HW 10

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 1

Show that (assuming sufficient smoothness of the domain and the data) u is a solution to the Dirichlet boundary value problem

$$-\Delta u = f$$

In Ω with B.C. $u = g$ on $\partial\Omega$ iff u is a minimizer of the energy functional, that is

$$E(u) = \min\left\{E(v) : v \in C^2\left(\bar{\Omega}\right)\right\} \text{ such that } u = g \text{ on } \partial\Omega$$

Here

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dA$$

(note, I will be using dA in the above integral assuming we are in $\mathbb{R}^2$. But the above can also be dV for $\mathbb{R}^3$ just as well and nothing will change in the derivation below. This is easier that writing dx and saying that x is a vector).

Solution

Since the proof is an iff, then we need to show both direction.

Forward direction Given that u solves

$$-\Delta u = f \tag{1}$$

with $u|_{\partial\Omega} = g$. Then we need to show that $E(v) \geq E(u)$ for all $v \in C^2\left(\bar{\Omega}\right)$ that also satisfy same B.C.

Multiplying both sides of (1) by $u - v$ and integrating over the domain gives

$$-\int_\Omega (\Delta u)(u - v)\, dA = \int_\Omega (u - v) f dA \tag{2}$$

For the left integral $\int_\Omega (\Delta u)(u - v)\, dA$, we will do integration by parts. Let $\Delta u \equiv dV, u - v = U$, then $\int_\Omega U dV = \int_{\partial\Omega} UV - \int_\Omega V dU$. Therefore $dU = \nabla(u - v)$ and $V = \nabla u$. After applying integration by parts the (2) now becomes

$$-\left(\int_{\partial\Omega} (u - v)\frac{\partial u}{\partial \boldsymbol{n}}\, dL - \int_\Omega \nabla u \cdot \nabla(u - v)\, dA\right) = \int_\Omega (u - v) f dA$$

But $\int_{\partial\Omega} (u - v)\frac{\partial u}{\partial n}\, dL = 0$ because $u = v$ on the boundary $\partial\Omega$ as both are g. The above now simplifies to

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla(u - v)\;\, dA &= \int_\Omega \left(uf - vf\right)\; dA \\
\int_\Omega \nabla u \cdot (\nabla u - \nabla v)\;\, dA &= \int_\Omega \left(uf - vf\right)\; dA \\
\int_\Omega |\nabla u|^2 - \nabla u \cdot \nabla v\, dA &= \int_\Omega \left(uf - vf\right)\; dA \\
\int_\Omega |\nabla u|^2 - \int_\Omega fu\, dA &= \int_\Omega (\nabla u \cdot \nabla v) - vf\, dA
\end{aligned}$$

Now we use Schwarz triangle inequality and write $\nabla u\cdot\nabla v\leq\frac{1}{2}\left(|\nabla u|^2+|\nabla v|^2\right)$. This comes from using $ab\leq\frac{1}{2}\left(a^2+b^2\right)$. Using this in the RHS of the above gives

$$
\begin{aligned}
\int_\Omega |\nabla u|^2\ dA-\int_\Omega fu\ dA &\leq \int_\Omega \frac{1}{2}\left(|\nabla u|^2+|\nabla v|^2\right)-fv\ dA\\
\int_\Omega |\nabla u|^2\ dA-\int_\Omega fu\ dA &\leq \int_\Omega \frac{1}{2}|\nabla u|^2 dA+\left(\frac{1}{2}\int_\Omega |\nabla v|^2-fv\ dA\right)\\
\int_\Omega \frac{1}{2}|\nabla u|^2 dA-\int_\Omega fu\ dA &\leq \frac{1}{2}\int_\Omega |\nabla v|^2-fv\ dA\\
\int_\Omega \frac{1}{2}|\nabla u|^2-fu\ dA &\leq \frac{1}{2}\int_\Omega |\nabla v|^2-fv\ dA
\end{aligned}
$$

But by definition $\int_\Omega\frac{1}{2}|\nabla u|^2-fu\ dA=E(u)$ and $\frac{1}{2}\int_\Omega|\nabla v|^2-fv\ dA=E(v)$, therefore the above becomes

$$E(u)\leq E(v)$$

Which is what we wanted to show. Now we will do the other direction.

<u>Reverse direction</u> Given that u minimizes energy among all test functions, i.e. given that $E(u)=\min E(w)$, then need to show that $-\Delta u=f$.

Consider $w=u+\varepsilon v$ where v is any test function $v\in C^2\left(\bar{\Omega}\right)$ and $v=g$ at $\partial\Omega$. Hence

$$\min\left(E(w)\right)=\min\left(E(u+\varepsilon v)\right)$$

Therefore $\min\left(E(u+\varepsilon v)\right)$ is achieved when $\varepsilon=0$, since this then gives $E(u)$ which by assumption is the minimum. Therefore

$$\frac{d}{d\varepsilon}E(u+\varepsilon v)=0$$

At $\varepsilon=0$. But the above can be written as the following, using the definition of energy

$$
\begin{aligned}
\frac{d}{d\varepsilon}\left(\int_\Omega\frac{1}{2}|\nabla(u+\varepsilon v)|^2-f(u+\varepsilon v)\ dA\right)&=0\\
\frac{d}{d\varepsilon}\left(\int_\Omega\frac{1}{2}\left(\nabla(u+\varepsilon v)\cdot\nabla(u+\varepsilon v)\right)-f(u+\varepsilon v)\ dA\right)&=0 \qquad (3)
\end{aligned}
$$

Expanding $\nabla(u+\varepsilon v)\cdot\nabla(u+\varepsilon v)$ gives

$$
\begin{aligned}
\nabla(u+\varepsilon v)\cdot\nabla(u+\varepsilon v)&=(\nabla u+\varepsilon\nabla v)\cdot(\nabla u+\varepsilon\nabla v)\\
&=|\nabla u|^2+2\varepsilon\nabla u\cdot\nabla v+\varepsilon^2|\nabla v|^2 \qquad (4)
\end{aligned}
$$

Substituting (4) into (3) gives

$$\frac{d}{d\varepsilon}\left(\int_\Omega v\left(|\nabla u|^2+2\varepsilon\nabla u\cdot\nabla v+\varepsilon^2|\nabla v|^2\right)-fu-\varepsilon fv\ dA\right)=0$$

Now we move the derivative inside the take derivative w.r.t. ε giving

$$\left(\int_\Omega\frac{1}{2}\left(2\nabla u\cdot\nabla v+2\varepsilon|\nabla v|^2\right)-fv\ dA\right)=0$$

Evaluate at $\varepsilon = 0$ the above becomes

$$\int_{\Omega} (\nabla u \cdot \nabla v)\, dA - \int_{\Omega} f v\, dA = 0$$

Integration by parts for the first integral. Let $\nabla u = U, dV = \nabla v$, then $\int_{\Omega} U dV = \int_{\partial\Omega} UV - \int_{\Omega} V dU$. Hence the above becomes

$$\left(\int_{\partial\Omega} v \frac{\partial u}{\partial \boldsymbol{n}}\, dL - \int_{\Omega} v \Delta u\, dA \right) - \int_{\Omega} f v\, dA = 0$$

But $v = 0$ at boundary $\partial\Omega$. The above simplifies to

$$-\int_{\Omega} v \Delta u\, dA - \int_{\Omega} f v\, dA = 0$$

$$\int_{\Omega} v \left(-\Delta u - f \right) dA = 0$$

Since the above is true for all v test function then this implies that $-\Delta u - f = 0$ or

$$-\Delta u = f$$

Which is what we wanted to show.

2 Problem 7.1.1 f

Find the Fourier transform of (f) $f(x) = \begin{cases} e^{-x}\sin x & x > 0 \\ 0 & x \leq 0 \end{cases}$

Solution

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} \sin x e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sin x e^{-ikx-x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sin x e^{-x(1+ik)} dx
\end{aligned} \tag{1}$$

Integration by parts. $\int u dv = uv - \int v du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \sin x, du = \cos x$. Hence

$$\begin{aligned}
I &= \int_{0}^{\infty} \sin x e^{-x(1+ik)} dx \\
&= \left[\sin x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^{\infty} - \int_0^{\infty} \cos x \frac{e^{-x(1+ik)}}{-(1+ik)} dx \\
&= \frac{-1}{1+ik} \left[\sin x e^{-x(1+ik)} \right]_0^{\infty} + \frac{1}{1+ik} \int_0^{\infty} \cos x e^{-x(1+ik)} dx
\end{aligned}$$

But $e^{-x(1+ik)} = e^{-x}e^{-ikx}$ and this goes to zero as $x \to \infty$ and since $\sin x = 0$ at $x = 0$ then the first term above is zero. The above reduces to

$$I = \frac{1}{1+ik} \int_0^{\infty} \cos x e^{-x(1+ik)} dx$$

Integration by parts. $\int u dv = uv - \int v du$. Let $dv = e^{-x(1+ik)}, v = \frac{e^{-x(1+ik)}}{-(1+ik)}, u = \cos x, du = -\sin x$. The above becomes

$$\begin{aligned}
I &= \frac{1}{1+ik} \left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^{\infty} - \int_0^{\infty} (-\sin x) \frac{e^{-x(1+ik)}}{-(1+ik)} dx \right) \\
&= \frac{1}{1+ik} \left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)} \right]_0^{\infty} - \frac{1}{1+ik} \int_0^{\infty} \sin x e^{-x(1+ik)} dx \right)
\end{aligned}$$

But $\int_0^\infty \sin x e^{-x(1+ik)} dx = I$. The above becomes

$$I = \frac{1}{1+ik}\left(\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)}\right]_0^\infty - \frac{1}{1+ik} I\right)$$
$$= \frac{1}{1+ik}\left[\cos x \frac{e^{-x(1+ik)}}{-(1+ik)}\right]_0^\infty - \left(\frac{1}{1+ik}\right)^2 I$$
$$I + \left(\frac{1}{1+ik}\right)^2 I = \frac{-1}{(1+ik)^2}\left[\cos x e^{-x(1+ik)}\right]_0^\infty$$

Now $\left[\cos x e^{-x(1+ik)}\right]_0^\infty = 0 - 1 = -1$. Hence the above reduces to

$$I\left(1 + \left(\frac{1}{1+ik}\right)^2\right) = \frac{1}{(1+ik)^2}$$
$$I = \frac{\frac{1}{(1+ik)^2}}{1 + \left(\frac{1}{1+ik}\right)^2}$$
$$= \frac{1}{1 + (1+ik)^2}$$
$$= \frac{1}{2 - k^2 + 2ik}$$

Therefore

$$\int_0^\infty \sin x e^{-x(1+ik)} dx = \frac{1}{2 - k^2 + 2ik}$$

Using (1) the Fourier transform becomes

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{2 - k^2 + 2ik}$$

This can be written as real and imaginary parts

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{\left(2 - k^2\right) - 2ik}{\left(\left(2 - k^2\right) + 2ik\right)\left(\left(2 - k^2\right) - 2ik\right)}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{\left(2 - k^2\right) - 2ik}{\left(2 - k^2\right)^2 + 4k^2}$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2 - k^2}{k^4 + 4} - i \frac{2k}{k^4 + 4}\right)$$

3 Problem 7.1.3 (a,b)

Find the inverse Fourier transform of the function $\frac{1}{k+c}$ when (a) $c = a$ is real (b) $c = ib$ is pure imaginary.

Solution

3.1 Part a

Using shifting property where $\mathscr{F}\left[f(x)\right] = \hat{f}(k)$ and let $\hat{f}(k) = \frac{1}{k}$ then by shifting property $\mathscr{F}\left[e^{iax} f(x)\right] = \hat{f}(k-a)$, (Theorem 7.4) therefore

$$\begin{aligned}\mathscr{F}\left[e^{-iax} f(x)\right] &= \hat{f}(k+a)\\ &= \frac{1}{k+a}\end{aligned} \tag{1}$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\operatorname{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{-iax}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k+a}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k+a}\right] = ie^{-iax}\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)$$

3.2 Part b

Using shifting property, given that $\mathscr{F}\left(f(x)\right) = \hat{f}(k)$, let $\hat{f}(k) = \frac{1}{k}$ then by shifting property (Theorem 7.4) $\mathscr{F}\left[e^{i(ib)x} f(x)\right] = \hat{f}(k-ib)$, then

$$\begin{aligned}\mathscr{F}\left[e^{bx} f(x)\right] &= \hat{f}(k+ib)\\ &= \frac{1}{k+ib}\end{aligned} \tag{1}$$

We now just need to find $f(x)$. From table of Fourier transforms on page 272, we see that $\mathscr{F}[\operatorname{sgn}(x)] = \frac{1}{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}$. Hence

$$\mathscr{F}\left[i\sqrt{\frac{\pi}{2}}\operatorname{sgn}(x)\right] = \frac{1}{k}$$

Therefore $f(x) = i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(x)$. Substituting this back into (1) gives

$$\mathscr{F}\left[ie^{bx}\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(x)\right] = \frac{1}{k+ib}$$

Or

$$\mathscr{F}^{-1}\left[\frac{1}{k+ib}\right] = ie^{bx}\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(x)$$

4 Problem 7.1.13

Prove the Shift Theorem 7.4 which is

Theorem 7.4: if $f(x)$ has Fourier transform $\hat{f}(k)$, then the Fourier transform of the shifted function $f(x-\xi)$ is $e^{-ik\xi}\hat{f}(k)$. Similarly the transform of the product function $e^{i\alpha x}f(x)$ for real α is the shifted transform $\hat{f}(k-\alpha)$ (note: using α in place of the strange second k that the book uses)

4.1 Part a

Showing if $f(x)$ has Fourier transform $\hat{f}(k)$, then Fourier transform of the shifted function $f(x-\xi)$ is $e^{-ik\xi}\hat{f}(k)$. From definition, the Fourier transform of $f(x-\xi)$ is given by

$$\mathscr{F}\left[f(x-\xi)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-\xi)\,e^{-ikx}dx$$

Let $x-\xi=u$. Then $\frac{du}{dx}=1$. The above becomes (limits do not change)

$$\begin{aligned}\mathscr{F}\left[f(x-\xi)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-ik(u+\xi)}du\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}e^{-ik\xi}du\\&=e^{-ik\xi}\overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{-iku}du}^{\hat{f}(k)}\end{aligned}$$

Therefore

$$\mathscr{F}\left[f(x-\xi)\right]=e^{-ik\xi}\hat{f}(k)$$

Which is what asked to show.

4.2 Part b

Showing that the Fourier transform of $e^{i\alpha x}f(x)$ is $\hat{f}(k-\alpha)$. From definition, the Fourier transform of $e^{i\alpha x}f(x)$ is

$$\begin{aligned}\mathscr{F}\left[e^{i\alpha x}f(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}f(x)\,e^{-ikx}dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-ix(k-\alpha)}dx\end{aligned}$$

But $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-ix(k-\alpha)}dx$ is $\hat{f}(k-\alpha)$ by replacing k with $k-\alpha$ in the definition of Fourier transform. Hence

$$\mathscr{F}\left[e^{i\alpha x}f(x)\right]=\hat{f}(k-\alpha)$$

Which is what asked to show.

5 Problem 7.1.20 (a)

The two-dimensional Fourier transform of a function $f\left(x,y\right)$ defined for $\left(x,y\right)\in\mathbb{R}^2$ is

$$\begin{aligned}\mathscr{F}\left[f\left(x,y\right)\right] &= \hat{f}\left(k,l\right)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(x,y\right)e^{-i\left(kx+ly\right)}dxdy\end{aligned}$$

(a) compute the Fourier transform of the following functions (i) $e^{-|x|-|y|}$, (iii) The delta function $\delta\left(x-\xi\right)\delta\left(y-\eta\right)$

(b) Show that if $f\left(x,y\right)=g\left(x\right)h\left(y\right)$ then $\hat{f}\left(k,l\right)=\hat{g}\left(k\right)\hat{h}\left(l\right)$

Solution

5.1 Part a

(i) The Fourier transform of $e^{-|x|-|y|}$ is

$$\begin{aligned}\hat{f}\left(k,l\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-|x|-|y|}e^{-i\left(kx+ly\right)}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-|x|}e^{-|y|}e^{-ikx}e^{-ily}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|y|}e^{-ily}\left(\int_{-\infty}^{\infty}e^{-|x|}e^{-ikx}dx\right)dy \qquad (1)\end{aligned}$$

But $\int_{-\infty}^{\infty}e^{-|x|}e^{-ikx}dx$ is the Fourier transform of $f\left(x\right)=e^{-|x|}$ with $\sqrt{2\pi}$ factor. In other words

$$\int_{-\infty}^{\infty}e^{-|x|}e^{-ikx}dx=\sqrt{2\pi}\hat{g}\left(k\right)$$

Where $\hat{g}\left(k\right)$ is used to indicate the Fourier transform of $e^{-|x|}$. Hence (1) becomes

$$\hat{f}\left(k,l\right)=\frac{\sqrt{2\pi}}{2\pi}\hat{f}_1\left(k\right)\int_{-\infty}^{\infty}e^{-|y|}e^{-ily}dy$$

But $\int_{-\infty}^{\infty}e^{-|y|}e^{-ily}dy=\sqrt{2\pi}\hat{h}\left(l\right)$ Where $\hat{h}\left(l\right)$ is used to indicate the Fourier transform of $e^{-|y|}$. The above becomes

$$\begin{aligned}\hat{f}\left(k,l\right) &= \frac{\sqrt{2\pi}}{2\pi}\hat{g}\left(k\right)\sqrt{2\pi}\hat{h}\left(l\right)\\ &= \hat{g}\left(k\right)\hat{h}\left(l\right) \qquad (2)\end{aligned}$$

So now we need to determine $\hat{g}(k)$ and $\hat{h}(l)$ and multiply the result.

$$\begin{aligned}
\hat{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x} e^{-ikx} dx + \int_{0}^{\infty} e^{-x} e^{-ikx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+x} dx + \int_{0}^{\infty} e^{-ikx-x} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-ikx+x}}{1-ik} \right]_{-\infty}^{0} + \left[\frac{e^{-ikx-x}}{-1-ik} \right]_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} \left[e^{-ikx} e^{x} \right]_{-\infty}^{0} - \frac{1}{1+ik} \left[e^{-ikx} e^{-x} \right]_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} (1-0) - \frac{1}{1+ik} (0-1) \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{(1+ik)+(1-ik)}{(1-ik)(1+ik)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+k^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}
\end{aligned}$$

Similarly

$$\begin{aligned}
\hat{h}(l) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|y|} e^{-ily} dy \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2}
\end{aligned}$$

Hence from (2) the Fourier transform of $e^{-|x|-|y|}$ is

$$\begin{aligned}
\hat{f}(k,l) &= \hat{g}(k)\,\hat{h}(l) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} \sqrt{\frac{2}{\pi}} \frac{1}{1+l^2} \\
&= \frac{2}{\pi} \frac{1}{\left(1+k^2\right)\left(1+l^2\right)}
\end{aligned}$$

(ii) The Fourier transform of $\delta(x-\xi)\delta(y-\eta)$. First we find the Fourier transform of $\delta(x-\xi)$ and then the Fourier transform of $\delta(y-\eta)$

$$\begin{aligned}\hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-\xi)e^{-ikx}dx\\ &= \frac{1}{\sqrt{2\pi}}e^{-ik\xi}\end{aligned}$$

And

$$\begin{aligned}\hat{h}(l) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(y-\eta)e^{-ily}dy\\ &= \frac{1}{\sqrt{2\pi}}e^{-il\eta}\end{aligned}$$

Hence the Fourier transform of the product $\delta(x-\xi)\delta(y-\eta)$ is (Using the product rule, which will be proofed in part b also).

$$\begin{aligned}\hat{f}(k,l) &= \hat{g}(k)\hat{h}(l)\\ &= \frac{1}{2\pi}e^{-ik\xi}e^{-il\eta}\end{aligned}$$

The above could be rewritten in terms of trig functions using Euler relation if needed.

5.2 Part b

By definition, the Fourier transform of $f(x,y)$ is

$$\hat{f}(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i(kx+ly)}dxdy$$

But $f(x,y) = g(x)h(y)$. Hence the above becomes

$$\begin{aligned}\hat{f}(k,l) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)e^{-i(kx+ly)}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)e^{-ikx}e^{-ily}dxdy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}h(y)e^{-ily}\left(\int_{-\infty}^{\infty}g(x)e^{-ikx}dx\right)dy\end{aligned}$$

But $\int_{-\infty}^{\infty}g(x)e^{-ikx}dx = \sqrt{2\pi}\hat{g}(k)$. The above reduces to

$$\hat{f}(k,l) = \frac{1}{2\pi}\sqrt{2\pi}\hat{g}(k)\int_{-\infty}^{\infty}h(y)e^{-ily}dy$$

But $\int_{-\infty}^{\infty}h(y)e^{-ily}dy = \sqrt{2\pi}\hat{h}(l)$. Hence the above becomes

$$\begin{aligned}\hat{f}(k,l) &= \frac{1}{2\pi}\sqrt{2\pi}\hat{g}(k)\sqrt{2\pi}\hat{h}(l)\\ &= \hat{g}(k)\hat{h}(l)\end{aligned}$$

Which is what asked to show.

6 Problem 7.2.2 (a)

Find the Fourier transform of (a) the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$

Solution

6.1 Part a

Using

$$1 + \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-z^2}dz \tag{1}$$

Taking Fourier transform of both sides, and using the known relation from tables which says

$$\mathscr{F}\left[\int_{-\infty}^{x} f(u)\,du\right] = \frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)$$

And using that Fourier transform of 1 is $\sqrt{2\pi}\delta(k)$ then (1) becomes

$$\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] = \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\hat{f}(k) + \pi\hat{f}(0)\,\delta(k)\right)$$

Where $\hat{f}(k)$ is the Fourier transform of e^{-u^2} (Gaussian) we derived in class as $e^{-u^2} \Leftrightarrow \frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}$. The above becomes

$$\begin{aligned}
\sqrt{2\pi}\delta(k) + \mathscr{F}[\operatorname{erf}(x)] &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \pi\left[\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}\right]_{k=0}\delta(k)\right)\\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \frac{\pi}{\sqrt{2}}\delta(k)\right)\\
&= \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}} + \sqrt{2\pi}\delta(k)
\end{aligned}$$

Therefore the above simplifies to

$$\begin{aligned}
\mathscr{F}[\operatorname{erf}(x)] &= \frac{2}{\sqrt{\pi}}\frac{1}{ik}\frac{1}{\sqrt{2}}e^{\frac{-k^2}{4}}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{ik}e^{\frac{-k^2}{4}}\\
&= -i\sqrt{\frac{2}{\pi}}\frac{1}{k}e^{\frac{-k^2}{4}}
\end{aligned}$$

7 Problem 7.2.3 (d)

Find the inverse Fourier transform of the following functions (d) $\frac{k^2}{k-i}$

Solution

Using property that

$$\begin{aligned}\mathscr{F}\left[f'(x)\right] &= ik\hat{f}(k)\\ \mathscr{F}\left[f''(x)\right] &= -k^2\hat{f}(k)\end{aligned} \tag{1}$$

Where in the above $\mathscr{F}\left[f(x)\right]=\hat{f}(k)$. Comparing the above with $\frac{k^2}{k-i}$, we see that

$$\hat{f}(k)=\frac{1}{k-i}$$

Hence we need to find inverse Fourier transform of $\frac{-1}{k-i}$ first in order to find $f(x)$, and then take second derivative of the result. Writing

$$\begin{aligned}\frac{1}{k-i} &= \frac{1}{i\left(\frac{k}{i}-1\right)}\\ &= \frac{1}{i(-ik-1)}\\ &= \frac{-1}{i(ik+1)}\\ &= i\frac{1}{(1+ik)}\end{aligned}$$

From table (page 272 in textbook) we see that

$$\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right]=\sqrt{2\pi}e^{-x}\sigma(x)$$

Using $a=1$ in the table entry. Where $\sigma(x)$ is the step function. Hence

$$i\mathscr{F}^{-1}\left[\frac{1}{(ik+1)}\right]=i\sqrt{2\pi}e^{-x}\sigma(x)$$

Therefore

$$f(x)=i\sqrt{2\pi}e^{-x}\sigma(x)$$

Now we take derivative of the above (using product rule)

$$f'(x)=-i\sqrt{2\pi}e^{-x}\sigma(x)+i\sqrt{2\pi}e^{-x}\delta(x)$$

Where $\delta(x)$ is added since derivative of $\sigma(x)$ has jump discontinuity at $x=0$. Taking one more derivative gives

$$\begin{aligned}f''(x) &= i\sqrt{2\pi}e^{-x}\sigma(x)-i\sqrt{2\pi}e^{-x}\delta(x)-i\sqrt{2\pi}e^{-x}\delta(x)+i\sqrt{2\pi}e^{-x}\delta'(x)\\ &= i\sqrt{2\pi}e^{-x}\sigma(x)-2i\sqrt{2\pi}e^{-x}\delta(x)+i\sqrt{2\pi}e^{-x}\delta'(x)\end{aligned}$$

Therefore

$$\mathscr{F}^{-1}\left[\frac{k^2}{k-i}\right] = i\sqrt{2\pi}e^{-x}\sigma(x) - 2i\sqrt{2\pi}e^{-x}\delta(x) + i\sqrt{2\pi}e^{-x}\delta'(x)$$

8 Problem 7.2.12

(a) Explain why the Fourier transform of a 2π periodic function $f(x)$ is a linear combinations of delta functions $\hat{f}(k) = \sum_{n=-\infty}^{\infty} c_n \delta(k-n)$ where c_n are the complex Fourier series coefficients (3.65) of $f(x)$ on $[-\pi, \pi]$

$$c_n = \left\langle f, e^{inx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{3.65}$$

(b) Find the Fourier transform of the following periodic functions (i) $\sin 2x$ (ii) $\cos^3 x$ (iii) The 2π periodic extension of $f(x) = x$ (iv) The sawtooth function $h(x) = x \bmod 1$. i.e. the fractional part of x

Solution

8.1 Part a

Since $f(x)$ is periodic, then its can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\left(\frac{2\pi}{T}\right)x}$$

But the period $T = 2\pi$ and the above simplifies to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

Taking the Fourier transform of the above gives

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{2}$$

Substituting (1) into (2) gives

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{-ix(k-n)} \right) dx
\end{aligned}$$

Changing the order of summation and integration

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} c_n e^{-ix(k-n)} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \left(\int_{-\infty}^{\infty} e^{-ix(k-n)} dx \right) \qquad (3)
\end{aligned}$$

But from tables we know that $\mathscr{F}(1) = \sqrt{2\pi}\delta(k)$. Which means that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} dx = \sqrt{2\pi}\delta(k)$$

Therefore, replacing k by $k-n$ in the above gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ix(k-n)}dx=\sqrt{2\pi}\delta\left(k-n\right)$$

$$\int_{-\infty}^{\infty}e^{-ix(k-n)}dx=\left(2\pi\right)\delta\left(k-n\right)\tag{4}$$

Substituting (4) into (3) gives

$$\begin{aligned}\hat{f}\left(k\right)&=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}c_n\left(2\pi\right)\delta\left(k-n\right)\\&=\sqrt{2\pi}\sum_{n=-\infty}^{\infty}c_n\delta\left(k-n\right)\end{aligned}$$

Note: The books seems to have a typo. It gives the above without the factor $\sqrt{2\pi}$ at the front.

8.2 Part b

(i) $\sin 2x$. Since this is periodic, then $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\left(2x\right)e^{-inx}dx$. For $n=2$ this gives $c_2=-\frac{i}{2}$ and for $n=-2$ it gives $c_{-2}=\frac{i}{2}$ and it is zero for all other n values due to orthogonality of $\sin$ functions. Using the above result obtained in part (a)

$$\begin{aligned}\hat{f}\left(k\right)&=\sqrt{2\pi}\sum_{n=-\infty}^{\infty}c_n\delta\left(k-n\right)\\&=\sqrt{2\pi}c_{-2}\delta\left(k+2\right)+\sqrt{2\pi}c_2\delta\left(k-2\right)\\&=\sqrt{2\pi}\frac{i}{2}\delta\left(k+2\right)-\sqrt{2\pi}\frac{i}{2}\delta\left(k-2\right)\\&=i\sqrt{\frac{\pi}{2}}\delta\left(k+2\right)-i\sqrt{\frac{\pi}{2}}\delta\left(k-2\right)\end{aligned}$$

(ii) $\cos^3 x$. Since this is periodic, then $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^3\left(x\right)e^{-inx}dx$. But $\cos^3\left(x\right)=\frac{1}{4}\cos\left(3x\right)+\frac{3}{4}\cos\left(x\right)$. Hence only $n=\pm1,n=\pm3$ will have coefficients and the rest are zero.

$$\begin{aligned}c_{-1}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3}{4}\cos\left(x\right)e^{ix}dx=\frac{3}{8}\\c_1&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3}{4}\cos\left(x\right)e^{-ix}dx=\frac{3}{8}\\c_{-3}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{4}\cos\left(3x\right)e^{-3ix}dx=\frac{1}{8}\\c_3&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{4}\cos\left(3x\right)e^{-3ix}dx=\frac{1}{8}\end{aligned}$$

Therefore, using result from part (a)

$$\begin{aligned}\hat{f}(k) &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta(k-n)\\ &= \sqrt{2\pi}\left(\frac{1}{8}\delta(k+3)+\frac{3}{8}\delta(k+1)+\frac{3}{8}\delta(k-1)+\frac{1}{8}\delta(k-3)\right)\\ &= \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\delta(k+3)+3\delta(k+1)+3\delta(k-1)+\delta(k-3)\right)\end{aligned}$$

(iii) The 2π periodic extension of $f(x) = x$

Since this is periodic, then

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx\\ &= \frac{2i}{n^2}\left(n\pi\cos(n\pi)-\sin(n\pi)\right)\\ &= \frac{2i}{n^2}\left(n\pi(-1)^n\right)\\ &= \frac{2i}{n}\pi(-1)^n\end{aligned}$$

Therefore, using result from part (a)

$$\begin{aligned}\hat{f}(k) &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta(k-n)\\ &= \sqrt{2\pi}\sum_{n=-\infty}^{\infty}\frac{2i}{n}\pi(-1)^n\delta(k-n)\\ &= 2i\pi\sqrt{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n}\delta(k-n) \qquad n\neq 0\end{aligned}$$

(iv) The sawtooth function

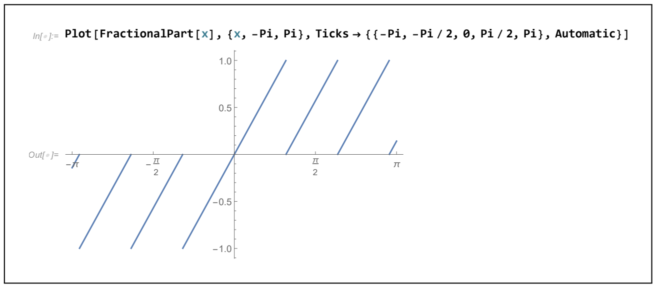

Figure 1: Plot of $f(x)$ (Fractional part of x)

9 Problem 7.3.4

Find a solution to the differential equation $-\frac{d^2u}{dx^2} + 4u = \delta(x)$ by using the Fourier transform

Solution

Taking Fourier transform of both sides gives

$$-(ik)^2\,\hat{u}(k) + 4\hat{u}(k) = \mathscr{F}[\delta(x)]$$
$$k^2\hat{u}(k) + 4\hat{u}(k) = \frac{1}{\sqrt{2\pi}}$$

Solving for $\hat{u}(k)$

$$\hat{u}(k)\left(k^2+4\right) = \frac{1}{\sqrt{2\pi}}$$
$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{k^2+4}$$

Finding inverse Fourier transform. From tables we see that $\mathscr{F}\left(e^{-a|x|}\right) = \sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}$. Using $a = 2$

$$\mathscr{F}\left[e^{-2|x|}\right] = \sqrt{\frac{2}{\pi}}\frac{2}{k^2+4}$$
$$\sqrt{\frac{\pi}{2}}\frac{1}{2}\mathscr{F}\left[e^{-2|x|}\right] = \frac{1}{k^2+4}$$
$$\sqrt{\frac{\pi}{2}}\mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{k^2+4}$$
$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{k^2+4}$$
$$\frac{1}{2}\mathscr{F}\left[\frac{1}{2}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{k^2+4}$$
$$\mathscr{F}\left[\frac{1}{4}e^{-2|x|}\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{k^2+4}$$

Therefore

$$u(x) = \frac{1}{4}e^{-2|x|}$$