MATH 4512, FINAL EXAM
December 20, 2018
SOLUTIONS

1. (25 points)

Use Laplace transform to find a solution of the following initial-value problem

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = \mathrm{e}^{3t}, \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = -1.$$

Let

$$Y(s) = \mathcal{L}\{y(t)\}, \qquad F(s) = \mathcal{L}\{\mathrm{e}^{3t}\} = \frac{1}{s-3}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2 - 6s + 9}\big((s-6) - 1 + F(s)\big) \\ &= \frac{1}{(s-3)^2}\left((s-3) - 4 + \frac{1}{s-3}\right) \\ &= \frac{1}{s-3} - \frac{4}{(s-3)^2} + \frac{1}{(s-3)^3}. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{L}\{t\mathrm{e}^{3t}\} &= -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{3t}\} = -\frac{d}{ds}\left(\frac{1}{s-3}\right) = \frac{1}{(s-3)^2} \\ \mathcal{L}\{t^2\mathrm{e}^{3t}\} &= -\frac{d}{ds}\mathcal{L}\{t\mathrm{e}^{3t}\} = -\frac{d}{ds}\left(\frac{1}{(s-3)^2}\right) = \frac{2}{(s-3)^3}, \end{aligned}$$

we further have

$$\begin{aligned} Y(s) &= \frac{1}{s-3} - \frac{4}{(s-3)^2} + \frac{1}{(s-3)^3} = \mathcal{L}\{\mathrm{e}^{3t}\} - 4\mathcal{L}\{t\mathrm{e}^{3t}\} + \frac{1}{2}\mathcal{L}\{t^2\mathrm{e}^{3t}\} \\ &= \mathcal{L}\{\mathrm{e}^{3t} - 4t\mathrm{e}^{3t} + \frac{1}{2}t^2\mathrm{e}^{3t}\} \end{aligned}$$

and consequently

$$y(t) = \mathrm{e}^{3t} - 4t\mathrm{e}^{3t} + \frac{1}{2}t^2\mathrm{e}^{3t} = (1 - 4t + \frac{1}{2}t^2)\mathrm{e}^{3t}.$$

2. (25 points)

Transform the differential equation

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = 0$$

into a system of differential equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{1}$$

(a) Determine stability of all solutions of (1).

(b) Find a general solution of (1).

(c) Find equilibrium points of (1) and examine their stability.

(d) Draw the phase portrait of (1).

Introducing $x(t) = u(t)$, $y(t) = \dot{u}(t)$, we have that

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= 2x(t) - y(t). \end{aligned}$$

Therefore

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

(a) The characteristic polynomial of the matrix A is

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = \lambda(1+\lambda) - 2 = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1).$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 1$. Since one eigenvalue has positive real part, all solutions of (1) are unstable.

(b) First we will determine eigenvectors corresponding to $\lambda_1 = -2$ and $\lambda_2 = 1$.
From

$$(A - \lambda_1 I)v = (A + 2I)v = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we can choose $v^1 = [1, -2]^\top$, while from

$$(A - \lambda_2 I)v = (A - I)v = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

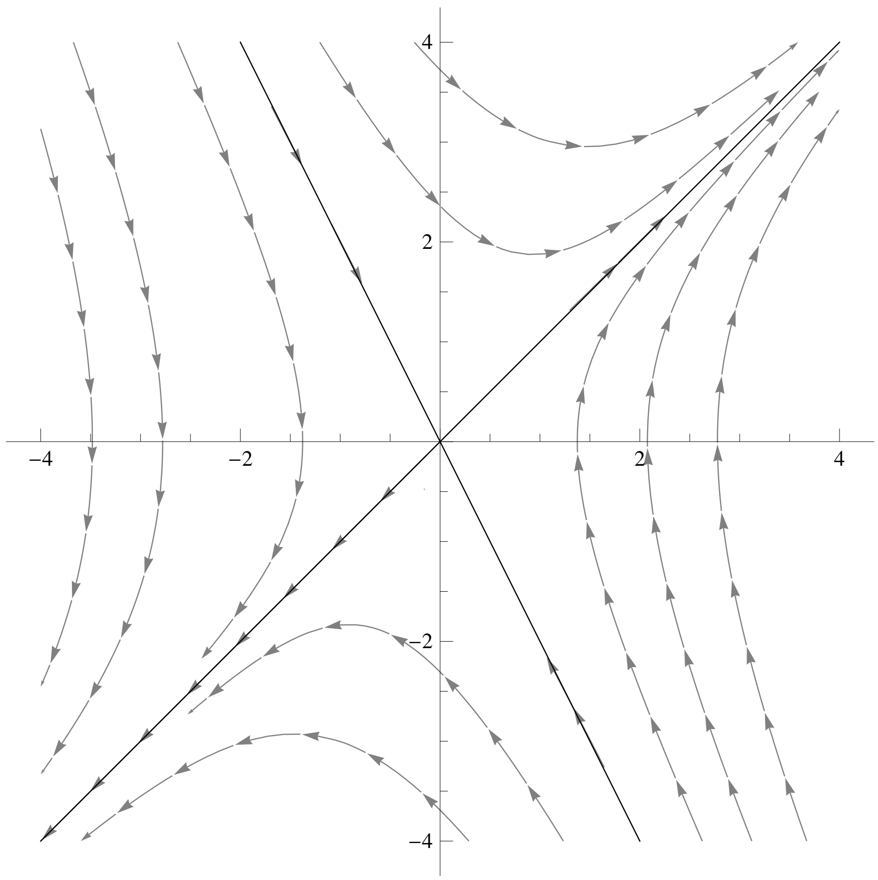
we can choose $v^2 = [1, 1]^\top$. The general solution of (1) has the form

$$c_1 \mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) The system (1) has only one equilibrium point $[0,0]^\top$. This constant solution is saddle since

$$-2 = \lambda_1 < 0 < \lambda_2 = 1.$$

(d) The phase portrait of (1):



3. (25 points)

Consider the system of nonlinear differential equations

$$\begin{aligned}\dot{x} &= y + 3yx^2 \\ \dot{y} &= 4x.\end{aligned}$$

(a) Find orbits of the system.

(b) Find orthogonal trajectories of the family of curves obtained in (a).

(a) Consider the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4x}{y + 3yx^2} = \frac{4x}{y(1+3x^2)}.$$

Since it is separable, we can solve it in the following way:

$$\begin{aligned} y\frac{dy}{dx} &= \frac{4x}{1+3x^2} \\ \frac{d}{dx}\frac{y^2}{2} &= \frac{4x}{1+3x^2} \\ \frac{y^2}{2} &= \int \frac{4x}{1+3x^2}dx = \frac{4}{6}\int \frac{ds}{s} = \frac{2}{3}\ln|1+3x^2| + c_1 \\ y^2 &= \frac{4}{3}\ln(1+3x^2) + c. \end{aligned}$$

The only equilibrium value is $[0,0]^\top$. Thus, the orbits of the given system are

- the equilibrium point $[0,0]^\top$,
- the curve $y^2 = \frac{4}{3}\ln(1+3x^2)$,
- the curves $y^2 = \frac{4}{3}\ln(1+3x^2) + c$, $c \neq 0$.

(b) Let $F(x,y,c) = \frac{4}{3}\ln(1+3x^2) - y^2 + c$. From

$$F_x = \frac{8x}{1+3x^2}, \qquad F_y = -2y,$$

we obtain that the orthogonal trajectories y need to satisfy

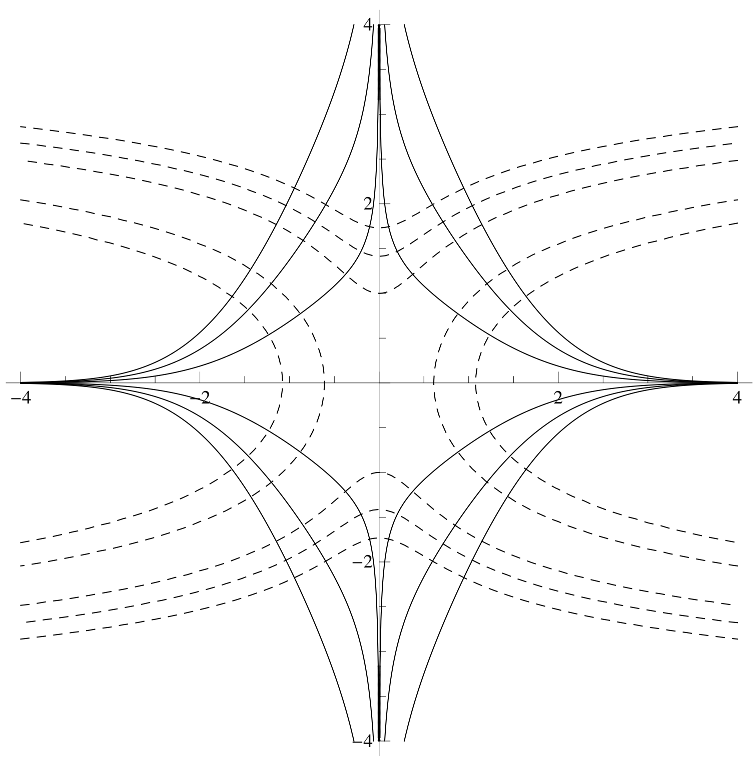
$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{y(1+3x^2)}{4x}.$$

This is a separable problem and we solve it as follows:

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= -\frac{1+3x^2}{4x} \\
\frac{d}{dx}(\ln|y|) &= -\frac{1+3x^2}{4x} \\
\ln|y| &= -\int \frac{1+3x^2}{4x}dx = -\frac{1}{4}\int\frac{dx}{x} - \frac{3}{4}\int x dx \\
\ln|y| &= -\frac{1}{4}\ln|x| - \frac{3x^2}{8} + c_1 \\
|y| &= c|x|^{-1/4}\exp(-3x^2/8).
\end{aligned}$$

Orthogonal trajectories are the curves that satisfy

$$|y| = c|x|^{-1/4}\exp(-3x^2/8).$$



Orbits $y^2 = \frac{4}{3}\ln(1+3x^2) + c$ (dashed) and
orthogonal trajectories $|y| = c|x|^{-1/4}\exp(-3x^2/8)$ (solid).

4. (25 points)

The charge $Q(t)$ on the capacitor within closed electric circuit satisfies the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t),$$

with an inductance L, a resistance R, a capacitance C, and a voltage source $E(t)$ at time t. If $L = 1\text{H}$, $R = 2\Omega$, $C = 0.2\text{F}$, and $E(t) = 17\cos(2t)\text{V}$, find charge $Q(t)$ that satisfies

$$Q(0) = 0\,\text{C}, \quad \frac{dQ}{dt}(0) = 9\,\text{A}.$$

Useful identities and properties:

$$\begin{aligned}
\sin(2\theta) &= 2\sin\theta\cos\theta, & \int e^{at}\sin(bt)dt &= \frac{e^{at}}{a^2+b^2}(a\sin(bt) - b\cos(bt)) + c,\\
\cos(2\theta) &= 2\cos^2\theta - 1, & \int e^{at}\cos(bt)dt &= \frac{e^{at}}{a^2+b^2}(a\cos(bt) + b\sin(bt)) + c.
\end{aligned}$$

With the given data, we are solving the following initial-value problem

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + 5Q = 17\cos(2t), \qquad Q(0) = 0, \quad \frac{dQ}{dt}(0) = 9.$$

The characteristic equation

$$r^2 + 2r + 5 = 0$$

has complex roots $r_1 = -1 + 2i$, $r_2 = -1 - 2i$. The functions

$$y_1(t) = e^{-t}\cos(2t), \qquad y_2(t) = e^{-t}\sin(2t),$$

form the fundamental set of solutions. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = 2e^{-2t}.$$

Now we search a particular solution ψ in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

With $E(t) = 17\cos(2t)$, we obtain

$$\begin{aligned}
u_1(t) &= -\int \frac{E(t)y_2(t)}{W[y_1,y_2](t)}dt = -\frac{17}{2}\int e^t\sin(2t)\cos(2t)dt = -\frac{17}{4}\int e^t\sin(4t)dt\\
&= -\frac{1}{4}e^t(\sin(4t) - 4\cos(4t))\\
u_2(t) &= \int \frac{E(t)y_1(t)}{W[y_1,y_2](t)}dt = \frac{17}{2}\int e^t\cos^2(2t)dt = \frac{17}{4}\int e^t(1+\cos(4t))dt\\
&= \frac{17}{4}e^t + \frac{1}{4}e^t(\cos(4t) + 4\sin(4t)).
\end{aligned}$$

The particular solution is

$$\begin{aligned}\psi(t) &= u_1(t)y_1(t) + u_2(t)y_2(t)\\ &= -\frac{1}{4}(\sin(4t) - 4\cos(4t))\cos(2t) + \frac{17}{4}\sin(2t) + \frac{1}{4}(\cos(4t) + 4\sin(4t))\sin(2t)\\ &= \cos(2t) + 4\sin(2t).\end{aligned}$$

Here we have used double-angle formulae.

The particular solution can be found using guessing

$$\psi(t) = a\cos(2t) + b\sin(2t), \qquad a, b \text{ are constants.}$$

Then from

$$\begin{aligned}\psi'(t) &= -2a\sin(2t) + 2b\cos(2t)\\ \psi''(t) &= -4a\cos(2t) - 4b\sin(2t)\end{aligned}$$

we get

$$17\cos(2t) = \psi''(t) + 2\psi'(t) + 5\psi(t) = (-4a + b)\sin(2t) + (a + 4b)\cos(2t).$$

Finally, $a = 1$, $b = 4$, and $\psi(t) = \cos(2t) + 4\sin(2t)$.

The general solution of the starting problem has the form

$$Q(t) = c_1y_1(t) + c_2y_2(t) + \psi(t) = c_1\mathrm{e}^{-t}\cos(2t) + c_2\mathrm{e}^{-t}\sin(2t) + \psi(t).$$

The initial condition $Q(0) = 0$, and $y_1(0) = 1$, $y_2(0) = 0$, $\psi(0) = 1$, imply

$$0 = Q(0) = c_1 + 1, \qquad c_1 = -1.$$

Now, from

$$\begin{aligned}y_1'(t) &= -\mathrm{e}^{-t}(\cos(2t) + 2\sin(2t))\\ y_2'(t) &= \mathrm{e}^{-t}(2\cos(2t) - \sin(2t))\\ \psi'(t) &= 8\cos(2t) - 2\sin(2t),\end{aligned}$$

the second condition $Q'(0) = 9$ further implies

$$9 = c_1y_1'(0) + c_2y_2'(0) + \psi'(0) = -c_1 + 2c_2 + 8, \qquad c_2 = 0.$$

The final solution (the charge on the capacitor at time t) is

$$\begin{aligned}Q(t) &= -y_1(t) + \psi(t)\\ &= -\mathrm{e}^{-t}\cos(2t) + \cos(2t) + 4\sin(2t)\\ &= (1 - \mathrm{e}^{-t})\cos(2t) + 4\sin(2t).\end{aligned}$$

This initial-value problem

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + 5Q = 17\cos(2t), \qquad Q(0) = 0, \quad \frac{dQ}{dt}(0) = 9,$$

can also be solved using Laplace transforms. Let

$$Y(s) = \mathcal{L}\{Q(t)\}, \qquad F(s) = \mathcal{L}\{17\cos(2t)\} = \frac{17s}{s^2+4}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2+2s+5}\big(9+F(s)\big) = \frac{1}{s^2+2s+5}\left(9+\frac{17s}{s^2+4}\right) \\ &= \frac{9s^2+17s+36}{(s^2+2s+5)(s^2+4)} = -\frac{s+1}{s^2+2s+5} + \frac{s+8}{s^2+4}, \end{aligned}$$

where in the last step we use partial fractions. Then

$$-\frac{s+1}{s^2+2s+5} = -\frac{s+1}{(s+1)^2+4} = -\mathcal{L}\{\mathrm{e}^{-t}\cos(2t)\}$$

$$\frac{s+8}{s^2+4} = \frac{s}{s^2+4} + 4\frac{2}{s^2+4} = \mathcal{L}\{\cos(2t)\} + 4\mathcal{L}\{\sin(2t)\},$$

and consequently

$$\begin{aligned} Y(s) &= -\mathcal{L}\{\mathrm{e}^{-t}\cos(2t)\} + \mathcal{L}\{\cos(2t)\} + 4\mathcal{L}\{\sin(2t)\} \\ Q(t) &= -\mathrm{e}^{-t}\cos(2t) + \cos(2t) + 4\sin(2t) \\ &= (1-\mathrm{e}^{-t})\cos(2t) + 4\sin(2t). \end{aligned}$$