HW 10

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 1

Show that (assuming sufficient smoothness of the domain and the data) u is a solution to the Dirichlet boundary value problem

$$-\Delta u = f$$

In Ω with B.C. $u = g$ on $\partial\Omega$ iff u is a minimizer of the energy functional, that is

$$E(u) = \min\left\{E(v) : v \in C^2(\bar{\Omega})\right\} \text{ such that } u = g \text{ on } \partial\Omega$$

Here

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dA$$

(note, I will be using dA in the above integral assuming we are in $\mathbb{R}^2$. But the above can also be dV for $\mathbb{R}^3$ just as well and nothing will change in the derivation below. This is easier that writing dx and saying that x is a vector).

<u>Solution</u>

Since the proof is an iff, then we need to show both direction.

<u>Forward direction</u> Given that u solves

$$-\Delta u = f \tag{1}$$

with $u|_{\partial\Omega} = g$. Then we need to show that $E(v) \geq E(u)$ for all $v \in C^2(\bar{\Omega})$ that also satisfy same B.C.

Multiplying both sides of (1) by $u - v$ and integrating over the domain gives

$$-\int_\Omega (\Delta u)(u - v)\, dA = \int_\Omega (u - v) f\, dA \tag{2}$$

For the left integral $\int_\Omega (\Delta u)(u - v)\, dA$, we will do integration by parts. Let $\Delta u \equiv dV, u - v = U$, then $\int_\Omega U dV = \int_{\partial\Omega} UV - \int_\Omega V dU$. Therefore $dU = \nabla(u - v)$ and $V = \nabla u$. After applying integration by parts the (2) now becomes

$$-\left(\int_{\partial\Omega} (u - v)\frac{\partial u}{\partial n}\, dL - \int_\Omega \nabla u \cdot \nabla(u - v)\, dA\right) = \int_\Omega (u - v) f\, dA$$

But $\int_{\partial\Omega} (u - v)\frac{\partial u}{\partial n}\, dL = 0$ because $u = v$ on the boundary $\partial\Omega$ as both are g. The above now simplifies to

$$\int_\Omega \nabla u \cdot \nabla(u - v)\, dA = \int_\Omega \left(uf - vf\right) dA$$

$$\int_\Omega \nabla u \cdot (\nabla u - \nabla v)\, dA = \int_\Omega \left(uf - vf\right) dA$$

$$\int_\Omega |\nabla u|^2 - \nabla u \cdot \nabla v\, dA = \int_\Omega \left(uf - vf\right) dA$$

$$\int_\Omega |\nabla u|^2 - \int_\Omega fu\, dA = \int_\Omega (\nabla u \cdot \nabla v) - vf\, dA$$

Now we use Schwarz triangle inequality and write $\nabla u \cdot \nabla v \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right)$. This comes from using $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$. Using this in the RHS of the above gives

$$\int_\Omega |\nabla u|^2\, dA - \int_\Omega fu\, dA \leq \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right) - fv\, dA$$

$$\int_\Omega |\nabla u|^2\, dA - \int_\Omega fu\, dA \leq \int_\Omega \frac{1}{2}|\nabla u|^2\, dA + \left(\frac{1}{2}\int_\Omega |\nabla v|^2 - fv\, dA\right)$$

$$\int_\Omega \frac{1}{2}|\nabla u|^2\, dA - \int_\Omega fu\, dA \leq \frac{1}{2}\int_\Omega |\nabla v|^2 - fv\, dA$$

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dA \leq \frac{1}{2}\int_\Omega |\nabla v|^2 - fv\, dA$$

But by definition $\int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dA = E(u)$ and $\frac{1}{2}\int_\Omega |\nabla v|^2 - fv\, dA = E(v)$, therefore the above

becomes

$$E(u) \leq E(v)$$

Which is what we wanted to show. Now we will do the other direction.

<u>Reverse direction</u> Given that u minimizes energy among all test functions, i.e. given that $E(u) = \min E(w)$, then need to show that $-\Delta u = f$.

Consider $w = u + \varepsilon v$ where v is any test function $v \in C^2(\bar{\Omega})$ and $v = g$ at $\partial\Omega$. Hence

$$\min(E(w)) = \min(E(u + \varepsilon v))$$

Therefore $\min(E(u + \varepsilon v))$ is achieved when $\varepsilon = 0$, since this then gives $E(u)$ which by assumption is the minimum. Therefore

$$\frac{d}{d\varepsilon} E(u + \varepsilon v) = 0$$

At $\varepsilon = 0$. But the above can be written as the following, using the definition of energy

$$\frac{d}{d\varepsilon}\left(\int_\Omega \frac{1}{2}|\nabla(u + \varepsilon v)|^2 - f(u + \varepsilon v)\ dA\right) = 0$$

$$\frac{d}{d\varepsilon}\left(\int_\Omega \frac{1}{2}(\nabla(u + \varepsilon v)\cdot\nabla(u + \varepsilon v)) - f(u + \varepsilon v)\ dA\right) = 0 \qquad (3)$$

Expanding $\nabla(u + \varepsilon v)\cdot\nabla(u + \varepsilon v)$ gives

$$\nabla(u + \varepsilon v)\cdot\nabla(u + \varepsilon v) = (\nabla u + \varepsilon\nabla v)\cdot(\nabla u + \varepsilon\nabla v)$$

$$= |\nabla u|^2 + 2\varepsilon\nabla u\cdot\nabla v + \varepsilon^2|\nabla v|^2 \qquad (4)$$

Substituting (4) into (3) gives

$$\frac{d}{d\varepsilon}\left(\int_\Omega v\left(|\nabla u|^2 + 2\varepsilon\nabla u\cdot\nabla v + \varepsilon^2|\nabla v|^2\right) - fu - \varepsilon fv\ dA\right) = 0$$

Now we move the derivative inside the take derivative w.r.t. ε giving

$$\left(\int_\Omega \frac{1}{2}\left(2\nabla u\cdot\nabla v + 2\varepsilon|\nabla v|^2\right) - fv\ dA\right) = 0$$

Evaluate at $\varepsilon = 0$ the above becomes

$$\int_\Omega (\nabla u\cdot\nabla v)\ dA - \int_\Omega fv\ dA = 0$$

Integration by parts for the first integral. Let $\nabla u = U, dV = \nabla v$, then $\int_\Omega U dV = \int_{\partial\Omega} UV - \int_\Omega V dU$. Hence the above becomes

$$\left(\int_{\partial\Omega} v\frac{\partial u}{\partial n}\ dL - \int_\Omega v\Delta u\ dA\right) - \int_\Omega fv\ dA = 0$$

But $v = 0$ at boundary $\partial\Omega$. The above simplifies to

$$-\int_\Omega v\Delta u\ dA - \int_\Omega fv\ dA = 0$$

$$\int_\Omega v\left(-\Delta u - f\right)dA = 0$$

Since the above is true for all v test function then this implies that $-\Delta u - f = 0$ or

$$-\Delta u = f$$

Which is what we wanted to show.

2 Problem 7.1.1 f

Find the Fourier transform of (f) $f(x) = \begin{cases} e^{-x}\sin x & x > 0 \\ 0 & x \leq 0 \end{cases}$

<u>Solution</u>

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}\sin xe^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \sin xe^{-ikx-x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \sin xe^{-x(1+ik)}dx \tag{1}
\end{aligned}$$

Integration by parts. $\int u\,dv = uv - \int v\,du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \sin x, du = \cos x$.
Hence

$$\begin{aligned}
I &= \int_{0}^{\infty} \sin xe^{-x(1+ik)}dx \\
&= \left[\sin x\frac{e^{-x(1+ik)}}{-(1+ik)}\right]_{0}^{\infty} - \int_{0}^{\infty}\cos x\frac{e^{-x(1+ik)}}{-(1+ik)}dx \\
&= \frac{-1}{1+ik}\left[\sin xe^{-x(1+ik)}\right]_{0}^{\infty} + \frac{1}{1+ik}\int_{0}^{\infty}\cos xe^{-x(1+ik)}dx
\end{aligned}$$

But $e^{-x(1+ik)} = e^{-x}e^{-ikx}$ and this goes to zero as $x \to \infty$ and since $\sin x = 0$ at $x = 0$ then the first term above is zero. The above reduces to

$$I = \frac{1}{1+ik}\int_{0}^{\infty}\cos xe^{-x(1+ik)}dx$$

Integration by parts. $\int u\,dv = uv - \int v\,du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \cos x, du = -\sin x$.
The above becomes

$$\begin{aligned}
I &= \frac{1}{1+ik}\left(\left[\cos x\frac{e^{-x(1+ik)}}{-(1+ik)}\right]_{0}^{\infty} - \int_{0}^{\infty}(-\sin x)\frac{e^{-x(1+ik)}}{-(1+ik)}dx\right) \\
&= \frac{1}{1+ik}\left(\left[\cos x\frac{e^{-x(1+ik)}}{-(1+ik)}\right]_{0}^{\infty} - \frac{1}{1+ik}\int_{0}^{\infty}\sin xe^{-x(1+ik)}dx\right)
\end{aligned}$$

But $\int_{0}^{\infty}\sin xe^{-x(1+ik)}dx = I$. The above becomes

$$\begin{aligned}
I &= \frac{1}{1+ik}\left(\left[\cos x\frac{e^{-x(1+ik)}}{-(1+ik)}\right]_{0}^{\infty} - \frac{1}{1+ik}I\right) \\
&= \frac{1}{1+ik}\left[\cos x\frac{e^{-x(1+ik)}}{-(1+ik)}\right]_{0}^{\infty} - \left(\frac{1}{1+ik}\right)^{2}I \\
I + \left(\frac{1}{1+ik}\right)^{2}I &= \frac{-1}{(1+ik)^{2}}\left[\cos xe^{-x(1+ik)}\right]_{0}^{\infty}
\end{aligned}$$

Now $\left[\cos x e^{-x(1+ik)}\right]_0^\infty = 0 - 1 = -1$. Hence the above reduces to

$$I\left(1 + \left(\frac{1}{1+ik}\right)^2\right) = \frac{1}{(1+ik)^2}$$

$$I = \frac{\frac{1}{(1+ik)^2}}{1 + \left(\frac{1}{1+ik}\right)^2}$$

$$= \frac{1}{1 + (1+ik)^2}$$

$$= \frac{1}{2 - k^2 + 2ik}$$

Therefore

$$\int_0^\infty \sin x e^{-x(1+ik)} dx = \frac{1}{2 - k^2 + 2ik}$$

Using (1) the Fourier transform becomes

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{2 - k^2 + 2ik}$$

This can be written as real and imaginary parts

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{\left(2 - k^2\right) - 2ik}{\left(\left(2 - k^2\right) + 2ik\right)\left(\left(2 - k^2\right) - 2ik\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\left(2 - k^2\right) - 2ik}{\left(2 - k^2\right)^2 + 4k^2}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2 - k^2}{k^4 + 4} - i\frac{2k}{k^4 + 4}\right)$$

3   Problem 7.1.3 (a,b)

Find the inverse Fourier transform of the function $\frac{1}{k+c}$ when (a) $c = a$ is real (b) $c = ib$ is pure imaginary.

<u>Solution</u>

3.1   Part a

Using shifting property where $\mathscr{F}\left[f(x)\right] = \hat{f}(k)$ and let $\hat{f}(k) = \frac{1}{k}$ then by shifting property $\mathscr{F}\left[e^{iax}f(x)\right] = \hat{f}(k - a)$, (Theorem 7.4) therefore

$$\mathscr{F}\left[e^{-iax}f(x)\right] = \hat{f}(k + a)$$

$$= \frac{1}{k + a} \tag{1}$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\text{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{-iax}\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)\right] = \frac{1}{k + a}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k + a}\right] = ie^{-iax}\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)$$

3.2   Part b

Using shifting property, given that $\mathscr{F}\left(f(x)\right) = \hat{f}(k)$, let $\hat{f}(k) = \frac{1}{k}$ then by shifting property (Theorem 7.4) $\mathscr{F}\left[e^{i(ib)x}f(x)\right] = \hat{f}(k - ib)$, then

$$\mathscr{F}\left[e^{bx}f(x)\right] = \hat{f}(k + ib)$$

$$= \frac{1}{k + ib} \tag{1}$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\text{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{bx}\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)\right] = \frac{1}{k + ib}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k + ib}\right] = ie^{bx}\sqrt{\frac{\pi}{2}}\,\text{sgn}(x)$$

4 Problem 7.1.13

Prove the Shift Theorem 7.4 which is

<u>Theorem 7.4</u>: if $f(x)$ has Fourier transform $\hat{f}(k)$, then the Fourier transform of the shifted function $f(x - \xi)$ is $e^{-ik\xi}\hat{f}(k)$. Similarly the transform of the product function $e^{i\alpha x}f(x)$ for real α is the shifted transform $\hat{f}(k - \alpha)$ (note: using α in place of the strange second k that the book uses)

4.1 Part a

Showing if $f(x)$ has Fourier transform $\hat{f}(k)$, then Fourier transform of the shifted function $f(x - \xi)$ is $e^{-ik\xi}\hat{f}(k)$. From definition, the Fourier transform of $f(x - \xi)$ is given by

$$\mathscr{F}\big[f(x - \xi)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x - \xi)e^{-ikx}dx$$

Let $x - \xi = u$. Then $\frac{du}{dx} = 1$. The above becomes (limits do not change)

$$\mathscr{F}\big[f(x - \xi)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-ik(u+\xi)}du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iku}e^{-ik\xi}du$$

$$= e^{-ik\xi}\overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iku}du}^{\hat{f}(k)}$$

Therefore

$$\mathscr{F}\big[f(x - \xi)\big] = e^{-ik\xi}\hat{f}(k)$$

Which is what asked to show.

4.2 Part b

Showing that the Fourier transform of $e^{i\alpha x}f(x)$ is $\hat{f}(k - \alpha)$. From definition, the Fourier transform of $e^{i\alpha x}f(x)$ is

$$\mathscr{F}\big[e^{i\alpha x}f(x)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}f(x)e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix(k-\alpha)}dx$$

But $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix(k-\alpha)}dx$ is $\hat{f}(k - \alpha)$ by replacing k with $k - \alpha$ in the definition of Fourier transform. Hence

$$\mathscr{F}\big[e^{i\alpha x}f(x)\big] = \hat{f}(k - \alpha)$$

Which is what asked to show.

5 Problem 7.1.20 (a)

The two-dimensional Fourier transform of a function $f(x,y)$ defined for $(x,y) \in \mathbb{R}^2$ is

$$\mathscr{F}[f(x,y)] = \hat{f}(k,l)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i(kx+ly)} dx dy$$

(a) compute the Fourier transform of the following functions (i) $e^{-|x|-|y|}$, (iii) The delta function $\delta(x-\xi)\delta(y-\eta)$

(b) Show that if $f(x,y) = g(x)h(y)$ then $\hat{f}(k,l) = \hat{g}(k)\hat{h}(l)$

Solution

5.1 Part a

(i) The Fourier transform of $e^{-|x|-|y|}$ is

$$\hat{f}(k,l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|x|-|y|} e^{-i(kx+ly)} dx dy$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|x|} e^{-|y|} e^{-ikx} e^{-ily} dx dy$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|y|} e^{-ily} \left(\int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \right) dy \qquad (1)$$

But $\int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx$ is the Fourier transform of $f(x) = e^{-|x|}$ with $\sqrt{2\pi}$ factor. In other words

$$\int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx = \sqrt{2\pi} \hat{g}(k)$$

Where $\hat{g}(k)$ is used to indicate the Fourier transform of $e^{-|x|}$. Hence (1) becomes

$$\hat{f}(k,l) = \frac{\sqrt{2\pi}}{2\pi} \hat{f}_1(k) \int_{-\infty}^{\infty} e^{-|y|} e^{-ily} dy$$

But $\int_{-\infty}^{\infty} e^{-|y|} e^{-ily} dy = \sqrt{2\pi} \hat{h}(l)$ Where $\hat{h}(l)$ is used to indicate the Fourier transform of $e^{-|y|}$. The above becomes

$$\hat{f}(k,l) = \frac{\sqrt{2\pi}}{2\pi} \hat{g}(k) \sqrt{2\pi} \hat{h}(l)$$
$$= \hat{g}(k) \hat{h}(l) \qquad (2)$$

So now we need to determine $\hat{g}(k)$ and $\hat{h}(l)$ and multiply the result.

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x} e^{-ikx} dx + \int_{0}^{\infty} e^{-x} e^{-ikx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+x} dx + \int_{0}^{\infty} e^{-ikx-x} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-ikx+x}}{1-ik} \right]_{-\infty}^{0} + \left[\frac{e^{-ikx-x}}{-1-ik} \right]_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} \left[e^{-ikx} e^{x} \right]_{-\infty}^{0} - \frac{1}{1+ik} \left[e^{-ikx} e^{-x} \right]_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik}(1-0) - \frac{1}{1+ik}(0-1) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{(1+ik)+(1-ik)}{(1-ik)(1+ik)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+k^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$$

Similarly

$$\hat{h}(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|y|} e^{-ily} dy$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2}$$

Hence from (2) the Fourier transform of $e^{-|x|-|y|}$ is

$$\hat{f}(k,l) = \hat{g}(k)\hat{h}(l)$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2}$$

$$= \frac{2}{\pi} \frac{1}{\left(1+k^2\right)\left(1+l^2\right)}$$

(ii) The Fourier transform of $\delta(x-\xi)\delta(y-\eta)$. First we find the Fourier transform of $\delta(x-\xi)$ and then the Fourier transform of $\delta(y-\eta)$

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-\xi) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik\xi}$$

And

$$\hat{h}(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(y-\eta) e^{-ily} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-il\eta}$$

Hence the Fourier transform of the product $\delta(x-\xi)\delta(y-\eta)$ is (Using the product rule, which will be proofed in part b also).

$$\hat{f}(k,l) = \hat{g}(k)\hat{h}(l)$$

$$= \frac{1}{2\pi} e^{-ik\xi} e^{-il\eta}$$

The above could be rewritten in terms of trig functions using Euler relation if needed.

5.2 Part b

By definition, the Fourier transform of $f(x, y)$ is

$$\hat{f}(k, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(kx+ly)} dx dy$$

But $f(x, y) = g(x) h(y)$. Hence the above becomes

$$\hat{f}(k, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) e^{-i(kx+ly)} dx dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) e^{-ikx} e^{-ily} dx dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(y) e^{-ily} \left(\int_{-\infty}^{\infty} g(x) e^{-ikx} dx \right) dy$$

But $\int_{-\infty}^{\infty} g(x) e^{-ikx} dx = \sqrt{2\pi} \hat{g}(k)$. The above reduces to

$$\hat{f}(k, l) = \frac{1}{2\pi} \sqrt{2\pi} \hat{g}(k) \int_{-\infty}^{\infty} h(y) e^{-ily} dy$$

But $\int_{-\infty}^{\infty} h(y) e^{-ily} dy = \sqrt{2\pi} \hat{h}(l)$. Hence the above becomes

$$\hat{f}(k, l) = \frac{1}{2\pi} \sqrt{2\pi} \hat{g}(k) \sqrt{2\pi} \hat{h}(l)$$

$$= \hat{g}(k) \hat{h}(l)$$

Which is what asked to show.

6   Problem 7.2.2 (a)

Find the Fourier transform of (a) the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$

<u>Solution</u>

6.1   Part a

Using

$$1 + \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-z^2} dz \tag{1}$$

Taking Fourier transform of both sides, and using the known relation from tables which says

$$\mathscr{F}\left[\int_{-\infty}^x f(u)\,du\right] = \frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)$$

And using that Fourier transform of 1 is $\sqrt{2\pi}\delta(k)$ then (1) becomes

$$\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)\right)$$

Where $\hat{f}(k)$ is the Fourier transform of e^{-u^2} (Gaussian) we derived in class as $e^{-u^2} \Leftrightarrow \frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}$. The above becomes

$$\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \pi\left[\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}\right]_{k=0}\delta(k)\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \frac{\pi}{\sqrt{2}}\delta(k)\right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \sqrt{2\pi}\delta(k)$$

Therefore the above simplifies to

$$\mathscr{F}[\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{ik}e^{\frac{-k^2}{4}}$$

$$= -i\sqrt{\frac{2}{\pi}}\frac{1}{k}e^{\frac{-k^2}{4}}$$

7   Problem 7.2.3 (d)

Find the inverse Fourier transform of the following functions (d) $\frac{k^2}{k-i}$

Solution

Using property that

$$\mathscr{F}\left[f'(x)\right] = ik\hat{f}(k)$$
$$\mathscr{F}\left[f''(x)\right] = -k^2\hat{f}(k) \tag{1}$$

Where in the above $\mathscr{F}\left[f(x)\right] = \hat{f}(k)$. Comparing the above with $\frac{k^2}{k-i}$, we see that

$$\hat{f}(k) = \frac{1}{k-i}$$

Hence we need to find inverse Fourier transform of $\frac{-1}{k-i}$ first in order to find $f(x)$, and then take second derivative of the result. Writing

$$\frac{1}{k-i} = \frac{1}{i\left(\frac{k}{i}-1\right)}$$

$$= \frac{1}{i(-ik-1)}$$

$$= \frac{-1}{i(ik+1)}$$

$$= i\frac{1}{(1+ik)}$$

From table (page 272 in textbook) we see that

$$\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right] = \sqrt{2\pi}e^{-x}\sigma(x)$$

Using $a = 1$ in the table entry. Where $\sigma(x)$ is the step function. Hence

$$i\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right] = i\sqrt{2\pi}e^{-x}\sigma(x)$$

Therefore

$$f(x) = i\sqrt{2\pi}e^{-x}\sigma(x)$$

Now we take derivative of the above (using product rule)

$$f'(x) = -i\sqrt{2\pi}e^{-x}\sigma(x) + i\sqrt{2\pi}e^{-x}\delta(x)$$

Where $\delta(x)$ is added since derivative of $\sigma(x)$ has jump discontinuity at $x = 0$. Taking one more derivative gives

$$f''(x) = i\sqrt{2\pi}e^{-x}\sigma(x) - i\sqrt{2\pi}e^{-x}\delta(x) - i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)$$

$$= i\sqrt{2\pi}e^{-x}\sigma(x) - 2i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)$$

Therefore

$$\mathscr{F}^{-1}\left[\frac{k^2}{k-i}\right] = i\sqrt{2\pi}e^{-x}\sigma(x) - 2i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)$$

8 Problem 7.2.12

(a) Explain why the Fourier transform of a 2π periodic function $f(x)$ is a linear combinations of delta functions $\hat{f}(k) = \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$ where c_n are the complex Fourier series coefficients (3.65) of $f(x)$ on $[-\pi, \pi]$

$$c_n = \left\langle f, e^{inx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{3.65}$$

(b) Find the Fourier transform of the following periodic functions (i) $\sin 2x$ (ii) $\cos^3 x$ (iii) The 2π periodic extension of $f(x) = x$ (iv) The sawtooth function $h(x) = x \bmod 1$. i.e. the fractional part of x

Solution

8.1 Part a

Since $f(x)$ is periodic, then its can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\left(\frac{2\pi}{T}\right)x}$$

But the period $T = 2\pi$ and the above simplifies to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

Taking the Fourier transform of the above gives

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{2}$$

Substituting (1) into (2) gives

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{-ix(k-n)} \right) dx$$

Changing the order of summation and integration

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} c_n e^{-ix(k-n)} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \left(\int_{-\infty}^{\infty} e^{-ix(k-n)} dx \right) \tag{3}$$

But from tables we know that $\mathscr{F}(1) = \sqrt{2\pi} \delta(k)$. Which means that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} dx = \sqrt{2\pi} \delta(k)$$

Therefore, replacing k by $k-n$ in the above gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix(k-n)} dx = \sqrt{2\pi} \delta(k-n)$$

$$\int_{-\infty}^{\infty} e^{-ix(k-n)} dx = (2\pi) \delta(k-n) \tag{4}$$

Substituting (4) into (3) gives

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n (2\pi) \delta(k-n)$$

$$= \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

Note: The books seems to have a typo. It gives the above without the factor $\sqrt{2\pi}$ at the front.

8.2   Part b

(i) $\underline{\sin 2x}$. Since this is periodic, then $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(2x) e^{-inx} dx$. For $n = 2$ this gives $c_2 = -\frac{i}{2}$ and for $n = -2$ it gives $c_{-2} = \frac{i}{2}$ and it is zero for all other n values due to orthogonality of $\sin$ functions. Using the above result obtained in part (a)

$$\hat{f}(k) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

$$= \sqrt{2\pi} c_{-2} \delta(k+2) + \sqrt{2\pi} c_2 \delta(k-2)$$

$$= \sqrt{2\pi} \frac{i}{2} \delta(k+2) - \sqrt{2\pi} \frac{i}{2} \delta(k-2)$$

$$= i\sqrt{\frac{\pi}{2}} \delta(k+2) - i\sqrt{\frac{\pi}{2}} \delta(k-2)$$

(ii) $\underline{\cos^3 x}$. Since this is periodic, then $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^3(x) e^{-inx} dx$. But $\cos^3(x) = \frac{1}{4} \cos(3x) + \frac{3}{4} \cos(x)$. Hence only $n = \pm 1, n = \pm 3$ will have coefficients and the rest are zero.

$$c_{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{3}{4} \cos(x) e^{ix} dx = \frac{3}{8}$$

$$c_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{3}{4} \cos(x) e^{-ix} dx = \frac{3}{8}$$

$$c_{-3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{4} \cos(3x) e^{-3ix} dx = \frac{1}{8}$$

$$c_3 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{4} \cos(3x) e^{-3ix} dx = \frac{1}{8}$$

Therefore, using result from part (a)

$$\hat{f}(k) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

$$= \sqrt{2\pi} \left(\frac{1}{8} \delta(k+3) + \frac{3}{8} \delta(k+1) + \frac{3}{8} \delta(k-1) + \frac{1}{8} \delta(k-3) \right)$$

$$= \frac{1}{4} \sqrt{\frac{\pi}{2}} \left(\delta(k+3) + 3\delta(k+1) + 3\delta(k-1) + \delta(k-3) \right)$$

(iii) The 2π periodic extension of $f(x) = x$

Since this is periodic, then

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx$$

$$= \frac{2i}{n^2} (n\pi \cos(n\pi) - \sin(n\pi))$$

$$= \frac{2i}{n^2} \left(n\pi (-1)^n \right)$$

$$= \frac{2i}{n} \pi (-1)^n$$

Therefore, using result from part (a)

$$\hat{f}(k) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

$$= \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \frac{2i}{n} \pi (-1)^n \delta(k-n)$$

$$= 2i\pi \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} \delta(k-n) \qquad n \neq 0$$
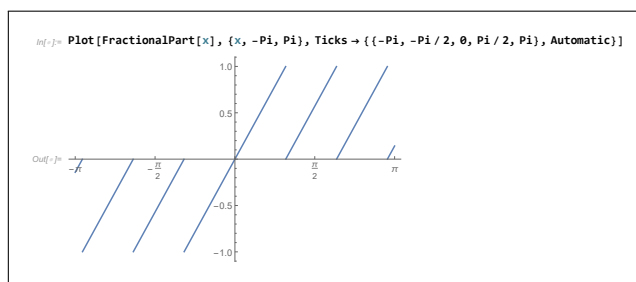
(iv) The sawtooth function

Figure 1: Plot of $f(x)$ (Fractional part of x)

9   Problem 7.3.4

Find a solution to the differential equation $-\frac{d^2u}{dx^2} + 4u = \delta(x)$ by using the Fourier transform

Solution

Taking Fourier transform of both sides gives

$$-(ik)^2 \hat{u}(k) + 4\hat{u}(k) = \mathscr{F}[\delta(x)]$$

$$k^2 \hat{u}(k) + 4\hat{u}(k) = \frac{1}{\sqrt{2\pi}}$$

Solving for $\hat{u}(k)$

$$\hat{u}(k)\left(k^2 + 4\right) = \frac{1}{\sqrt{2\pi}}$$

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + 4}$$

Finding inverse Fourier transform. From tables we see that $\mathscr{F}\left(e^{-a|x|}\right) = \sqrt{\frac{2}{\pi}} \frac{a}{k^2 + a^2}$. Using $a = 2$

$$\mathscr{F}\left[e^{-2|x|}\right] = \sqrt{\frac{2}{\pi}} \frac{2}{k^2 + 4}$$

$$\sqrt{\frac{\pi}{2}} \frac{1}{2} \mathscr{F}\left[e^{-2|x|}\right] = \frac{1}{k^2 + 4}$$

$$\sqrt{\frac{\pi}{2}} \mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{k^2 + 4}$$

$$\frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + 4}$$

$$\frac{1}{2} \mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + 4}$$

$$\mathscr{F}\left[\frac{1}{4}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + 4}$$

Therefore

$$u(x) = \frac{1}{4}e^{-2|x|}$$