pi x) + m \qquad (4)
$$

But m is the average of the integral of (2) which is, where T the period is 2, gives

$$
\begin{aligned}
m &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (1 - \cos x)\, dx \\
&= \frac{1}{2} \int_{-1}^{1} (1 - \cos x)\, dx \\
&= \frac{1}{2} (x - \sin x)_{-1}^{1} \\
&= \frac{1}{2} (1 - \sin(1) - (-1 - \sin(-1))) \\
&= \frac{1}{2} (1 - \sin(1) - (-1 + \sin(1))) \\
&= \frac{1}{2} (2 - 2\sin(1)) \\
&= 1 - \sin(1)
\end{aligned}
$$

Substituting this value for m back into (4) gives

$$\int_0^x \sum_{n=1}^{\infty} -\frac{2n\pi(-1)^n}{n^2\pi^2-1}\sin(1)\sin(\pi ns)\,ds = (1-\sin(1)) + \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(1)}{n^2\pi^2-1}\cos(n\pi x) \tag{5}$$

Now the Fourier series for (2) which is $1-\cos(x)$ is found to compare it to (5) above to see they match in order to see if term by term integration was justified or not above. Since $1-\cos(x)$ is even, then only a_n are not zero.

$$\begin{aligned} a_0 &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} 1-\cos(x)\,dx \\ &= \int_{-1}^{1} 1-\cos(x)\,dx \\ &= 2-2\sin(1) \end{aligned}$$

And

$$\begin{aligned} a_n &= \int_{-1}^{1}(1-\cos(x))\cos(n\pi x)\,dx \\ &= \int_{-1}^{1}\cos(n\pi x)\,dx - \int_{-1}^{1}\cos(x)\cos(n\pi x)\,dx \\ &= 2\int_0^1\cos(n\pi x)\,dx - 2\int_0^1\cos(x)\cos(n\pi x)\,dx \end{aligned} \tag{6}$$

The first integral in (6)

$$\begin{aligned} 2\int_0^1\cos(n\pi x)\,dx &= \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 \\ &= \frac{1}{n\pi}\sin(n\pi) \\ &= 0 \end{aligned} \tag{7}$$

The second integral in (6) is $2\int_0^1\cos(x)\cos(n\pi x)\,dx$. Integration by parts. $u=\cos x, dv = \cos(n\pi x), du = -\sin x, v = \frac{\sin(n\pi x)}{n\pi}$. Therefore

$$\begin{aligned} 2\int_0^1\cos(x)\cos(n\pi x)\,dx &= 2\left(\left[\cos x\frac{\sin(n\pi x)}{n\pi}\right]_0^1 + \int_0^1\sin x\frac{\sin(n\pi x)}{n\pi}dx\right) \\ &= 2\left(\frac{1}{n\pi}\overbrace{[\cos x\sin(n\pi x)]_0^1}^{0} + \frac{1}{n\pi}\int_0^1\sin x\sin(n\pi x)\,dx\right) \\ &= \frac{2}{n\pi}\int_0^1\sin x\sin(n\pi x)\,dx \end{aligned}$$

Integration by parts. $u=\sin x, dv=\sin(n\pi x), du=\cos x, v=\frac{-\cos(n\pi x)}{n\pi}$. The above becomes

$$\begin{aligned} 2\int_0^1\cos(x)\cos(n\pi x)\,dx &= \frac{2}{n\pi}\left(\frac{-1}{n\pi}[\sin x\cos(n\pi x)]_0^1 + \frac{1}{n\pi}\int_0^1\cos x\cos(n\pi x)\,dx\right) \\ &= \frac{2}{n\pi}\left(\frac{-1}{n\pi}[\sin(1)\cos(n\pi)] + \frac{1}{n\pi}\int_0^1\cos x\cos(n\pi x)\,dx\right) \\ &= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] + \frac{2}{n^2\pi^2}\int_0^1\cos x\cos(n\pi x)\,dx \end{aligned}$$

Moving the integral in the RHS to the left side gives

$$\begin{aligned} 2\int_0^1\cos(x)\cos(n\pi x)\,dx - \frac{2}{n^2\pi^2}\int_0^1\cos x\cos(n\pi x) &= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] \\ \left(2-\frac{2}{n^2\pi^2}\right)\int_0^1\cos(x)\cos(n\pi x)\,dx &= \frac{-2}{n^2\pi^2}\left[\sin(1)(-1)^n\right] \\ \int_0^1\cos(x)\cos(n\pi x)\,dx &= \frac{\frac{-1}{n^2\pi^2}\left[\sin(1)(-1)^n\right]}{\left(1-\frac{1}{n^2\pi^2}\right)} \\ &= \frac{-\left(\sin(1)(-1)^n\right)}{n^2\pi^2-1} \end{aligned} \tag{8}$$

Substituting (7,8) back into (6) gives

$$a_n = \frac{2\sin(1)(-1)^n}{n^2\pi^2 - 1}$$

Hence the Fourier series for $1 - \cos(x)$ over $-1 \leq x \leq 1$ is

$$\begin{aligned}
1 - \cos(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \\
&\sim \frac{(2 - 2\sin(1))}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(1)}{n^2\pi^2 - 1} \cos(n\pi x) \\
&\sim (1 - \sin(1)) + \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(1)}{n^2\pi^2 - 1} \cos(n\pi x) \qquad (9)
\end{aligned}$$

Comparing (9) and (5), shows they are the same. This shows that integration term by term was justified. This is because $\sin x$ is continuous and odd, hence its mean is zero. Then by Theorem 3.20 it can be integrated term by term.

10 Problem 3.5.5 (a,f,i)

Which of the following sequence of functions converge pointwise to the zero function for all $x \in \Re$? Which converges uniformly ?

(a) $-\frac{x^2}{n^2}$ (f) $|x-n|$ (i) $\begin{cases} \frac{x}{n} & |x|<1 \\ \frac{1}{nx} & |x| \geq 1 \end{cases}$

solution

10.1 Part a

Let $f_n(x) = -\frac{x^2}{n^2}$. At $x = 0$ then $f_n(0) = 0$. And for $x \neq 0$ then, if we fix x at say x^* and increase n, then $\lim_{n\to\infty} f_n(x^*) = 0$. Hence it converges pointwise for the zero function for all x because for any x, we fix it and do the same as above, which goes to zero for that x.

For uniform convergence, it means that for any x we can find large enough n such that all $f_n(x)$ are inside a tube, of some diameter $< \varepsilon$ around the zero function. But since x is not bounded, then $f_n(x)$ can be as large as we want. So not possible to find n larger enough to bound all $f_n(x)$ for all x $x \in \Re$ to be $< \varepsilon$ from the zero function.

Hence not uniform convergent. The difference between this and the pointwise case earlier, is that here n we find, should work for all x at the same time.

10.2 Part f

Let $f_n(x) = |x-n|$. At any x, $\lim_{n\to\infty} |x-n|$ is positive. By fixing $x = x^*$, then $f_n(x^*)$ this will keep increasing as n increases. Hence not pointwise convergent to the zero function. Therefore also not uniform convergent since uniform convergence implies pointwise convergence.

10.3 Part i

At $x = 0, f_n(x) = 0$. And for $|x| < 1, \lim_{n\to\infty} \frac{x}{n} \to 0$ since $|x| < 1$. Hence for $|x| < 1$ it converges pointwise to zero. For $|x| \geq 1$, by fixing $x = x^*$, then $\lim_{n\to\infty} \frac{1}{nx^*} \to 0$ also. Hence converges pointwise to zero for all $x \in \Re$.

For uniform convergence, $\max |f_n(x)| = \frac{1}{n}$ which is at $x = 1$. And $\max |f_n(x)| \to 0$ as $n \to \infty$. Hence we could always find n which will make all $f_n(x)$ within ε from each others at any x by increasing n. Hence uniform convergent

11 Problem 3.5.7 (b,d,f)

Does the convergence of $v_n(x)$ converges pointwise to the zero function for all $x \in \Re$? Does it converge uniformly?

(b) $v_n(x) = \begin{cases} 1 & n < x < n+1 \\ 0 & \text{otherwise} \end{cases}$ (d) $v_n(x) = \begin{cases} \frac{1}{n} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$

(f) $v_n(x) = \begin{cases} n^2x^2 - 1 & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

solution

11.1 Part b

This is a pulse width 1 that keeps moving to the right as n increases. All other values are zero. Hence as $n \to \infty$, the pulse will move to ∞ and all values will be zero. Therefore converges pointwise. Since Max of $v_n(x)$ is 1, then it is not not uniform convergent since for $\varepsilon < 1$, we can not bound $v_n(x)$ for all values for all x to be inside the tube around zero function with width $\varepsilon < 1$.

11.2 Part d

$n < x < 2n$ is a pulse that moves to the right, but its width also increases as it moves. It height also decreases as it moves, keeping the area of the pulse 1 all the time. Fixing x at x^* the pulse will eventually become zero height at that x. Therefore converges pointwise to the zero function.

For uniform convergence, $\max|v_n(x)| = \frac{1}{n}$ and $\max|v_n(x)| \to 0$ as $n \to \infty$. Hence we could always find n which will make all $f_n(x)$ within ε from each others at any x by increasing n. Hence uniform convergent

11.3 Part f

As n increases, the range where x is not zero becomes smaller around $x = 0$. The value of $v_n(x)$ can be written as

$$v_n(x) = n^2 e^{2\ln x} - 1$$

As $x \to 0$ from either side, which what happens when $n \to \infty$, then $v_n(x) \to -1$. Hence it does not go to zero at $x = 0$. Therefore not pointwise convergent. It follows that not uniform convergent since uniform convergent implies pointwise convergent.