HW 2

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 2.2.17

(a) Solve the initial value problem $u_t - xu_x = 0, u(0,x) = \frac{1}{1+x^2}$. (b) Graph the solution at times $t = 0,1,2,3$. (c) What is $\lim_{t\to\infty} u(t,x)$?

Solution

1.1 Part a

The characteristic curves equations is given by

$$\frac{dx}{dt} = -x$$

Integrating this results in $\ln|x| = -t + C$ or $x = \xi e^{-t}$. Hence the characteristic variable is

$$\xi(x,t) = xe^t$$

u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$u(t,\xi) = F(\xi)$$
$$u(t,x) = F\left(xe^t\right) \tag{1}$$

Where F is arbitrary function determined from initial conditions. Using initial conditions at $t = 0$, the above becomes

$$\frac{1}{1+x^2} = F(x)$$

Using the above in (1) gives the final solution as

$$u(t,x) = \frac{1}{1+(xe^t)^2} \tag{2}$$

1.2 Part b

The following are some plots and the code used.

```
In[ ]:= p = Grid[Partition[Table[Quiet@Plot[u[x, time], {x, -5, 5},
                    PlotRange → {All, {0, 1.1}},
                    AxesLabel → {Style["x", 12], Style["u", 14]},
                    BaseStyle → 12,
                    ImageSize → 400, PlotStyle → Red, GridLines → Automatic,
                    GridLinesStyle → LightGray,
                    PlotLabel → Row[{"time = ", padIt2[time, {1, 1}], " seconds"}]],
              {time, {0, 1, 2, 3}}
            ], 2], Spacings → {1, 1}, Frame → All]
```

Figure 1: Source code

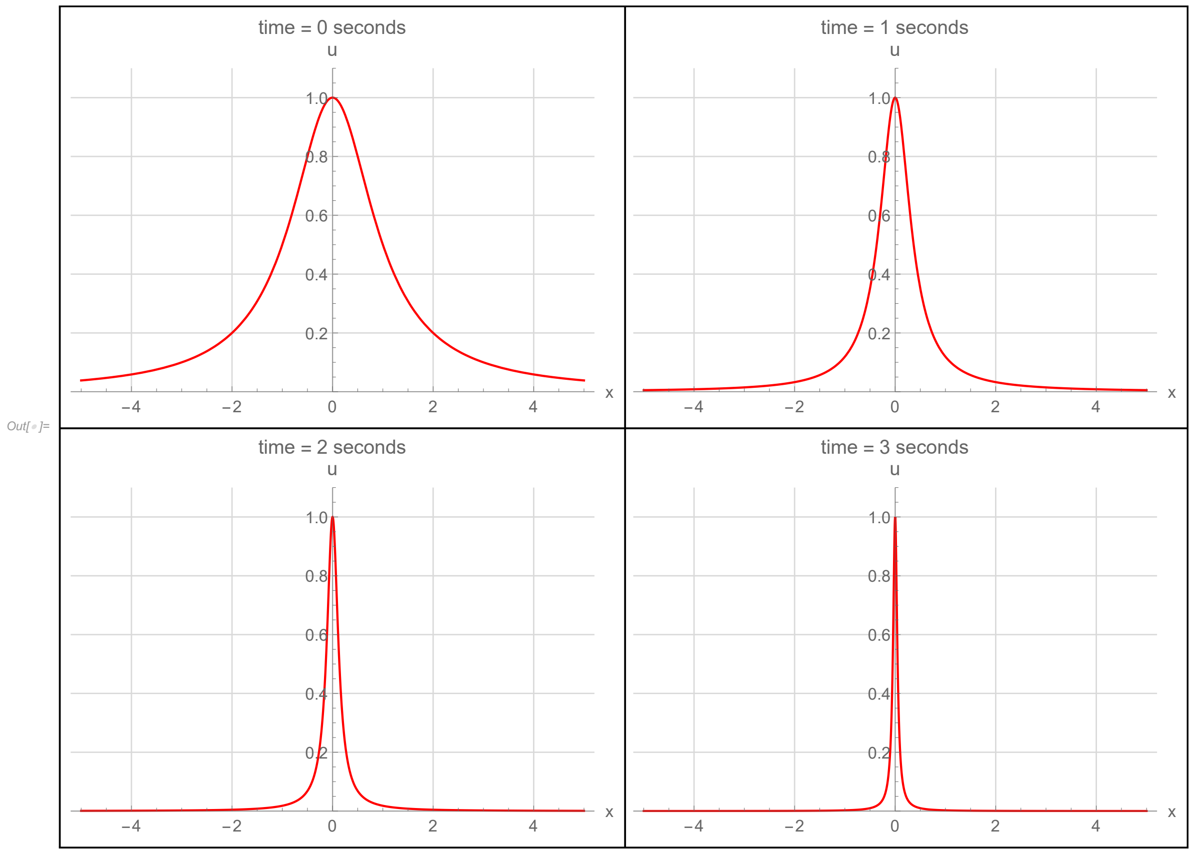

Figure 2: Solution at different times

1.3 Part c

From the solution in (2), when $x = 0$, then $\lim_{t\to\infty} u(t,0) = 1$. But when $x \neq 0$, then $\lim_{t\to\infty} u(t,x) = 0$. Therefore

$$\lim_{t\to\infty} u(t,x) = \begin{matrix} 1 & x = 0 \\ 0 & x \neq 0 \end{matrix}$$

Hence the solution is discontinuous at $x = 0$ in the limit as $t \to \infty$.

2 Problem 2.2.18

Suppose the initial data $u(0,x) = f(x)$ of the nonuniform transport equation (2.28), which is $u_t + \left(x^2-1\right)u_x = 0$ is continuous and satisfies $f(x) \to 0$ as $|x| \to \infty$. What is the limiting solution profile $u(t,x)$ as (a) $t \to \infty$ (b) $t \to -\infty$?

Solution

The characteristic curves equations is given by $\frac{dx}{dt} = \left(x^2-1\right)$. Integrating this results in

$$\begin{aligned}
\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| &= t + C_3 \\
\ln\left|\frac{x-1}{x+1}\right| &= 2t + C_2 \\
\frac{x-1}{x+1} &= \xi e^{2t} \\
\xi &= \frac{x-1}{x+1}e^{-2t}
\end{aligned}$$

u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$\begin{aligned}
u &= F(\xi) \\
&= F\left(\frac{x-1}{x+1}e^{-2t}\right) \qquad (1)
\end{aligned}$$

Where F is arbitrary function which is determined from initial conditions. From initial conditions the above becomes

$$f(x) = F\left(\frac{x-1}{x+1}\right)$$

Let $\frac{x-1}{x+1} = z$. Hence $(x-1) = z(x+1)$ or $x - 1 - z - zx = 0$ or $x(1-z) - 1 - z = 0$ or $x = \frac{1+z}{1-z}$. Therefore

$$f\left(\frac{1+z}{1-z}\right) = F(z)$$

Therefore (1) can now be written as

$$u(t,x) = f\left(\frac{1+\left(\frac{x-1}{x+1}e^{-2t}\right)}{1-\left(\frac{x-1}{x+1}e^{-2t}\right)}\right) \qquad (2)$$

2.1 Part (a)

As $t \to \infty$ then solution (2) becomes

$$\begin{aligned}
\lim_{t\to\infty} u(t,x) &= f\left(\frac{1+0}{1-0}\right) \\
&= f(1)
\end{aligned}$$

2.2 Part (b)

And as $t \to -\infty$ then

$$\begin{aligned}
\lim_{t\to-\infty} u(t,x) &= f\left(\frac{+\infty}{-\infty}\right) \\
&= f(-1)
\end{aligned}$$

3 Problem 2.2.26

Consider the transport equation $\frac{\partial u}{\partial t}+c(t,x)\frac{\partial u}{\partial x}=0$ with time varying wave speed. Define the corresponding characteristic ODE to be $\frac{dx}{dt}=c(t,x)$, the graphs of whose solutions $x(t)$ are the characteristic curves. (a) Prove that any solution $u(t,x)$ to the PDE is constant on each characteristic curve. (b) Suppose that the general solution to the characteristic equation is written in the form $\xi(t,x)=k$, where k is an arbitrary constant. Prove that $\xi(t,x)$ defines a characteristic variable, meaning that $u(t,x)=f(\xi(t,x))$ is a solution to the time-varying transport equation for any continuously differentiable scalar function $f\in C^1$.

Solution

3.1 Part (a)

Let $x(t)$ be the solution to characteristic ODE $\frac{dx}{dt}=c(t,x)$. Then

$$\begin{aligned}\frac{d}{dt}(u(t,x(t))) &= \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}\\ &= \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}c(t,x)\end{aligned}$$

But $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}c(t,x)=0$, since this is the given PDE above. The above now reduces to

$$\frac{d}{dt}(u(t,x(t)))=0$$

Which implies that $u(t,x(t))$ is constant on the characteristic curves.

3.2 Part (b)

$$\begin{aligned}\frac{\partial}{\partial t}f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial}{\partial t}(\xi(t,x))\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}\frac{dx}{dt}\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}c(t,x)\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial}{\partial x}f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial}{\partial x}\xi(t,x)\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}\frac{dt}{dx}+\frac{\partial \xi}{\partial x}\frac{dx}{dx}\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial x}\right)\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}f(\xi(t,x))+c(t,x)\frac{\partial}{\partial x}f(\xi(t,x)) &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}c(t,x)\right)+c(t,x)\frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial x}\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}c(t,x)+c(t,x)\frac{\partial \xi}{\partial x}\right)\\ &= \frac{df}{d\xi(t,x)}\left(\frac{\partial \xi}{\partial t}+2\frac{\partial \xi}{\partial x}c(t,x)\right)\end{aligned}$$

But $\xi(t,x)$ is constant k. Hence $\frac{df}{d\xi(t,x)}=0$. Therefore RHS above is zero, and the above reduces to

$$\frac{\partial}{\partial t}f(\xi(t,x))+c(t,x)\frac{\partial}{\partial x}f(\xi(t,x))=0$$

This shows that $f(\xi(t,x))$ satisfies the given transport PDE. Hence it is a solution. Or $u(t,x)=f(\xi(t,x))$.

4 Problem 2.2.29

Consider the first-order PDE $u_t + (1 - 2t)\, u_x = 0$. Use exercise 2.2.26 to: (a) Find and sketch the characteristic curves. (b) Write down the general solution. (c) Solve the initial value problem with $u(0, x) = \frac{1}{1+x^2}$. (d) Describe the behavior of your solution $u(t, x)$ from part (c) as $t \to \infty$. What about $t \to -\infty$?

Solution

4.1 Part (a)

The characteristic curves are given by $\frac{dx}{dt} = (1 - 2t)$. Therefore

$$
\begin{aligned}
x(t) &= t - t^2 + \xi \\
\xi &= x - \left(t - t^2\right)
\end{aligned}
$$

The following is plot of characteristic curves for different ξ values.

```
Table[Plot[t - t^2 + k, {t, 0, 4}, PlotRange → {All, {-10, 7}},
    AxesLabel → {"t", "x(t)"}, BaseStyle → 14],
  {k, 0, 5, 1}];
Show[%]
```

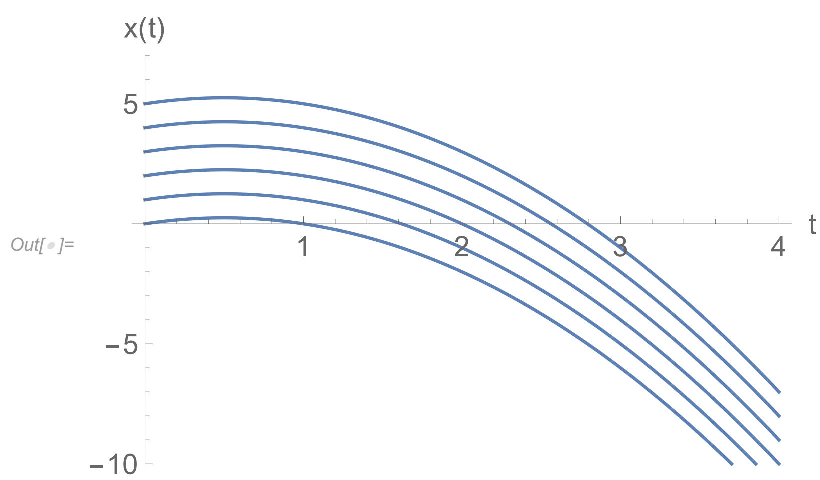

Figure 3: Plot of some characteristic curves

4.2 Part (b)

solution u on the characteristic curves is an arbitrary function of the characteristic variable. Hence

$$
\begin{aligned}
u(t, x) &= F(\xi) \\
&= F\left(x - \left(t - t^2\right)\right) \\
&= F\left(x - t + t^2\right) \qquad (1)
\end{aligned}
$$

Where F is arbitrarily function.

4.3 Part (c)

At $t = 0$ the above solution becomes

$$
\frac{1}{1 + x^2} = F(x) \tag{2}
$$

Therefore using (2) in (1), then (1) becomes

$$
u(t, x) = \frac{1}{1 + \left(x - t + t^2\right)^2} \tag{3}
$$

4.4 Part (d)

The solution in (3) shows that

$$\lim_{t\to\infty} u(t,x) = \frac{1}{\infty} = 0$$

Also

$$\lim_{t\to-\infty} u(t,x) = \frac{1}{\infty} = 0$$

Hence the solution vanishes for large t.

5 Problem 2.4.2

(a) Solve the wave equation $u_{tt} = u_{xx}$ when the initial displacement is the box function $u(0,x) = \begin{cases} 1 & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$, while the initial velocity is zero. (b) Sketch the resulting solution at several times.

Solution

5.1 Part (a)

d'Alembert solution of the wave equation is given by

$$u(t,x) = \frac{1}{2}\left(f(x-ct)+f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Where c is the wave speed which is $c=1$ in this problem and $f(x) = u(0,x)$ and $g(x) = u_t(0,x) = 0$. The above simplifies to

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left(f(x-t)+f(x+t)\right) \\ &= \frac{1}{2}\left(\begin{cases} 1 & 1<x-t<2 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 1<x+t<2 \\ 0 & \text{otherwise} \end{cases}\right) \\ &= \frac{1}{2}\left(\begin{cases} 1 & 1+t<x<2+t \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 1-t<x<2-t \\ 0 & \text{otherwise} \end{cases}\right) \end{aligned}$$

Complete split of the box function into two separate halves happens at $t=0.5$ because when $t=0.5$ in the above gives

$$u(t,x) = \frac{1}{2}\left(\begin{cases} 1 & 1.5<x<2.5 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & 0.5<x<1.5 \\ 0 & \text{otherwise} \end{cases}\right)$$

This shows that just after $t=0.5$, there is no longer a common region between $1.5<x<2.5$ and $0.5<x<1.5$.

Hence for $\underline{t>0.5}$ the solution u will be $\frac{1}{2}$ when $1+t<x<2+t$ or when $1-t<x<2-t$ and will be zero otherwise.

But when $t<0.5$, there will still be a common region before the full split. Some region is till common, and some region is not. For example, picking $t=0.25$, then there is a common region between $1.25<x<2.25$ and $0.75<x<1.75$. In this case the common region is $1.25<x<1.75$. Over this region, $u=1$. But over the non common region $u=\frac{1}{2}$ when $0.75<x<1.25$ and $u=\frac{1}{2}$ for $0.1.75<x<2.25$ and $u=0$ otherwise. In terms of t the above can be written as

When $t \geq \frac{1}{2}$ then the solution is

$$u = \frac{1}{2}\begin{cases} \frac{1}{2} & 1-t<x<2-t \\ \frac{1}{2} & 1+t<x<2+t \\ 0 & \text{otherwise} \end{cases}$$

When $t<\frac{1}{2}$

$$u = \frac{1}{2}\begin{cases} 1 & 1+t<x<2-t \\ \frac{1}{2} & 1-t<x<1+t \\ \frac{1}{2} & 2-t<x<2+t \\ 0 & \text{otherwise} \end{cases}$$

It it easier to do all of this using the computer by plotting the solution for different times.

5.2 Part (b)

The following are plots of the motion of the wave for several times.

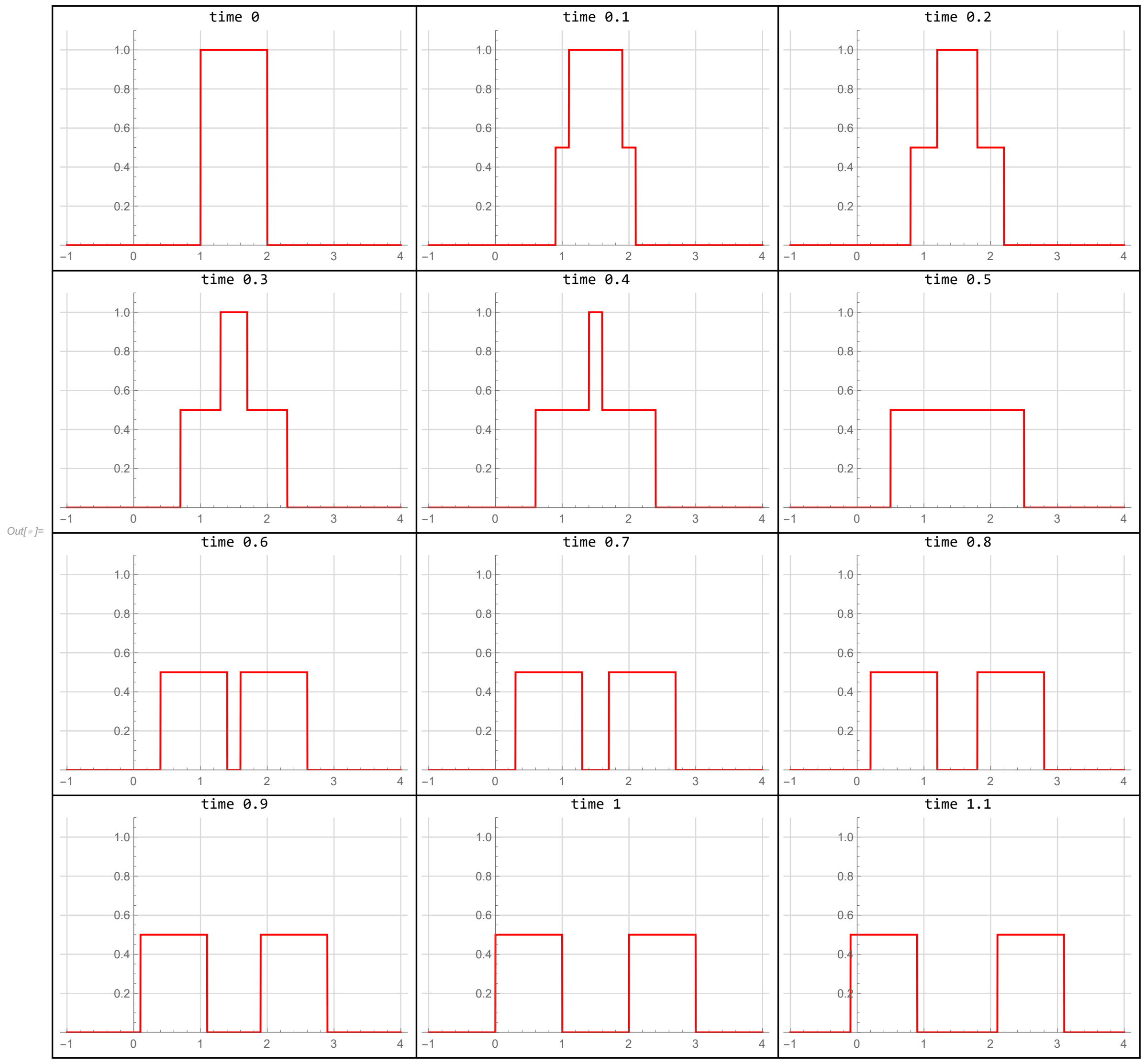

Figure 4: Plots for several times

```
In[∘]:= u[x_, t_] := 1/2 (Piecewise[{{1, 1 < x - t < 2}, {0, True}}] + Piecewise[{{1, 1 < x + t < 2}, {0, True}}]);
plots = Table[Grid[{{Row[{"time ", t}]}},
                    {Plot[u[x, t], {x, -1, 4}, Exclusions → None, ImageSize → 300,
                           PlotPoints → 40,
                           PerformanceGoal → "Quality", PlotStyle → Red,
                           GridLines → Automatic, GridLinesStyle → LightGray,
                           PlotRange → {All, {0, 1.1}}]}
                }], {t, {0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1, 1.1}}];
Grid[Partition[plots, 3], Frame → All]
```

Figure 5: Code used

6 Problem 2.4.3

Answer 2.4.2 when the initial velocity is the box function while the initial displacement is zero.

Solution

6.1 Part (a)

d'Alembert solution of the wave equation is

$$u(t,x) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Where c is the wave speed which is $c=1$ in this problem and $f(x)=0$ and $g(x)=u_t(0,x)=f(x)$ which is the box function given in the last problem. The above becomes

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_{x-t}^{x+t} f(s)\,ds \\ &= \frac{1}{2}\int_{s=x-t}^{s=x+t} \begin{cases} 1 & 1<s<2 \\ 0 & \text{otherwise} \end{cases} ds \end{aligned}$$

6.2 Part (b)

The following are plots of the motion of the wave for several times of the above solution

```
In[ ]:= u[x_, t_] := 1/2 Integrate[Piecewise[{{1, 1 < s < 2}, {0, True}}], {s, x - t, x + t}];
plots = Table[Grid[{{Row[{"time ", t}]},
            {Plot[u[x, t], {x, -1, 4}, Exclusions → None, ImageSize → 300,
                PlotPoints → 40,
                PerformanceGoal → "Quality", PlotStyle → Red,
                GridLines → Automatic, GridLinesStyle → LightGray,
                PlotRange → {All, {0, 1.1}}]}
        }], {t, {0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1, 1.1}}];
Grid[Partition[plots, 3], Frame → All]
```

Figure 6: Code used

Out[ ]=

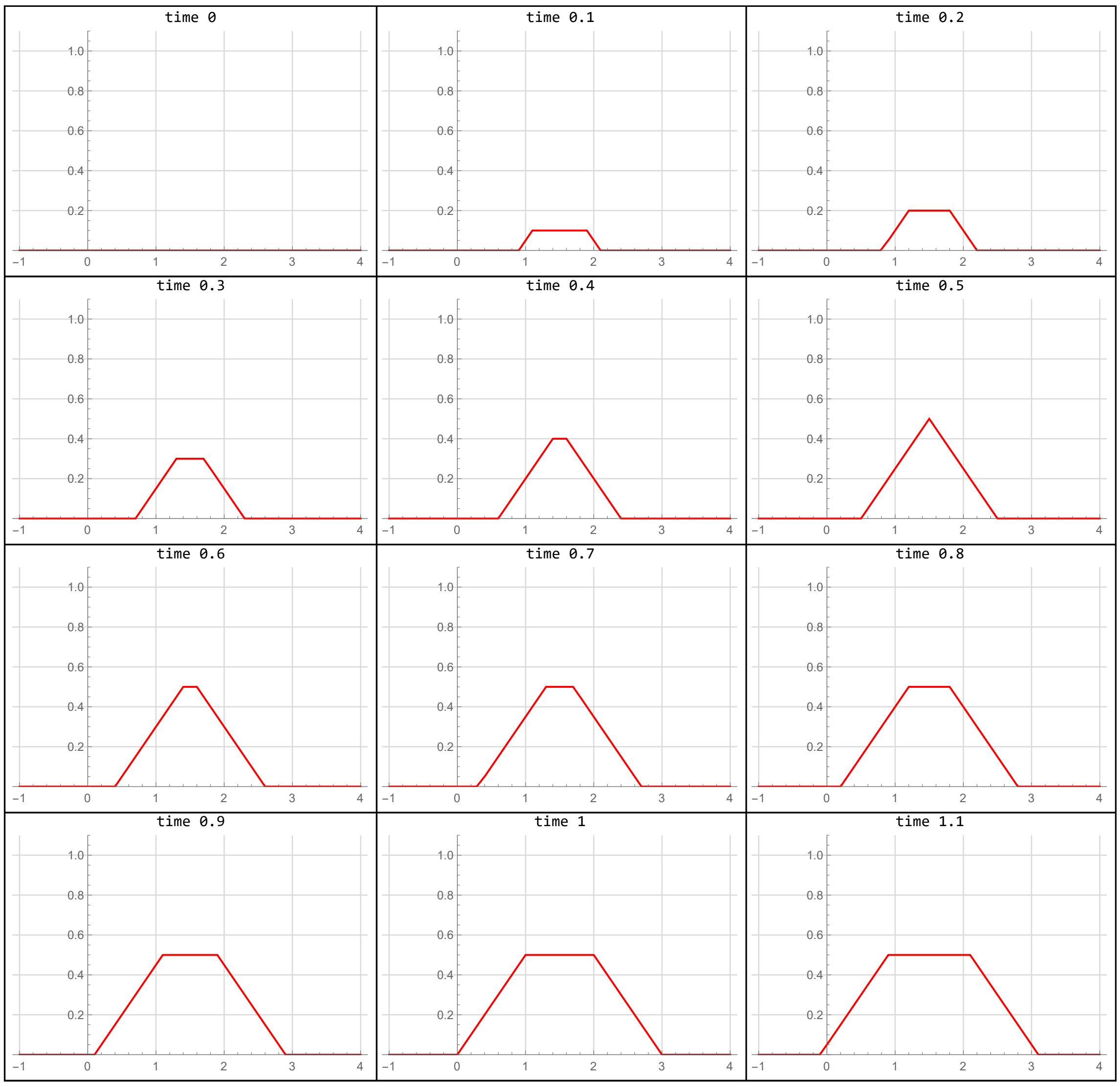

Figure 7: Plots for several times

7 Problem 2.4.4

Write the following solutions to the wave equation $u_{tt} = u_{xx}$ in d'Alembert form (2.82) which is $u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$. Hint: What is the appropriate initial data? (b) $\cos 2x \sin 2t$. (d) $t^2 + x^2$

Solution

7.1 Part(b)

Since $c = 1$, the solution becomes

$$\cos 2x \sin 2t = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Let $f(x) = u(0,x) = 0$. The above solution simplifies to

$$\begin{aligned} 2\cos 2x \sin 2t &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ \cos 2x \sin 2t &= \frac{1}{4}\int_{x-t}^{x+t} g(s)\,ds \qquad (1) \end{aligned}$$

We now need to determine $g(s)$ to satisfy the above. By fundamental theorem of calculus

$$\frac{1}{4}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{4}\left[g'(x+t) - g'(x-t)\right] \qquad (2)$$

Let $g(x) = 2\cos 2x$. Now we need to verify that this will satisfy equation (1). Expanding RHS of (2) gives

$$\begin{aligned} g'(x+t) - g'(x-t) &= 2(-\sin(2(x+t)) + \sin(2(x-t))) \\ &= 2(\sin(2x-2t) - \sin(2x+2t)) \end{aligned}$$

But $\sin(A-B) = \sin A\cos B - \cos A \sin B$ and $\sin(A+B) = \sin A\cos B + \cos A\sin B$. Substituting these in the above, where $A = 2x, B = 2t$, the above becomes

$$\begin{aligned} g'(x+t) - g'(x-t) &= 2(\sin 2x\cos 2t - \cos 2x \sin 2t - (\sin 2x \cos 2t + \cos 2x \sin 2t)) \\ &= 2(\sin 2x \cos 2t - \cos 2x \sin 2t - \sin 2x\cos 2t - \cos 2x \sin 2t) \\ &= 4\cos 2x \sin 2t \qquad (3) \end{aligned}$$

Substituting (3) into (1) gives

$$\begin{aligned} \cos 2x \sin 2t &= \frac{1}{4}(4\cos 2x \sin 2t) \\ &= \cos 2x \sin 2t \end{aligned}$$

Verified.

Hence if initial condition is $f(x) = 0$ and if $g(x) = 2\cos 2x$, then the solution using d'Alembert form will be the one given $u(t,x) = 2\cos 2x \sin 2t$ which is what we are asked to show. Therefore

$$\begin{aligned} \cos 2x \sin 2t &= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ u(0,x) &= 0 \\ u_t(0,x) &= 2\cos 2x \end{aligned}$$

7.2 Part(d)

Since $c = 1$, the solution becomes

$$t^2 + x^2 = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Let $g(x) = u_t(0,x) = 0$. The above reduces to

$$t^2 + x^2 = \frac{1}{2}\left(f(x-t) + f(x+t)\right)$$

Assuming $f(x)=x^2$, now we will see if this assumption generates the solution needed. The RHS above now becomes

$$\begin{aligned}
\frac{1}{2}\left(f(x-t)+f(x+t)\right) &= \frac{1}{2}\left((x-t)^2+(x+t)^2\right)\\
&= \frac{1}{2}\left(\left(x^2+t^2-2xt\right)+\left(x^2+t^2+2xt\right)\right)\\
&= \frac{1}{2}\left(x^2+t^2+x^2+t^2\right)\\
&= t^2+x^2
\end{aligned}$$

Verified.

Hence by setting $g(x)=0$ and $f(x)=x^2$ the given solution is obtained. Therefore

$$\begin{aligned}
t^2+x^2 &= \frac{1}{2}\left(f(x-t)+f(x+t)\right)+\frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds\\
u(0,x) &= x^2\\
u_t(0,x) &= 0
\end{aligned}$$

8 Problem 2.4.10

Suppose $u(t,x)$ solves the initial value problem $u_{tt} = 4u_{xx} + \sin(\omega t)\cos(x)$, $u(0,x) = 0$, $u_t(0,x) = 0$. Is $h(t) = u(t,0)$ a periodic function?

Solution

The solution is given by eq (2.96) in the textbook (since $f(x) = 0$ and $g(x) = 0$ and $c^2 = 4$ or $c = 2$) as the following

$$u(t,x) = \frac{1}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(s,y)\, dyds$$

But here $F(s,y) = \sin(\omega s)\cos(y)$. Therefore, using the book example 2.19, where we just need to change $\sin x$ to $\cos x$ in the solution shown, then the above integral gives

$$\begin{aligned} u(t,x) &= \frac{1}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \sin(\omega s)\cos(y)\, dyds \\ &= \begin{cases} \frac{\sin(\omega t) - \omega \sin t}{1-\omega^2}\cos x & 0 < \omega \neq 1 \\ \frac{\sin t - t\cos t}{2}\cos x & \omega = 1 \end{cases} \end{aligned}$$

At $x = 0$, then

$$h(t) = u(t,0) = \begin{cases} \frac{\sin(\omega t) - \omega \sin t}{1-\omega^2} & 0 < \omega \neq 1 \\ \frac{\sin t - t\cos t}{2} & \omega = 1 \end{cases}$$

Therefore $h(t)$ is periodic only if $\omega = \frac{p}{q} \neq 1$ is a rational number.

9 Problem 2.4.11

(a) Write down an explicit formula for the solution to initial value problem $u_{tt} = 4u_{xx}, u(0,x) = \sin x, u_t(0,x) = \cos x$ for $-\infty < x < \infty, t \geq 0$. (b) True of False: The solution is a periodic function of t. (c) Now solve the forced initial value problem $u_{tt} = 4u_{xx} + \cos 2t, u(0,x) = \sin x, u_t(0,x) = \cos x$ for $-\infty < x < \infty, t \geq 0$. (d) True of False: The forced equation exhibits resonance. Explain. (e) Does the answer to part (d) change if the forcing function is $\sin 2t$?

Solution

9.1 Part (a)

Using d'Alembert formula where $u(0,x) = f(x) = \sin x$ and $u_t(0,x) = g(x) = \cos x$, then the solution is

$$u(t.x) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

But $c = 2, f(x) = \sin x, g(x) = \cos x$, then the above becomes

$$\begin{aligned}
u(t,x) &= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(s)\,ds \\
&= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}[\sin(s)]_{x-2t}^{x+2t} \\
&= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}(\sin(x+2t) - \sin(x-2t)) \\
&= \frac{1}{2}\sin(x-2t) + \frac{1}{2}\sin(x+2t) + \frac{1}{4}\sin(x+2t) - \frac{1}{4}\sin(x-2t) \\
&= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t)
\end{aligned}$$

9.2 Part (b)

True.

If we can find a common multiple between $x - 2t$ and $x + 2t$ then the solution is periodic. i.e. if $F_1(z)$ has period p_1 and $F_2(z)$ has period p_2, then if we can find positive integers a_1, a_2 such that $a_1 p_1 = a_2 p_2 = r$, then r is the period of $F_1(x) + F_2(x)$.

In this problem, $F_1 = \sin(x-2t), F_2 = \sin(x+2t)$. But both of these have period 2π. Hence $p_1 = 2\pi, p_2 = 2\pi$. Therefore choosing $a_1 = 1, a_2 = 1$, then $r = 2\pi$. The period of sum.

9.3 Part (c)

When the PDE becomes $u_{tt} = 4u_{xx} + \cos 2t$, then we need to add forcing solution part of the solution. Hence the solution now becomes, using 2.97 in the book as (using $c = 2$)

$$u(t,x) = \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(s)\,ds + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} F\big(s,y\big)\,dyds$$

Where $F\left(s,y\right) = \cos\left(2t\right)$. Hence the above becomes (using result from part (a) for the non forcing part) as

$$\begin{aligned}
u\left(t,x\right) &= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\int_{x-2(t-s)}^{x+2(t-s)}\cos\left(2s\right)dyds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\cos\left(2s\right)\int_{x-2(t-s)}^{x+2(t-s)}dyds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\cos\left(2s\right)\left(\left(x+2\left(t-s\right)\right)-\left(x-2\left(t-s\right)\right)\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\cos\left(2s\right)\left(x+2t-2s-\left(x-2t+2s\right)\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\cos\left(2s\right)\left(x+2t-2s-x+2t-2s\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\cos\left(2s\right)\left(4t-4s\right)ds
\end{aligned}$$

But $\frac{1}{4}\int_0^t\cos\left(2s\right)\left(4t-4s\right)ds = \frac{\sin^2 t}{2}$. Hence the above solution becomes

$$u\left(t,x\right) = \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{\sin^2 t}{2}$$

Which can also be written as

$$\begin{aligned}
u\left(t,x\right) &= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\cos\left(2t\right)\right)\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}-\frac{1}{4}\cos\left(2t\right)
\end{aligned}$$

9.4 Part (d)

False. No resonance. Solution is periodic. There is no term in the solution which is being multiplied by t. Hence solution do not grow with time which indicates no resonance.

9.5 Part (e)

If the PDE now becomes $u_{tt} = 4u_{xx}+\sin 2t, u\left(0,x\right) = \sin x, u_t\left(0,x\right) = \cos x$, then the solution becomes

$$\begin{aligned}
u\left(t,x\right) &= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\int_{x-2(t-s)}^{x+2(t-s)}\sin\left(2s\right)dyds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\sin\left(2s\right)\int_{x-2(t-s)}^{x+2(t-s)}dyds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\sin\left(2s\right)\left(\left(x+2\left(t-s\right)\right)-\left(x-2\left(t-s\right)\right)\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\sin\left(2s\right)\left(x+2t-2s-\left(x-2t+2s\right)\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\sin\left(2s\right)\left(x+2t-2s-x+2t-2s\right)ds\\
&= \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\int_0^t\sin\left(2s\right)\left(4t-4s\right)ds
\end{aligned}$$

But $\frac{1}{4}\int_0^t\sin\left(2s\right)\left(4t-4s\right)ds = \frac{1}{4}\left(2t-\sin\left(2t\right)\right)$. Hence the solution now becomes

$$u\left(t,x\right) = \frac{1}{4}\sin\left(x-2t\right)+\frac{3}{4}\sin\left(x+2t\right)+\frac{1}{4}\left(2t-\sin\left(2t\right)\right)$$

We see now that resonance now occurs due to above term $\frac{1}{2}t$ in the solution. This means as t increases, the solution will keep increasing with no limit.

10 Problem 2.4.13

Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$. The total energy

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

Represents the sum of kinetic and potential energies of the displacement $u(t,x)$ at time t. Suppose that $\Delta u \to 0$ sufficiently rapidly as $x \to \pm\infty$; more precisely, one can find $\alpha > \frac{1}{2}$ and $C(t) > 0$ such that $|u_t(t,x)|, |u_x(t,x)| \leq \frac{C(t)}{|x|^{\alpha}}$ for each fixed t and all sufficiently large $|x| \gg 0$. For such solutions establish the law of conservation of energy by showing that $E(t)$ is finite and constant. Hint: You do not need the formula for the solution.

Solution

To show $E(t)$ is constant, it is sufficient to show that $\frac{d}{dt}E(t) = 0$. From above

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx$$

Moving $\frac{d}{dt}$ inside the integral (assuming solution is piecewise smooth), the above becomes

$$\frac{d}{dt}E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d}{dt}u_t^2 + c^2\frac{d}{dt}u_x^2\right) dx$$

But $\frac{d}{dt}u_t^2 = 2u_t u_{tt}$ and $\frac{d}{dt}u_x^2 = 2u_x u_{xt}$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} \frac{1}{2}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right) dx \\ &= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} dx\end{aligned}$$

But $u_{tt} = c^2 u_{xx}$ from the PDE itself. The above now simplifies to

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} c^2 u_t u_{xx} + c^2 u_x u_{xt} dx \\ &= c^2 \int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} dx\end{aligned}$$

But $u_t u_{xx} + u_x u_{xt} = \frac{d}{dx}(u_t u_x)$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= c^2 \int_{-\infty}^{\infty} \frac{d}{dx}(u_t u_x)\, dx \\ &= c^2 \int_{-\infty}^{\infty} d(u_t u_x) \\ &= c^2 \left[u_t u_x\right]_{-\infty}^{\infty}\end{aligned}$$

But the problem says that as $x \to \pm\infty$ then $u_x \to 0$. It also say that $|u_t|$ is bounded. This shows that the RHS above is zero. Therefore $\frac{d}{dt}E(t) = 0$ or $\underline{E(t) \text{ is constant}}$. The fact constant is bounded is seen by noting that the problems says that $|u_x|$ and $|u_t|$ are bounded. This completes the proof.

11 Problem 2.4.15

The telegraph equation $u_{tt} + au_t = c^2 u_{xx}$ with $a > 0$, models the vibration of a string under frictional damping. (a) Show that, under the decay assumption of exercise 2.4.13, the wave energy (2.98)

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

of a classical solution is a nonincreasing function of t. (b) Prove uniqueness of such solutions to the initial value problem for the telegraph equation.

Solution

11.1 Part (a)

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx$$

Moving $\frac{d}{dt}$ inside the integral (assuming solution is smooth), the above becomes

$$\frac{d}{dt}E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d}{dt}u_t^2 + c^2\frac{d}{dt}u_x^2\right) dx$$

But $\frac{d}{dt}u_t^2 = 2u_t u_{tt}$ and $\frac{d}{dt}u_x^2 = 2u_x u_{xt}$. The above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} \frac{1}{2}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right) dx \\ &= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} dx\end{aligned}$$

But $u_{tt} = c^2 u_{xx} - au_t$ from the PDE itself, hence the above simplifies to

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{-\infty}^{\infty} u_t\left(c^2 u_{xx} - au_t\right) + c^2 u_x u_{xt} dx \\ &= \int_{-\infty}^{\infty} c^2 u_t u_{xx} - au_t^2 + c^2 u_x u_{xt} dx \\ &= c^2\int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} dx - a\int_{-\infty}^{\infty} u_t^2 dx\end{aligned}$$

But $\int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} dx = \frac{d}{dx}(u_t u_x)$, then the above becomes

$$\begin{aligned}\frac{d}{dt}E(t) &= c^2\int_{-\infty}^{\infty} \frac{d}{dx}(u_t u_x)\, dx - a\int_{-\infty}^{\infty} u_t^2 dx \\ &= c^2\int_{-\infty}^{\infty} d(u_t u_x)\, dx - a\int_{-\infty}^{\infty} u_t^2 dx \\ &= c^2 [u_t u_x]_{-\infty}^{\infty} - a\int_{-\infty}^{\infty} u_t^2 dx\end{aligned}$$

As in the previous problem $[u_t u_x]_{-\infty}^{\infty} = 0$ since $u_x \to 0$ for $x \to \pm\infty$. Then the above now reduces to

$$\frac{d}{dt}E(t) = -a\int_{-\infty}^{\infty} u_t^2 dx$$

But $\int_{-\infty}^{\infty} u_t^2 dx$ is either zero or positive because the integrand is always positive.

Hence $\frac{d}{dt}E(t)$ is negative quantity because $a > 0$. This shows that rate of change of energy is either zero or negative and can not be positive. This means $E(t)$ is non increasing which is what we are asked to show.

11.2 Part (b)

Let $u_1(t,x)$ and $u_2(t,x)$ be two different solutions to same $u_{tt} + au_t = c^2 u_{xx}$ with same initial data. Let $w(t,x) = u_1(t,x) - u_2(t,x)$. Therefore

$$w_{tt} + aw_t = c^2 w_{xx}$$

Applying the energy formula to $w(t,x)$ shows that

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left((w_t)^2 + c^2 (w_x)^2\right) dx$$
$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left((w_t)^2 + c^2 (w_x)^2\right) dx$$

Following same steps in problem 2.4.13, the above becomes zero. Which means that $\frac{dE}{dt} = 0$ or $E(t)$ is constant. But $E(-\infty) = E(\infty) = 0$ which means that $E(t) = 0$. In other words

$$\int_{-\infty}^{\infty} \frac{1}{2}\left((w_t)^2 + c^2 (w_x)^2\right) dx = 0$$

But since the integrand is positive, then this means $w_t = 0$ and $w_x = 0$. But this implies that $w(t,x)$ is itself a constant.

We now need to show that this constant is zero. i.e. to show that $w(t,x) = 0$ to finish the proof.

Since $w(0,x) = 0$, because this is the initial data, which is the difference between the initial data of the two solutions u_1, u_2 which is the same, hence the difference of the initial data is zero.

But if $w(0,x) = 0$ and $w(t,x)$ is constant, it must be that $w(t,x) = 0$ for all time and space.

But since $w(t,x) = u_1(t,x) = u_2(t,x)$ then

$$u_1(t,x) = u_2(t,x)$$

Which mean that the solution to the telegraph PDE is unique.