HW 1

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 1.8a

Find all quadratic polynomial solutions of the 3D Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Solution

A quadratic polynomial in variables x, y, z is

$$u = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 xz + a_{10} yz \tag{1}$$

Hence $u_x = a_2 + 2a_5 x + a_8 y + a_9 z$ which implies that $u_{xx} = 2a_5$. Similarly $u_y = a_3 + 2a_6 y + a_8 x + a_{10} z$, therefore $u_{yy} = 2a_6$. And finally $u_z = a_4 + 2a_7 z + a_9 x + a_{10} y$ and $u_{zz} = 2a_7$. Substituting these results in the Laplace equation gives above result in

$$\begin{aligned} 2a_5 + 2a_6 + 2a_7 &= 0 \\ a_5 + a_6 + a_7 &= 0 \end{aligned}$$

Therefore $a_5 = -(a_6 + a_7)$. Using this relation back in (1) gives

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 z - (a_6 + a_7) x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 xz + a_{10} yz \\ &= a_1 + a_2 x + a_3 y + a_4 z + a_6 \left(-x^2 + y^2\right) + a_7 \left(-x^2 + z^2\right) + a_8 xy + a_9 xz + a_{10} yz \end{aligned}$$

Which can be written as

$$u\left(x, y, z\right) = A_1 + A_2 x + A_3 y + A_4 z + A_5 \left(y^2 - x^2\right) + A_6 \left(z^2 - x^2\right) + A_7 xy + A_8 xz + A_9 yz$$

2 Problem 1.7

Find all real solutions to 2D Laplace equation $u_{xx} + u_{yy} = 0$ of the form $u = \log\left(p\left(x,y\right)\right)$ where $p\left(x,y\right)$ is a quadratic polynomial.

Solution

A quadratic polynomial $p\left(x,y\right)$ in variables x,y is

$$p\left(x,y\right) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$$

Therefore

$$u\left(x,y\right) = \log\left(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy\right)$$

Hence

$$u_x = \frac{a_2 + 2a_4 x + a_6 y}{p\left(x,y\right)}$$

and

$$u_{xx} = \frac{2a_4}{p\left(x,y\right)} - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)^2} \tag{1}$$

Similarly

$$u_y = \frac{a_3 + 2a_5 y + a_6 x}{p\left(x,y\right)}$$

And

$$u_{yy} = \frac{2a_5}{p\left(x,y\right)} - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)^2} \tag{2}$$

Substituting (1,2) into $u_{xx} + u_{yy} = 0$ gives

$$\left(\frac{2a_4}{p\left(x,y\right)} - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)^2}\right) + \left(\frac{2a_5}{p\left(x,y\right)} - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)^2}\right) = 0$$

$$2a_4 - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2}{p\left(x,y\right)} + 2a_5 - \frac{\left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)} = 0$$

$$2a_4 + 2a_5 - \frac{\left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2}{p\left(x,y\right)} = 0$$

Or

$$\left(2a_4 + 2a_5\right) p\left(x,y\right) = \left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2$$

But $p\left(x,y\right) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$. Hence the above becomes

$$\left(2a_4 + 2a_5\right)\left(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy\right) = \left(a_2 + 2a_4 x + a_6 y\right)^2 + \left(a_3 + 2a_5 y + a_6 x\right)^2$$

Expanding and comparing coefficients gives

$$2x^2 a_4^2 + 2x^2 a_4 a_5 + 2a_6 a_4 xy + 2a_6 a_5 xy + 2a_2 a_4 x + 2a_2 x a_5 + 2y^2 a_4 a_5 + 2y^2 a_5^2 + 2a_3 a_4 y + 2a_3 a_5 y + 2a_1 a_4 + 2a_1 a_5 = \\ 4x^2 a_4^2 + x^2 a_6^2 + 4a_4 a_6 xy + 4a_5 a_6 xy + 4x a_2 a_4 + 2a_3 a_6 x + 4y^2 a_5^2 + y^2 a_6^2 + 2a_2 a_6 y + 4a_3 a_5 y + a_2^2 + a_3^2$$

Simplifying

$$2a_4 a_5 x^2 + 2a_2 a_5 x + 2a_4 a_5 y^2 + 2a_3 a_4 y + 2a_1 a_4 + 2a_1 a_5 = \\ 2x^2 a_4^2 + a_6^2 x^2 + 2a_4 a_6 xy + 2a_5 a_6 xy + 2a_2 a_4 x + 2a_3 a_6 x + 2a_5^2 y^2 + a_6^2 y^2 + 2a_2 a_6 y + 2a_3 a_5 y + a_2^2 + a_3^2$$

Comparing coefficients of terms that contain no x,y and coefficients of x, y, xy, x^2, y^2 gives the following equations in order

$$
\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2\\
2a_2a_5 &= 2a_2a_4 + 2a_3a_6\\
2a_3a_4 &= 2a_2a_6 + 2a_3a_5\\
0 &= 4a_4a_6\\
2a_4a_5 &= 2a_4^2 + a_6^2\\
2a_4a_5 &= 2a_5^2 + a_6^2
\end{aligned}
$$

Equation $0 = 4a_4a_6$ above implies that $a_4 = 0$ or $a_6 = 0$ or both are zero. But if both are zero, there is no solution. On the other hand, if $a_4 = 0$, then this also leads to no solution as all equations reduce to $0 = 0$. Therefore only choice left is $\underline{a_6 = 0}$. Now the above equations become

$$
\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2\\
2a_2a_5 &= 2a_2a_4\\
2a_3a_4 &= 2a_3a_5\\
0 &= 0\\
2a_4a_5 &= 2a_4^2\\
2a_4a_5 &= 2a_5^2
\end{aligned}
$$

Or

$$
\begin{aligned}
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2\\
a_5 &= a_4\\
a_4 &= a_5\\
0 &= 0\\
a_5 &= a_4\\
a_4 &= a_5
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_4 &= a_5 && (3)\\
a_6 &= 0 && (4)\\
2a_1a_4 + 2a_1a_5 &= a_2^2 + a_3^2
\end{aligned}
$$

Since $a_4 = a_5$ then

$$
\begin{aligned}
2a_1a_5 + 2a_1a_5 &= a_2^2 + a_3^2\\
a_5 &= \frac{a_2^2 + a_3^2}{2a_1} && (5)
\end{aligned}
$$

Using (3,4,5) in $p(x, y) = a_1 + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy$ gives

$$
\begin{aligned}
p(x, y) &= a_1 + a_2x + a_3y + a_5x^2 + a_5y^2\\
&= a_1 + a_2x + a_3y + a_5\left(x^2 + y^2\right)\\
&= a_1 + a_2x + a_3y + \frac{a_2^2 + a_3^2}{2a_1}\left(x^2 + y^2\right)
\end{aligned}
$$

Only <u>three arbitrary constants</u> are needed. Let $a_1 = a, a_2 = b, a_3 = c$ the above becomes

$$
p(x, y) = a + bx + cy + \frac{b^2 + c^2}{2a}\left(x^2 + y^2\right)
$$

And the solution becomes

$$
u(x, y) = \log\left(a + bx + cy + \frac{b^2 + c^2}{2a}\left(x^2 + y^2\right)\right)
$$

3 Problem 1.13

Find all solutions $u = f(r)$ of the 3D Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ that depends only on radial coordinates $r = \sqrt{x^2 + y^2 + z^2}$

Solution

The Laplacian in 3D in spherical coordinates is

$$\nabla^2 u\left(r,\theta,\phi\right) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta + u_{\theta\theta}\right) + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}$$

The above shows that the terms that depend only on r makes the laplacian

$$\nabla^2 u(r) = u_{rr} + \frac{2}{r}u_r$$

Hence the PDE $\nabla^2 u(r) = 0$ becomes an ODE now since there is only one dependent variable giving

$$u''(r) + \frac{2}{r}u'(r) = 0$$

Let $v = u'(r)$ and the above becomes

$$v'(r) + \frac{2}{r}v(r) = 0$$

This is linear first order ODE. The integrating factor is $I = e^{\int \frac{2}{r}dr} = e^{2\ln r} = r^2$. Therefore the above becomes $\frac{d}{dr}\left(vr^2\right) = 0$ or $vr^2 = C_1$ or $v(r) = \frac{C_1}{r^2}$. Therefore

$$u' = \frac{C_1}{r^2}$$
$$du = \frac{C_1}{r^2}dr$$

Integrating gives the solution

$$u = -\frac{C_1}{r} + C_2$$

The above is the required solution. Hence

$$\boxed{f(r) = -\frac{C_1}{r} + C_2}$$

Where C_1, C_2 are arbitrary constants.

4 Problem 1.20

The displacement $u(t,x)$ of a forced violin string is modeled by the PDE $u_{tt} = 4u_{xx} + F(t,x)$. When the string is subjected to the external force $F(t,x) = \cos x$, the solution is $u(t,x) = \cos(x-2t) + \frac{1}{4}\cos x$, while when $F(t,x) = \sin x$, the solution is $u(t,x) = \sin(x-2t) + \frac{1}{4}\sin x$. Find a solution when the forcing function is (a) $\cos x - 5\sin x$, (b) $\sin(x-3)$

Solution

4.1 Part (a)

Since the PDE is linear, superposition can be used. When the input is $F(t,x) = \cos x - 5\sin x$ then the solution is

$$\begin{aligned} u(t,x) &= \left(\cos(x-2t) + \frac{1}{4}\cos x\right) - 5\left(\sin(x-2t) + \frac{1}{4}\sin x\right) \\ &= \cos(x-2t) + \frac{1}{4}\cos x - 5\sin(x-2t) - \frac{5}{4}\sin x \end{aligned}$$

4.2 Part (b)

Since the PDE is linear, superposition can be used. When the input is $F(t,x) = \sin(x-3)$ then the solution same as when the input is $\sin x$ but shifted by 3. Hence

$$u(t,x) = \sin((x-3)-2t) + \frac{1}{4}\sin(x-3)$$

5 Problem 1.27b

Solve the following inhomogeneous linear ODE $5u'' - 4u' + 4u = e^x \cos x$

Solution

First the homogeneous solution u_h is found, then a particular solution u_p is found. The general solution will be the sum of both $u = u_h + u_p$. Since this is a constant coefficient ODE, the characteristic equation is $5\lambda^2 - 4\lambda + 4 = 0$. The roots are $\lambda_1 = \frac{2}{5} + \frac{4}{5}i, \lambda_1 = \frac{2}{5} - \frac{4}{5}i$, which implies the solution is

$$u_h(x) = e^{\frac{2}{5}x}\left(c_1 \cos\left(\frac{4}{5}x\right) + c_2 \sin\left(\frac{4}{5}x\right)\right)$$

Using the method of undetermined coefficients, and since the forcing function is $e^x \cos x$, then let

$$u_p = Ae^x(B\cos x + C\sin x) \tag{1}$$

Hence

$$\begin{aligned}
u_p' &= Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x) &(2)\\
u_p'' &= Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x) + Ae^x(-B\sin x + C\cos x) + Ae^x(-B\cos x - C\sin x)\\
&= Ae^x(B\cos x + C\sin x - B\sin x + C\cos x - B\sin x + C\cos x - B\cos x - C\sin x)\\
&= Ae^x(-B\sin x + C\cos x - B\sin x + C\cos x)\\
&= Ae^x(-2B\sin x + 2C\cos x) &(3)
\end{aligned}$$

Substituting (1,2,3) back into the original ODE gives

$$\begin{aligned}
5Ae^x(-2B\sin x + 2C\cos x) - 4\left(Ae^x(B\cos x + C\sin x) + Ae^x(-B\sin x + C\cos x)\right) + 4Ae^x(B\cos x + C\sin x) &= e^x\cos x\\
Ae^x(-10B\sin x + 10C\cos x) - Ae^x(4B\cos x + 4C\sin x) - Ae^x(-4B\sin x + 4C\cos x) + Ae^x(4B\cos x + 4C\sin x) &= e^x\cos x\\
Ae^x(-10B\sin x + 10C\cos x - 4B\cos x - 4C\sin x + 4B\sin x - 4C\cos x + 4B\cos x + 4C\sin x) &= e^x\cos x
\end{aligned}$$

Hence

$$Ae^x(6C\cos x - 6B\sin x) = e^x\cos x$$

Comparing coefficients shows that

$$\begin{aligned}
A &= 1\\
B &= 0\\
C &= \frac{1}{6}
\end{aligned}$$

Hence from (1)

$$u_p = e^x\frac{\sin x}{6}$$

Therefore the general solution is

$$\begin{aligned}
u(x) &= u_h(x) + u_p(x)\\
&= e^{\frac{2}{5}x}\left(c_1\cos\left(\frac{4}{5}x\right) + c_2\sin\left(\frac{4}{5}x\right)\right) + e^x\frac{\sin x}{6}
\end{aligned}$$

6 Problem 2.1.6

Solve the PDE $\frac{\partial^2 u}{\partial x \partial y} = 0$ for $u(x, y)$

Solution

Integrating once w.r.t x gives

$$\frac{\partial u}{\partial y} = F(y)$$

Where $F(y)$ acts as the constant of integration, but since this is a PDE, it becomes an arbitrary function of y only. Integrating the above again w.r.t. y gives

$$u = \int F(y)\, dy + G(x)$$

Where $G(x)$ is an arbitrary function of x only. If we let $\int F(y)\, dy = H(y)$ where $H(y)$ is the antiderivative for the indefinite integral which depends on y only. Then the above can be written as

$$\boxed{u(x, y) = H(y) + G(x)}$$

To verify, from the above $\frac{\partial u}{\partial y} = H'(y)$ and hence

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{d}{dx}\left(H'(y)\right)\\ &= 0\end{aligned}$$

7 Problem 2.2.2

Solve the following initial value problems and graph the solutions at $t = 1, 2, 3$

a $u_t - 3u_x = 0, u(0, x) = e^{-x^2}$

b $u_t + 2u_x = 0, u(-1, x) = \frac{x}{1+x^2}$

c $u_t + u_x + \frac{1}{2}u = 0, u(0, x) = \arctan(x)$

d $u_t - 4u_x + u = 0, u(0, x) = \frac{1}{1+x^2}$

Solution

7.1 Part a

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -3$ in this problem. Hence characteristic lines are

$$x = x_0 - 3t$$

Where x_0 means the same as $x(0)$, i.e. $x(t)$ at time $t = 0$. Since $c = -3$ then

$$\xi = x + 3t$$

Let

$$u(t, x) \equiv v(t, \xi)$$

$u_t - 3u_x = 0$ is now transformed to $v(t, \xi)$ as follows

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial v}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} \\ &= \frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial \xi} \end{aligned} \tag{1}$$

And

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} \\ &= 0 + \frac{\partial v}{\partial \xi} \\ &= \frac{\partial v}{\partial \xi} \end{aligned} \tag{2}$$

Substituting (1,2) in $u_t - 3u_x = 0$ gives the transformed PDE as

$$\begin{aligned} \frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial \xi} - 3\frac{\partial v}{\partial \xi} &= 0 \\ \frac{\partial v}{\partial t} &= 0 \end{aligned}$$

Integrating w.r.t ξ gives the solution in $v(t, \xi)$ space as

$$v(t, \xi) = F(\xi)$$

Where $F(\xi)$ is an arbitrary continuous function of ξ. Transforming back to $u(t, x)$ gives

$$u(t, x) = F(x + 3t) \tag{3}$$

At $t = 0$ the above becomes

$$e^{-x_0^2} = F(x_0)$$

This means that (3) becomes (since $x = x_0 + ct$ or $x = x_0 - 3t$ or $x_0 = x + 3t$)

$$u(t, x) = e^{-(x+3t)^2}$$

7.2 Part b

$$\begin{aligned} u_t + 2u_x &= 0 \\ u(-1, x) &= \frac{x}{1 + x^2} \end{aligned}$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 2$ in this problem. Hence characteristic lines are

$$x = x_0 + 2t$$

And

$$\xi = x - 2t$$

Let $u(t,x) \equiv v(t,\xi)$. Then $u_t + 2u_x = 0$ is transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$\frac{\partial v}{\partial t} = 0$$

Integrating w.r.t ξ gives the solution

$$v(t,\xi) = F(\xi)$$

Where $F(\xi)$ is an arbitrary continuous function of ξ. Transforming back to $u(t,x)$ results in

$$u(t,x) = F(x - 2t) \tag{3}$$

At $t = -1$ the above becomes

$$\frac{x_0}{1 + x_0^2} = F(x_0 + 2)$$

Let $x_0 + 2 = z$. Then $x_0 = z - 2$. And the above becomes

$$\frac{z-2}{1 + (z-2)^2} = F(z)$$

This means that (3) becomes

$$\begin{aligned} u(t,x) &= \frac{(x-2t) - 2}{1 + ((x-2t) - 2)^2} \\ &= \frac{x - 2t - 2}{1 + (x - 2t - 2)^2} \end{aligned}$$

7.3 Part c

$$\begin{aligned} u_t + u_x + \frac{1}{2}u &= 0 \qquad (1) \\ u(0,x) &= \arctan(x) \end{aligned}$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 1$ in this problem. Hence characteristic lines are given by solution to

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ x(t) &= x_0 + t \end{aligned}$$

And

$$\begin{aligned} \xi &= x - ct \\ &= x - t \end{aligned}$$

Then $u_t + u_x$ are transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$u_t + u_x = \frac{\partial v}{\partial t}$$

Substituting the above into (1) gives (where now v is used in place of u).

$$\frac{\partial v}{\partial t} + \frac{1}{2}v = 0$$

This is now first order ODE since it only depends on t. Therefore $v' + \frac{1}{2}v = 0$. This is linear in v. Hence the solution is $\frac{d}{dt}\left(ve^{\int \frac{1}{2}dt}\right) = 0$ or $ve^{\frac{1}{2}t} = F(\xi)$ where F is arbitrary function of ξ. Hence

$$v(t,\xi) = e^{\frac{-1}{2}t}F(\xi)$$

Converting back to $u(t,x)$ gives

$$u(t,x) = e^{\frac{-t}{2}} F(x-t) \tag{2}$$

At $t = 0$ the above becomes

$$\arctan(x_0) = F(x_0)$$

From the above then (2) can be written as

$$u(t,x) = e^{\frac{-t}{2}} \arctan(x-t)$$

7.4 Part d

$$u_t - 4u_x + u = 0$$
$$u(0,x) = \frac{1}{1+x^2}$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -4$ in this problem. Hence characteristic lines are

$$x = x_0 - 4t$$

And

$$\xi = x + 4t$$

Then $u_t - 4u_x$ are transformed to $v(t,\xi)$ as was done in part (a) (will not be repeated) which results in

$$u_t - 4u_x = \frac{\partial v}{\partial t}$$

Substituting the above into (1) gives (where now v is used in place of u).

$$\frac{\partial v}{\partial t} + v = 0$$

This is now first order ODE since it only depends on t. Therefore $v' + v = 0$. This is linear in v. Hence the solution is $\frac{d}{dt}\left(ve^{\int dt}\right) = 0$ or $ve^t = F(\xi)$ where F is arbitrary function of ξ. Hence

$$v(t,\xi) = e^{-t} F(\xi)$$

Converting to $u(t,x)$ gives

$$u(t,x) = e^{-t} F(x+4t) \tag{2}$$

At $u(0,x) = \frac{1}{1+x^2}$ the above becomes

$$\frac{1}{1+x_0^2} = F(x_0)$$

From the above then (2) can be written as

$$u(t,x) = \frac{e^{-t}}{1+(x+4t)^2}$$

8 Problem 2.2.3

Graph some of the characteristic lines for the following equation and write down the formula for the general solution

(b) $u_t + 5u_x = 0$, (d) $u_t - 4u_x + u = 0$

Solution

8.1 Part b

$$u_t + 5u_x = 0$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 5$ in this problem. Hence characteristic lines are

$$\boxed{x(t) = x_0 + 5t} \tag{1}$$

And

$$\xi = x - 5t$$

Then $u_t - 5u_x = 0$ is transformed to $v(t, \xi)$ as was done in earlier (will not be repeated) which results in

$$u_t - 5u_x = \frac{\partial v}{\partial t}$$

Therefore $\frac{\partial v}{\partial t} = 0$ which has the general solution $v(t, \xi) = F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t, x)$ gives

$$\boxed{u(t, x) = F(x - 5t)}$$

On the characteristic lines given by (1) the solution $u(t, x)$ is constant. The slope of the characteristic lines is 5 and intercept is x_0. The following is a plot of few lines using different values of x_0.

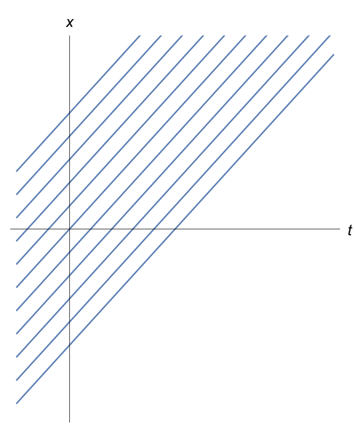

Figure 1: Showing some characteristic lines for part b

8.2 Part d

$$u_t - 4u_x + u = 0$$

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = -4$ in this problem. Hence characteristic lines are

$$\boxed{x(t) = x_0 - 4t} \tag{1}$$

And

$$\xi = x + 4t$$

Then $u_t - 4u_x$ is transformed to $v(t, \xi)$ as was done in earlier (will not be repeated) which results in

$$u_t - 4u_x = \frac{\partial v}{\partial t}$$

Therefore the original PDE becomes $\frac{\partial v}{\partial t} + v = 0$, where u is replaced by v. This is linear first order ODE which has the solution $v(t,\xi) = e^{-t}F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t,x)$ gives the general solution as

$$u(t,x) = e^{-t}F(x+4t)$$

The following is a plot of few characteristic lines $x = x_0 - 4t$ using different values of x_0.

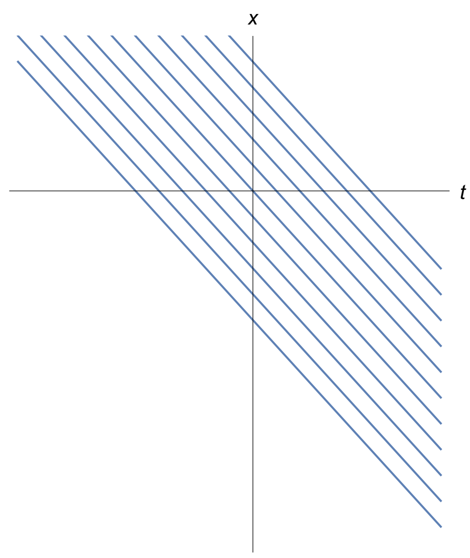

Figure 2: Showing some characteristic lines for part d

9 Problem 2.2.5

Solve $u_t + 2u_x = \sin x, u(0,x) = \sin x$

Solution

Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. But $c = 2$ in this problem. Hence characteristic lines are

$$\boxed{x = x_0 + 2t} \tag{1}$$

And

$$\xi = x - 2t$$

Then $u_t + 2u_x$ is transformed to $v(t,\xi)$ as was done in earlier (will not be repeated) which results in

$$u_t + 2u_x = \frac{\partial v}{\partial t}$$

Substituting this into the original PDE gives

$$\frac{\partial v(t,\xi)}{\partial t} = \sin(\xi + 2t)$$

Integrating w.r.t t gives

$$\begin{aligned} v(t,\xi) &= \int \sin(\xi + 2t)\, dt + F(\xi) \\ &= -\frac{\cos(\xi + 2t)}{2} + F(\xi) \end{aligned}$$

Transforming back to $u(t,x)$ gives

$$\begin{aligned} u(t,x) &= -\frac{\cos(x - 2t + 2t)}{2} + F(x - 2t) \\ &= \frac{-1}{2}\cos(x) + F(x - 2t) \end{aligned} \tag{1}$$

When $t = 0$, $u(0,x) = \sin x$, therefore the above becomes

$$\begin{aligned} \sin x_0 &= F(x_0) - \frac{1}{2}\cos x_0 \\ F(x_0) &= \sin x_0 + \frac{1}{2}\cos x_0 \end{aligned}$$

Therefore the solution (1) becomes

$$\begin{aligned} u(t,x) &= \left(\sin(x - 2t) + \frac{1}{2}\cos(x - 2t)\right) - \frac{1}{2}\cos x \\ &= \sin(x - 2t) + \frac{1}{2}\cos(x - 2t) - \frac{1}{2}\cos x \end{aligned}$$

10 Problem 2.2.9

(a) Prove that if the initial data is bounded, $|f(x)| \le M$ for all $x \in \mathbb{R}$, then the solution to the damped transport equation (2.14) $u_t + cu_x + au = 0$ with $a > 0$ satisfies $u(t,x) \to 0$ as $t \to \infty$.
(b) Find a solution to (2.14) that is defined for all (t,x) but does not satisfy $u(t,x) \to 0$ as $t \to \infty$.

Solution

10.1 Part(a)

$u_t + cu_x + au = 0$ is solved to show what is required. Let ξ be the characteristic variable defined such that $\xi = x - ct$. Where characteristic lines are given by $x = x_0 + ct$. Hence characteristic lines are

$$x = x_0 + ct \tag{1}$$

And

$$\xi = x - ct$$

Then $u_t + cu_x$ is transformed to $v(t,\xi)$ as was done in earlier (will not be repeated) which results in

$$u_t + cu_x = \frac{\partial v}{\partial t}$$

Substituting this into the original PDE gives

$$\frac{\partial v}{\partial t} + av = 0$$

Where u is replaced by v. This can be viewed as first order linear ODE since it depends on t only. Its solution is $v(t,\xi) = e^{-at}F(\xi)$ where F is arbitrary function of ξ. Transforming back to $u(t,x)$ gives

$$u(t,x) = e^{-at}F(x-ct) \tag{1}$$

At $t = 0$ initial data is $f(x)$. Hence the above becomes at $t = 0$

$$f(x) = F(x)$$

Hence (1) now becomes

$$u(t,x) = e^{-at}f(x-ct) \tag{2}$$

But since $|f(x)|$ is bounded, and since $a > 0$ then $e^{-at} \to 0$ as $t \to \infty$. Which implies the solution itself $u(t,x)$ goes to zero as well. This is the reason why initial data needed to be bounded for this to happen.

10.2 Part(b)

Keeping $a > 0$. If initial data have the form $f(x)e^{-bx}$ where $|b| > a$, then at $t = 0$ the solution found in (1) becomes

$$f(x_0)e^{-bx_0} = F(x_0)$$

Then the solution (2) now becomes, after replacing x_0 by $x - ct$

$$\begin{aligned} u(t,x) &= e^{-at}e^{-b(x-ct)}f(x-ct) \\ &= e^{-at+bct}e^{-bx}f(x-ct) \\ &= e^{(bc-a)t}e^{-bx}f(x-ct) \end{aligned}$$

The problem is asking to show that this does not go to zero for all $x \in \mathbb{R}$ as $t \to \infty$. Since $|b| > a$ then $bc - a$ is positive quantity (c is assumed positive)[1].

Therefore $e^{(bc-a)t}$ will blow up as $t \to \infty$. And therefore the whole solution will not go to zero. For any x, no matter how large x is, a large enough t can be found to make the product $e^{(bc-a)t}e^{-bx}$ blow up.

[1] If c was negative then initial data could be choosen to be $f(x)e^{bx}$ where $|b| > a$ which will lead to same result.