MATH 4512 – Midterm exam #3 November 20, 2019

(SOLUTIONS)

Problem 1. (35 points)
Solve the initial-value problem

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t), \qquad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution:

1st approach (The eigenvalue-eigenvector method):
The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda(2+\lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2.$$

The matrix A has one eigenvalue $\lambda = -1$ with multiplicity 2.
In order to find $x^1(t)$, first we need to find a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda I)v = 0$, i.e.

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of this system imply $v_1 = v_2$. We can choose $v = [1, 1]^\top$ and obtain

$$x^1(t) = \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For finding $x^2(t)$, we search for a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda I)^2 v = 0$ and $(A - \lambda I)v \neq 0$. Since

$$(A - \lambda I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

for any vector $v \in \mathbb{R}^2$ we have that $(A - \lambda I)^2 v = 0$. We can choose $v = [1, 0]^\top$ since

$$(A - \lambda I)v = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then

$$x^2(t) = \mathrm{e}^{-t}\left(v + t(A - \lambda I)v\right) = \mathrm{e}^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \mathrm{e}^{-t} \begin{bmatrix} 1-t \\ -t \end{bmatrix}.$$

The general solution is

$$x(t) = c_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \begin{bmatrix} 1-t \\ -t \end{bmatrix}.$$

From the initial condition $x(0) = [2,1]^\top$ we obtain

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}.$$

Then $c_1 = c_2 = 1$ and the final solution is

$$x(t) = \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathrm{e}^{-t} \begin{bmatrix} 1-t \\ -t \end{bmatrix} = \mathrm{e}^{-t} \begin{bmatrix} 2-t \\ 1-t \end{bmatrix}.$$

2nd approach (Laplace transforms):
We will determine $X(s) = \mathcal{L}(x(t))$ from the condition $(sI - A)X(s) = x(0)$, i.e.

$$\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

From the first equation we have $(s+2)X_1(s) - X_2(s) = 2$ and $X_2(s) = (s+2)X_1(s) - 2$. The second equation is now

$$1 = X_1(s) + sX_2(s) = X_1(s) + s(s+2)X_1(s) - 2s = X_1(s)(s^2 + 2s + 1) - 2s.$$

Then

$$X_1(s) = \frac{2s+1}{(s+1)^2} = \frac{2}{s+1} - \frac{1}{(s+1)^2}.$$

Using

$$\frac{1}{s+1} = \mathcal{L}\{\mathrm{e}^{-t}\}$$
$$\frac{1}{(s+1)^2} = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{-t}\} = \mathcal{L}\{t\mathrm{e}^{-t}\},$$

we further derive

$$X_1(s) = 2\mathcal{L}\{\mathrm{e}^{-t}\} - \mathcal{L}\{t\mathrm{e}^{-t}\} = \mathcal{L}\{2\mathrm{e}^{-t} - t\mathrm{e}^{-t}\}.$$

Therefore

$$x_1(t) = (2-t)\mathrm{e}^{-t}.$$

Now

$$\begin{aligned} X_2(s) &= (s+2)X_1(s) - 2 = (s+2)\frac{2s+1}{(s+1)^2} - 2 = \frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ &= \mathcal{L}\{\mathrm{e}^{-t}\} - \mathcal{L}\{t\mathrm{e}^{-t}\} = \mathcal{L}\{(1-t)\mathrm{e}^{-t}\}, \end{aligned}$$

and $x_2(t) = (1-t)\mathrm{e}^{-t}$. The final solution is

$$x(t) = \mathrm{e}^{-t} \begin{bmatrix} 2-t \\ 1-t \end{bmatrix}.$$

Problem 2. (35 points)
Transforming the second-order differential equation

$$y''(t) - 4y'(t) + 5y(t) = 0$$

into a system of first-order differential equations, find its solution that satisfies

$$y(\pi) = 0, \qquad y'(\pi) = -1.$$

Solution:
Introducing

$$x_1(t) = y(t) \qquad \text{and} \qquad x_2(t) = y'(t),$$

the differential equation becomes

$$x_2'(t) = -5x_1(t) + 4x_2(t).$$

Therefore we obtain the following initial-value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(\pi) \\ x_2(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The system matrix

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

has the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) + 5 = \lambda^2 - 4\lambda + 5,$$

with the roots $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$ as eigenvalues of A. For determining $x^1(t)$ and $x^2(t)$, it is sufficient to consider just $\lambda_1 = 2 + i$.
A complex eigenvector $v = [v_1, v_2]^\top$ that corresponds to λ_1 satisfies $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2-i & 1 \\ -5 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $v_2 = (2 + i)v_1$. Thus, the vector v has the form

$$v = \begin{bmatrix} v_1 \\ (2+i)v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

and we can choose $v_1 = 1$. Then a complex-valued solution of the system is

$$\phi(t) = \mathrm{e}^{(2+i)t} \begin{bmatrix} 1 \\ 2+i \end{bmatrix} = \mathrm{e}^{2t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

$$= \mathrm{e}^{2t} \begin{bmatrix} \cos t + i \sin t \\ 2\cos t - \sin t + i(2\sin t + \cos t) \end{bmatrix}.$$

Taking $x^1(t) = \mathrm{Re}(\phi(t))$ and $x^2(t) = \mathrm{Im}(\phi(t))$, we obtain a general solution of the form

$$x(t) = c_1 \mathrm{e}^{2t} \begin{bmatrix} \cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 \mathrm{e}^{2t} \begin{bmatrix} \sin t \\ 2\sin t + \cos t \end{bmatrix}.$$

The initial condition $x(\pi) = [0, -1]^\top$ implies

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = x(\pi) = c_1 \mathrm{e}^{2\pi} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \mathrm{e}^{2\pi} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Hence $c_1 = 0$ and $c_2 = \mathrm{e}^{-2\pi}$. The solution of the initial value problem is

$$x(t) = \mathrm{e}^{2(t-\pi)} \begin{bmatrix} \sin t \\ 2\sin t + \cos t \end{bmatrix},$$

while the solution of the second-order differential equation with $y(\pi) = 0$, $y'(\pi) = -1$, is

$$y(t) = x_1(t) = \mathrm{e}^{2(t-\pi)} \sin t.$$

Problem 3. (30 points)
For the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

determine e^{At}.

Solution:

We will determine e^{At} from the relation

$$\mathrm{e}^{At} = X(t)X(0)^{-1},$$

where $X(t)$ is a fundamental matrix solution of the system $\dot{x}(t) = Ax(t)$. The characteristic polynomial of the system matrix A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda).$$

The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.
In order to find $x^1(t)$, first we need to find a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The second equation implies $v_1 = v_2$. We can choose $v = [1, 1]^\top$ and obtain

$$x^1(t) = \mathrm{e}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For finding $x^2(t)$, we search for a vector $v = [v_1, v_2]^\top$ such that $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From both equations we get $v_1 = 0$. Choosing $v_2 = 1$, we obtain

$$x^2(t) = \mathrm{e}^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The fundamental matrix solution of the system $\dot{x}(t) = Ax(t)$ is

$$X(t) = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t & \mathrm{e}^{2t} \end{bmatrix}.$$

The inverse matrix of

$$X(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is

$$X(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore

$$\mathrm{e}^{At} = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t - \mathrm{e}^{2t} & \mathrm{e}^{2t} \end{bmatrix}.$$