Exercise 6 / Section 3.12

$$\dot{x} = \begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} , \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 & -2 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{R_3 + R_1} \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} .$$

$$= (1-\lambda)^2(3-\lambda) + \cancel{(1-\lambda)} - 2(1-\lambda)^2 - \cancel{(1-\lambda)} = (1-\lambda)^2(3-\lambda-2) = (1-\lambda)^3 = 0$$

System matrix has one eigenvalue $\lambda = 1$ of multiplicity 3.

1) finding $x^1(t)$:

$$(A - \lambda I) v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left.\begin{aligned} v_1 + v_3 &= 0 \\ 2v_1 + v_2 + 2v_3 &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} v_3 &= -v_1 \\ v_2 &= 0 \end{aligned}$$

choose $v^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $x^1(t) = e^{\lambda t} v^1 = e^{t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2) finding $x^2(t)$:

$$(A - \lambda I)^2 v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} v = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 + v_3 = 0 \quad \to \quad v_1 = -v_3$

v_2 - arbitrary , choose $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$(A - \lambda I) v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x^2(t) = e^{\lambda t}\left(v + t(A - \lambda I) v\right) = e^t \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) = e^t \begin{bmatrix} -t \\ 1 \\ t \end{bmatrix}$$

3) finding $x^3(t)$:

$$(A - \lambda I)^3 v = (A - \lambda I)^2 (A - \lambda I) v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} v$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ v is arbitrary vector such that $(A - \lambda I)^2 v \neq 0$

choose $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $(A - \lambda I)^2 v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

we need also $(A - \lambda I) v = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$$x^3(t) = e^{\lambda t}\left(v + t(A - \lambda I) v + \frac{t^2}{2}(A - \lambda I)^2 v\right)$$

$$x^3(t) = e^t\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\1\\2\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) = e^t\begin{bmatrix}-\frac{t^2}{2}-2t+1\\ t\\ \frac{t^2}{2}+2t\end{bmatrix}$$

4) fund. matrix solution

$$X(t) = e^t\begin{bmatrix}1 & -t & -\frac{t^2}{2}-2t+1\\ 0 & 1 & t\\ -1 & t & \frac{t^2}{2}+2t\end{bmatrix}$$

5) matrix $e^{At} = X(t)\,X(0)^{-1}$

$$X(0) = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc}\overbrace{1 \quad 0 \quad 0}^{I} & & & \overbrace{0 \quad 0 \quad -1}^{X(0)^{-1}} & & \\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{array}\right]$$

$$e^{At} = X(t)\,X(0)^{-1} = e^t\begin{bmatrix}1 & -t & -\frac{t^2}{2}-2t+1\\ 0 & 1 & t\\ -1 & t & \frac{t^2}{2}+2t\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

$$= e^t\begin{bmatrix}-\frac{t^2}{2}-2t+1 & -t & -\frac{t^2}{2}-2t\\ t & 1 & t\\ \frac{t^2}{2}+2t & t & \frac{t^2}{2}+2t+1\end{bmatrix}$$

6) Since $t_0 = 0$ and $x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the solution of the IVP is

$$x(t) = e^{At} \int_0^t e^{-As} f(s)\, ds \quad , \quad \text{with} \quad f(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^s .$$

$$e^{-As} f(s) = e^{-s} \begin{bmatrix} -\frac{s^2}{2} + 2s + 1 & s & -\frac{s^2}{2} + 2s \\ -s & 1 & -s \\ \frac{s^2}{2} - 2s & -s & \frac{s^2}{2} - 2s + 1 \end{bmatrix} \cdot e^s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{s^2}{2} + 2s + 1 \\ -s \\ \frac{s^2}{2} - 2s \end{bmatrix}$$

$$\int_0^t e^{-As} f(s)\, ds = \begin{bmatrix} -\frac{t^3}{6} + t^2 + t \\ -\frac{t^2}{2} \\ \frac{t^3}{6} - t^2 \end{bmatrix} = t \begin{bmatrix} -\frac{t^2}{6} + t + 1 \\ -\frac{t}{2} \\ \frac{t^2}{6} - t \end{bmatrix}$$

$$x(t) = e^{At} \int_0^t e^{-As} f(s)\, ds = t e^t \begin{bmatrix} -\frac{t^2}{2} - 2t + 1 & -t & -\frac{t^2}{2} - 2t \\ t & 1 & t \\ \frac{t^2}{2} + 2t & t & \frac{t^2}{2} + 2t + 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{t^2}{6} + t + 1 \\ -\frac{t}{2} \\ \frac{t^2}{6} - t \end{bmatrix}$$

$$= t e^t \begin{bmatrix} -\frac{t^2}{6} - t + 1 \\ \frac{t}{2} \\ \frac{t^2}{6} + t \end{bmatrix}$$

Exercise 8 / Section 3.12 Find the solution of the initial-value problem

$$y''' + y' = \sec t \cdot \tan t$$
$$y(0) = y'(0) = y''(0) = 0$$

<u>Solution</u>: First we transform the differential equation into a system. Let

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t).$$

Then $y''' + y' = x_3' + x_2 = \sec t \cdot \tan t$, and

$$x_1'(t) = x_2(t), \qquad x_1(0) = 0$$
$$x_2'(t) = x_3(t), \qquad x_2(0) = 0$$
$$x_3'(t) = -x_2(t) + \sec t \cdot \tan t, \qquad x_3(0) = 0.$$

The matrix form of this system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \sec t \cdot \tan t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda = -\lambda(\lambda^2+1) = 0$$

Eigenvalues of the system matrix are $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$.

$\lambda_1 = 0$: $$(A - \lambda_1 I)v = Av = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$
$v_3 = 0$
v_1 - arbitrary

choose $v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ - then $x^1(t) = e^{\lambda_1 t} v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = i$:

$$(A - \lambda_2 I)v = \begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\longrightarrow$ $-iv_1 + v_2 = 0$, $v_1 = -iv_2$

$\searrow$ $-iv_2 + v_3 = 0$, $v_3 = iv_2$

$$v = \begin{bmatrix} -iv_2 \\ v_2 \\ iv_2 \end{bmatrix} = v_2 \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}, \quad \text{choose } v^2 = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$$

complex-valued solution is $\phi(t) = e^{\lambda_2 t} v^2 = e^{it} \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$

$$= (\cos t + i \sin t) \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} \sin t - i \cos t \\ \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

$$x^2(t) = \operatorname{Re}(\phi(t)) = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}, \quad x^3(t) = \operatorname{Im}(\phi(t)) = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$$

---

fundamental matrix solution is

$$X(t) = \begin{bmatrix} 1 & \sin t & -\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{X(0)^{-1}}$$

$$e^{At} = X(t)\,X(0)^{-1} = \begin{bmatrix} 1 & \sin t & -\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

$$x(t) = e^{A(t-t_0)}x^0 + e^{At}\int_{t_0}^{t} e^{-As} f(s)\,ds = e^{At}\int_0^t e^{-As} f(s)\,ds$$

$$e^{-As} f(s) = \begin{bmatrix} 1 & -\sin s & 1-\cos s \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \sec s \cdot \tan s \end{bmatrix} = \begin{bmatrix} 1-\cos s \\ -\sin s \\ \cos s \end{bmatrix} \cdot \sec s \cdot \tan s$$

$$\int_0^t (1-\cos s)\sec s \cdot \tan s\,ds = \int_0^t \sec s \cdot \tan s\,ds - \int_0^t \tan s\,ds$$

$$= \left(\sec s + \ln|\cos s|\right)\Big|_0^t = \sec t - 1 + \ln|\cos t|$$

$$-\int_0^t \sin s \cdot \sec s \cdot \tan s\,ds = \int_0^t (1 - \sec^2 s)\,ds = t - \tan s\Big|_0^t = t - \tan t$$

$$\int_0^t \cos s \cdot \sec s \cdot \tan s \, ds = \int_0^t \tan s \, ds = -\ln|\cos s| \Big|_0^t = -\ln|\cos t|$$

$$\Rightarrow \int_0^t e^{-As} f(s)\,ds = \begin{bmatrix} \sec t - 1 + \ln|\cos t| \\ t - \tan t \\ -\ln|\cos t| \end{bmatrix}$$

$$e^{At}\int_0^t e^{-As} f(s)\,ds = \begin{bmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} \sec t - 1 + \ln|\cos t| \\ t - \tan t \\ -\ln|\cos t| \end{bmatrix}$$

$$= \begin{bmatrix} -1 + t\cdot\sin t + \cos t\,(1 + \ln|\cos t|) \\ t\cos t - \sin t\,(1 + \ln|\cos t|) \\ -t\sin t + \sin t\cdot\tan t - \cos t\cdot\ln|\cos t| \end{bmatrix}$$

* Solution of the starting IVP is

$$y(t) = x_1(t) = -1 + t\sin t + \cos t\,(1 + \ln|\cos t|).$$

* THE SOLUTION IN THE BOOK SEEMS TO BE INCORRECT *

Exercise 7/Section 3.13 Using Laplace transforms solve

$$\dot{x} = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 4e^t \cos t \\ 0 \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution: Matrix $sI - A$ is $\begin{bmatrix} s-4 & -5 \\ 2 & s+2 \end{bmatrix}$.

The system for finding $X(s) = \mathcal{L}\{x(t)\}$ is

$$(sI - A)\,X(s) = F(s) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{where} \quad F(s) = \begin{bmatrix} \mathcal{L}\{4e^t \cos t\} \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} s-4 & -5 \\ 2 & s+2 \end{bmatrix} \cdot \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) + 1 \\ 1 \end{bmatrix}, \quad F_1(s) = \mathcal{L}\{4e^t \cos t\} = 4 \cdot \frac{s-1}{(s-1)^2 + 1}$$

$$(s-4)X_1(s) - 5X_2(s) = F_1(s) + 1$$

$$2X_1(s) + (s+2)X_2(s) = 1 \quad \Rightarrow \quad X_1(s) = \frac{1}{2}\left(1 - (s+2)X_2(s)\right)$$

$$\frac{s-4}{2}\left(1 - (s+2)X_2(s)\right) - 5X_2(s) = F_1(s) + 1 \qquad (\text{mult. by } 2)$$

$$-(s-4)(s+2)X_2(s) - 10X_2(s) = 2(F_1(s) + 1) - (s-4)$$

$$X_2(s)\left(-s^2 - 2s + 4s + 8 - 10\right) = 2(F_1(s) + 1) - (s-4)$$

$$X_2(s)\left(-s^2 + 2s - 2\right) = 2(F_1(s) + 1) - (s-4)$$

$$X_2(s) = \frac{s-4}{s^2-2s+2} - 2\,\frac{F_1(s)+1}{s^2-2s+2} = \frac{s-6}{s^2-2s+2} - 2\,\frac{F_1(s)}{s^2-2s+2}$$

Notice that s^2-2s+2 has complex roots $1 \pm i$. Therefore we will use $s^2-2s+2 = (s-1)^2+1$.

The first term in $X_2(s)$ is $\frac{s-6}{s^2-2s+2}$. It can be transformed as

$$\frac{s-6}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1} - \frac{5}{(s-1)^2+1} = \mathcal{L}\{e^t \cos t\} - 5\,\mathcal{L}\{e^t \sin t\}.$$

Continue with the second term in $X_2(s)$:

$$-2\,\frac{F_1(s)}{s^2-2s+2} = -2\cdot\frac{1}{(s-1)^2+1}\cdot\frac{4(s-1)}{(s-1)^2+1} = -8\,\frac{s-1}{((s-1)^2+1)^2}$$

From $\frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{1}{2}\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{1}{2}\mathcal{L}\{-t\sin t\}$

$$= \mathcal{L}\left\{\frac{t}{2}\sin t\right\},$$

we further derive $-8\,\frac{s-1}{((s-1)^2+1)^2} = -8\,\mathcal{L}\left\{e^t \cdot \frac{t}{2}\sin t\right\}$

$$= \mathcal{L}\{-4te^t \sin t\}.$$

Then $X_2(s) = \mathcal{L}\{e^t \cos t - 5e^t \sin t - 4te^t \sin t\} = \mathcal{L}\{x_2(t)\}$.

$$X_1(s) = \frac{1}{2}\Big(1-(s+2)X_2(s)\Big) = \frac{1}{2}\left(1 - \frac{(s+2)(s-6)}{s^2-2s+2} + 2(s+2)\frac{F_1(s)}{s^2-2s+2}\right)$$

$$X_1(s) = \frac{1}{2}\left(\frac{s^2-2s+2-(s^2-4s-12)}{s^2-2s+2} + 2(s+2)\frac{4(s-1)}{((s-1)^2+1)^2} \right)$$

$$= \frac{s+7}{(s-1)^2+1} + 4\frac{(s-1)(s+2)}{((s-1)^2+1)^2}$$

The first term in $X_1(s)$ we write as

$$\frac{s+7}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{8}{(s-1)^2+1} = \mathcal{L}\{e^t \cos t\} + 8\mathcal{L}\{e^t \sin t\}.$$

The second term we transform into

$$4\frac{(s-1)(s+2)}{((s-1)^2+1)^2} = 4\frac{(s-1)(s-1+3)}{((s-1)^2+1)^2} = 4\frac{(s-1)^2}{((s-1)^2+1)^2} + 12\frac{s-1}{((s-1)^2+1)^2}$$

$$= 4\frac{1}{(s-1)^2+1} - 4\frac{1}{((s-1)^2+1)^2} + 12\frac{s-1}{((s-1)^2+1)^2}$$

$$= 4\mathcal{L}\{e^t \sin t\} - 4\mathcal{L}\{\tfrac{1}{2}e^t(\sin t - t\cos t)\}$$

$$+ 12\mathcal{L}\{e^t \cdot \tfrac{1}{2} t \sin t\}$$

Here we have used $\frac{1}{(s^2+1)^2} = \mathcal{L}\{\sin t * \sin t\} = \mathcal{L}\{\tfrac{1}{2}(\sin t - t\cos t)\}$

and thus $\frac{1}{((s-1)^2+1)^2} = \mathcal{L}\{\tfrac{1}{2}e^t(\sin t - t\cos t)\}.$

Finally, $X_1(s) = \mathcal{L}\{e^t\cos t + 8e^t \sin t + 4e^t \sin t - 2e^t(\sin t - t\cos t)$

$+ 6e^t t \sin t\} =$

$= \mathcal{L}\{10e^t \sin t + e^t \cos t + 6te^t \sin t + 2te^t \cos t\} = \mathcal{L}\{x_1(t)\}$