MATH 4512 – Midterm exam #2 October 26, 2018

(SOLUTIONS)

Problem 1. (25 points)
Find the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 4y = \cos(2t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution:
Consider the homogeneous problem

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation $r^2 + 4 = 0$ ($a = 1, b = 0, c = 4$) has complex roots $r_1 = 2i$ and $r_2 = -2i$. Since $-b/2a = 0$, the fundamental set of solutions consists of the functions

$$y_1(t) = \cos(2t) \qquad \text{and} \qquad y_2(t) = \sin(2t).$$

Their Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = 2\cos^2(2t) + 2\sin^2(2t) = 2.$$

1st approach: A particular solution can be obtained from

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{\cos(2t)}{2}\sin(2t)dt, \qquad u_2(t) = \int \frac{\cos(2t)}{2}\cos(2t)dt.$$

The function u_1 is

$$u_1(t) = -\frac{1}{2}\int \sin(2t)\cos(2t)dt = -\frac{1}{4}\int \sin(4t)dt = \frac{1}{16}\cos(4t).$$

The function u_2 is

$$u_2(t) = \frac{1}{2}\int \cos^2(2t)dt = \frac{1}{4}\int (1 + \cos(4t))dt = \frac{1}{4}\left(t + \frac{1}{4}\sin(4t)\right).$$

Thus

$$\psi(t) = \frac{1}{16}\cos(4t)\cos(2t) + \frac{1}{4}\left(t + \frac{1}{4}\sin(4t)\right)\sin(2t) = \frac{1}{16}\cos(2t) + \frac{t}{4}\sin(2t).$$

The solution of the initial-value problem has the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\cos(2t) + c_2\sin(2t) + \psi(t).$$

From the condition $y(0) = 0$, we get

$$0 = y(0) = c_1 y_1(0) + c_2 y_2(0) + \psi(0) = c_1 + \frac{1}{16} \qquad \text{and} \qquad c_1 = -\frac{1}{16}.$$

Since

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \psi'(t)$$
$$\psi'(t) = -\frac{1}{8}\sin(2t) + \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t) = \frac{1}{8}\sin(2t) + \frac{t}{2}\cos(2t),$$

the second initial condition $y'(0) = 0$ further implies

$$0 = y'(0) = 2c_2 + \psi'(0) = 2c_2 \qquad \text{and} \qquad c_2 = 0.$$

The solution of the starting problem is now

$$y(t) = -\frac{1}{16}y_1(t) + \psi(t) = -\frac{1}{16}\cos(2t) + \frac{1}{16}\cos(2t) + \frac{t}{4}\sin(2t) = \frac{t}{4}\sin(2t).$$

2nd approach (guessing): Consider the complex-valued problem $\frac{d^2y}{dt^2} + 4y = \mathrm{e}^{2it}$ and guess its particular solution $\phi(t) = At\mathrm{e}^{2it}$. From

$$\phi'(t) = A\mathrm{e}^{2it} + 2iAt\mathrm{e}^{2it}, \qquad \phi''(t) = 4iA\mathrm{e}^{2it} - 4At\mathrm{e}^{2it},$$

we get

$$\mathrm{e}^{2it} = \phi''(t) + 4\phi(t) = 4iA\mathrm{e}^{2it} - 4At\mathrm{e}^{2it} - 4At\mathrm{e}^{2it} = 4iA\mathrm{e}^{2it}.$$

Thus $A = 1/(4i) = -i/4$ and

$$\phi(t) = -\frac{i}{4}t\mathrm{e}^{2it} = -\frac{i}{4}t(\cos(2t) + i\sin(2t)) = \frac{1}{4}t\sin(2t) - \frac{i}{4}t\cos(2t).$$

The particular solution of the starting problem is $\psi(t) = \mathrm{Re}(\phi(t)) = \frac{t}{4}\sin(2t)$. In the first approach we derived the general form of the solution

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \psi(t).$$

Applying the initial condition, we obtain $0 = y(0) = c_1$. From

$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t),$$

we now get $0 = y'(0) = 2c_2$, i.e. $c_2 = 0$. The general solution of the IVP is $y(t) = \psi(t) = \frac{t}{4}\sin(2t)$.

3rd approach (Laplace transforms): Let $Y(s) = \mathcal{L}\{y(t)\}$. Then

$$Y(s) = \frac{1}{s^2+4}\mathcal{L}\{\cos(2t)\} = \frac{1}{s^2+4}\frac{s}{s^2+4} = \frac{s}{(s^2+4)^2}.$$

We can write now

$$Y(s) = \frac{1}{2}\frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{1}{2}\mathcal{L}\{\sin(2t)\}\mathcal{L}\{\cos(2t)\} = \frac{1}{2}\mathcal{L}\{\sin(2t) * \cos(2t)\},$$

which will give us $y(t) = \frac{1}{2}\sin(2t) * \cos(2t)$. One can calculate this convolution and get $y(t) = \frac{t}{4}\sin(2t)$. Instead, notice the following

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = -\frac{2s}{(s^2+4)^2}.$$

Therefore

$$Y(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = -\frac{1}{4}\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\frac{1}{4}\frac{d}{ds}\mathcal{L}\{\sin(2t)\} = -\frac{1}{4}\mathcal{L}\{-t\,\sin(2t)\},$$

and consequently $y(t) = \frac{t}{4}\sin(2t)$.

Problem 2. (25 points)
Find a function $g(t)$, $t \geq 0$, such that

$$\mathcal{L}\{g(t)\} = \frac{s^2}{(s^2+9)^2}, \qquad s > 0.$$

Solution:
First we have that

$$\frac{s^2}{(s^2+9)^2} = s \cdot \frac{s}{(s^2+9)^2} = -\frac{s}{6}\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\frac{s}{6}\frac{d}{ds}\mathcal{L}\{\sin(3t)\}$$
$$= -\frac{s}{6}\mathcal{L}\{-t\sin(3t)\} = s\,\mathcal{L}\{\frac{1}{6}t\sin(3t)\}.$$

If we introduce $H(s) = \mathcal{L}\{h(t)\}$ with $h(t) = \frac{1}{6}t\sin(3t)$, then

$$\frac{s^2}{(s^2+9)^2} = s\,H(s) = \mathcal{L}\{h'(t)\} + h(0) = \mathcal{L}\{h'(t)\}.$$

This gives us

$$g(t) = h'(t) = \frac{1}{6}\sin(3t) + \frac{t}{2}\cos(3t).$$

We could have also start from

$$\frac{s^2}{(s^2+9)^2} = \frac{s}{s^2+9}\cdot\frac{s}{s^2+9} = \mathcal{L}\{\cos(3t)\}\mathcal{L}\{\cos(3t)\} = \mathcal{L}\{\cos(3t)*\cos(3t)\}.$$

The convolution $g(t) = \cos(3t)*\cos(3t)$ is

$$\cos(3t)*\cos(3t) = \int_0^t \cos(3t-3u)\cos(3u)du = \int_0^t (\cos(3t)\cos(3u)+\sin(3t)\sin(3u))\cos(3u)du$$
$$= \cos(3t)\int_0^t \cos^2(3u)du + \sin(3t)\int_0^t \sin(3u)\cos(3u)du$$
$$= \frac{1}{2}\cos(3t)\int_0^t (1+\cos(6u))du + \frac{1}{2}\sin(3t)\int_0^t \sin(6u)du$$
$$= \frac{1}{2}\cos(3t)\left(t+\frac{1}{6}\sin(6t)\right) + \frac{1}{2}\sin(3t)\frac{1}{6}(-\cos(6t)+1)$$
$$= \frac{t}{2}\cos(3t) + \frac{1}{12}\sin(6t)\cos(3t) - \frac{1}{12}\sin(3t)\cos(6t) + \frac{1}{12}\sin(3t)$$
$$= \frac{t}{2}\cos(3t) + \frac{1}{12}\sin(3t) + \frac{1}{12}\sin(3t) = \frac{t}{2}\cos(3t) + \frac{1}{6}\sin(3t).$$

Problem 3. (50 points)
Consider the following initial-value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t\,\mathrm{e}^t, \qquad y(0) = y'(0) = 0. \tag{P}$$

(i) Find fundamental set of solutions for the homogeneous differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0.$$

(ii) Find a particular solution of the initial-value problem (P).

(iii) Using the results from (i) and (ii), find the solution of (P) that satisfies the given initial conditions.

(iv) Solve the problem (P) using Laplace transforms.

Solution:
(i) The characteristic equation $r^2 - 2r + 1 = 0$ has one real root $r = 1$. The fundamental set of solutions is

$$y_1(t) = \mathrm{e}^t, \qquad y_2(t) = t\,\mathrm{e}^t,$$

with the Wronskian

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \mathrm{e}^t(\mathrm{e}^t + t\,\mathrm{e}^t) - t\,\mathrm{e}^{2t} = \mathrm{e}^{2t}.$$

(ii) The particular solution can be obtained from

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{t\,\mathrm{e}^t}{\mathrm{e}^{2t}} t\,\mathrm{e}^t dt = -\int t^2 dt = -\frac{t^3}{3}$$
$$u_2(t) = \int \frac{t\,\mathrm{e}^t}{\mathrm{e}^{2t}} \mathrm{e}^t dt = \int t dt = \frac{t^2}{2}.$$

Thus

$$\psi(t) = -\frac{t^3}{3}\mathrm{e}^t + \frac{t^3}{2}\mathrm{e}^t = \frac{t^3}{6}\mathrm{e}^t.$$

We can also guess particular solution as $\psi(t) = t^2(A_1t + A_0)\mathrm{e}^t$ and obtain the same function ($A_0 = 0$, $A_1 = 1/6$).
(iii) The solution of the problem (P) has the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\mathrm{e}^t + c_2t\,\mathrm{e}^t + \frac{t^3}{6}\mathrm{e}^t.$$

From the condition $y(0) = 0$, we immediately get $c_1 = 0$. Since

$$y'(t) = c_1\mathrm{e}^t + c_2(1+t)\mathrm{e}^t + \frac{3t^2 + t^3}{6}\mathrm{e}^t,$$

the second initial condition $y'(0) = 0$ further implies $c_2 = 0$. The solution of (P) is now

$$y(t) = \psi(t) = \frac{t^3}{6}\mathrm{e}^t.$$

(iv) Let $Y(s) = \mathcal{L}\{y(t)\}$. Applying $y(0) = y'(0) = 0$, we have that

$$Y(s) = \frac{1}{s^2 - 2s + 1}\mathcal{L}\{t\,\mathrm{e}^t\} = \frac{1}{(s-1)^2}\left(-\frac{d}{ds}F(s)\right)$$

where $F(s) = \mathcal{L}\{\mathrm{e}^t\} = (s-1)^{-1}$, $s > 1$. From

$$\frac{d}{ds}F(s) = -\frac{1}{(s-1)^2},$$

we conclude

$$Y(s) = \frac{1}{(s-1)^4}.$$

On lectures we showed

$$\mathcal{L}\{t^n\mathrm{e}^{at}\} = \frac{n!}{(s-a)^{n+1}}, \qquad n \in \mathbb{N}.$$

With $a = 1$, $n = 3$, this implies

$$Y(s) = \frac{1}{(s-1)^4} = \frac{1}{3!}\mathcal{L}\{t^3\mathrm{e}^t\} = \mathcal{L}\{\frac{1}{6}t^3\mathrm{e}^t\},$$

and finally

$$y(t) = \frac{1}{6}t^3\mathrm{e}^t.$$