MATH 4512, FINAL EXAM
December 18, 2019
SOLUTIONS

1. (16 points)

(a) (8 points) Find $\mathcal{L}\{t\ \sin t\}$.

(b) (8 points) Using the result from (a), find a function $f(t)$ such that

$$\mathcal{L}\{f(t)\} = \frac{2s-4}{(s^2-4s+5)^2}.$$

(a) From $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we obtain

$$\mathcal{L}\{t\sin t\} = -\mathcal{L}\{-t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

(b) Notice that

$$\frac{2s-4}{(s^2-4s+5)^2} = \frac{2(s-2)}{((s-2)^2+1)^2} = F(s-2),$$

where

$$F(s) = \frac{2s}{(s^2+1)^2} = \mathcal{L}\{t\sin t\}.$$

Then

$$\mathcal{L}\{f(t)\} = \frac{2s-4}{(s^2-4s+5)^2} = F(s-2) = \mathcal{L}\{\mathrm{e}^{2t}\,t\sin t\},$$

and $f(t) = t\,\mathrm{e}^{2t}\,\sin t$.

2. (18 points)

(a) (3 points) Write an initial-value problem describing vibrations of a small object of mass 1 kg attached to a spring with spring constant 9 N/m, and immersed in a viscous medium with damping constant 6 N s/m. At time $t = 0$, the mass, which is hanging in rest, is acted upon by an external force $F(t) = \cos t$ N.

(b) (8 points) Find a particular solution $\psi(t)$ of the differential equation from (a).

(c) (7 points) Solve the initial-value problem from (a).

(a) Here $m = 1$, $k = 9$, $c = 6$, and $F(t) = \cos t$. The IVP describing position y of this object in dependence of time t, with initial conditions $y(0) = y'(0) = 0$, is

$$y''(t) + 6y'(t) + 9y(t) = \cos t, \qquad y(0) = y'(0) = 0.$$

(b) The characteristic equation for $y''(t) + 6y'(t) + 9y(t) = 0$ is $r^2 + 6r + 9 = (r + 3)^2 = 0$ has a double root $r = -3$.

We will use guessing for the particular solution $\phi(t)$ of the complex-valued problem

$$y''(t) + 6y'(t) + 9y(t) = \mathrm{e}^{i\,t}.$$

Let $\phi(t) = A\,\mathrm{e}^{i\,t}$. Then $\phi'(t) = A\,i\,\mathrm{e}^{i\,t}$, $\phi''(t) = -A\,\mathrm{e}^{i\,t}$, and

$$\mathrm{e}^{i\,t} = \phi''(t) + 6\phi'(t) + 9\phi(t) = (-A + 6A\,i + 9A)\mathrm{e}^{i\,t} = (8 + 6i)A\,\mathrm{e}^{i\,t}.$$

We obtain

$$A = \frac{1}{8 + 6i} = \frac{8 - 6i}{100} = \frac{4}{50} - \frac{3}{50}i,$$

and

$$\phi(t) = \left(\frac{4}{50} - \frac{3}{50}i\right)\mathrm{e}^{i\,t} = \left(\frac{4}{50} - \frac{3}{50}i\right)(\cos t + i \sin t)$$

$$= \frac{4}{50}\cos t + \frac{3}{50}\sin t + i\left(\frac{4}{50}\sin t - \frac{3}{50}\cos t\right).$$

The particular solution $\psi(t)$ of the differential equation $y''(t) + 6y'(t) + 9y(t) = \cos t$ is

$$\psi(t) = \operatorname{Re}\phi(t) = \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

(c) The general solution is

$$y(t) = (c_1 + c_2\,t)\mathrm{e}^{-3t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

From $y(0) = 0$ we obtain

$$0 = y(0) = c_1 + \frac{4}{50}, \qquad c_1 = -\frac{4}{50} = -\frac{2}{25}.$$

Since

$$y'(t) = c_2\, \mathrm{e}^{-3t} - 3(c_1 + c_2\, t)\mathrm{e}^{-3t} - \frac{4}{50}\sin t + \frac{3}{50}\cos t,$$

the initial condition $y'(0) = 0$ implies

$$0 = y'(0) = c_2 - 3c_1 + \frac{3}{50}, \qquad c_2 = 3c_1 - \frac{3}{50} = -\frac{15}{50} = -\frac{3}{10}.$$

The solution of the IVP is

$$y(t) = \left(-\frac{2}{25} - \frac{3}{10}\, t\right) \mathrm{e}^{-3t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t.$$

3. (32 points) Consider the linear system of differential equations

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

(a) (5 points) Determine stability of all solutions to $\dot{x} = Ax$.

(b) (10 points) Find the general solution to $\dot{x} = Ax$.

(c) (10 points) Find e^{At}.

(d) (7 points) Solve the initial-value problem

$$\dot{x} = Ax, \qquad x(0) = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}.$$

(a) The characteristic polynomial of the matrix A is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= -(1+\lambda)(1 - 2\lambda + \lambda^2 + 4) = -(1+\lambda)(\lambda^2 - 2\lambda + 5).$$

Eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = 1 + 2i$, and $\lambda_3 = 1 - 2i$. Since both λ_2 and λ_3 have positive real part, all solutions of the system $\dot{x} = Ax$ are unstable.

(b) Eigenvector for $\lambda_1 = -1$ satisfies $(A - \lambda_1 I)v = 0$. Then

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Adding second equation $2v_1 + 2v_2 - 2v_3 = 0$ and third equation $3v_1 + 2v_2 + 2v_3 = 0$, we obtain

$$5v_1 + 4v_2 = 0, \qquad v_1 = -\frac{4}{5}\, v_2.$$

From second equation it follows

$$v_3 = v_1 + v_2 = -\frac{4}{5}\, v_2 + v_2 = \frac{1}{5}\, v_2.$$

Every eigenvector corresponding to λ_1 has the form

$$v = \begin{bmatrix} -4/5 \\ 1 \\ 1/5 \end{bmatrix} v_2$$

and we can choose

$$v^1 = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}.$$

For eigenvalue $\lambda_2 = 1 + 2i$, we solve $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} -2-2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first equation immediately gives $v_1 = 0$, while from the second it follows $v_3 = -iv_2$. Then

$$v = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} v_2,$$

and we can choose

$$v = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}.$$

The complex-valued solution $\mathrm{e}^{\lambda_2 t} v$ can be written as

$$\mathrm{e}^{\lambda_2 t} v = \mathrm{e}^{(1+2i)t} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \mathrm{e}^{t}(\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$= \mathrm{e}^{t} \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix} + i\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}.$$

The general solution to $\dot{x} = Ax$ is

$$x(t) = c_1\,\mathrm{e}^{-t} \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + c_2\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix} + c_3\,\mathrm{e}^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}.$$

(c) The fundamental matrix solutions $X(t)$ for this system is

$$X(t) = \begin{bmatrix} -4\mathrm{e}^{-t} & 0 & 0 \\ 5\mathrm{e}^{-t} & \mathrm{e}^{t}\cos 2t & \mathrm{e}^{t}\sin 2t \\ \mathrm{e}^{-t} & \mathrm{e}^{t}\sin 2t & -\mathrm{e}^{t}\cos 2t \end{bmatrix}.$$

Then

$$X(0) = \begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

From

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_3]{-\frac{1}{4}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & 0 & 5/4 & 1 & 0 \\ 0 & 0 & 1 & -1/4 & 0 & -1 \end{array}\right]$$

we obtain

$$X(0)^{-1} = \begin{bmatrix} -1/4 & 0 & 0 \\ 5/4 & 1 & 0 \\ -1/4 & 0 & -1 \end{bmatrix}.$$

Finally,

$$e^{At} = X(t)X(0)^{-1} = \begin{bmatrix} e^{-t} & 0 & 0 \\ -\frac{5}{4}e^{-t} + \frac{5}{4}e^{t}\cos 2t - \frac{1}{4}e^{t}\sin 2t & e^{t}\cos 2t & -e^{t}\sin 2t \\ -\frac{1}{4}e^{-t} + \frac{5}{4}e^{t}\sin 2t + \frac{1}{4}e^{t}\cos 2t & e^{t}\sin 2t & e^{t}\cos 2t \end{bmatrix}.$$

(d) The initial-value problem

$$\dot{x} = Ax, \qquad x(0) = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}.$$

can be solved using the formula $x(t) = e^{At}x(0)$, or from the initial condition

$$\begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4c_1 \\ 5c_1 + c_2 \\ c_1 - c_3 \end{bmatrix}.$$

Then $c_1 = -1$, $c_2 = -5 - 5c_1 = 0$, and $c_3 = c_1 = -1$. The solution to the IVP is

$$x(t) = -e^{-t} \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} - e^{t} \begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix} = \begin{bmatrix} 4e^{-t} \\ -5e^{-t} - e^{t}\sin 2t \\ -e^{-t} + e^{t}\cos 2t \end{bmatrix}.$$

4. (22 points) Consider the autonomous nonlinear system of differential equations

$$\dot{x} = 4y$$
$$\dot{y} = 2x + xy^2.$$

(a) (7 points) Find orbits of the system.

(b) (7 points) Determine stability of equilibrium solutions of the system.

(c) (8 points) Write the nonlinear system as

$$\dot{z} = Az + g(z), \qquad z = \begin{bmatrix} x \\ y \end{bmatrix},$$

and draw the phase portrait of $\dot{z} = Az$.

(a) The differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2x + xy^2}{4y} = \frac{x(2+y^2)}{4y}$$

is separable, and we can solve it in the following way:

$$\int \frac{y}{2+y^2}\,dy = \frac{1}{4}\int x\,dx$$
$$\frac{1}{2}\ln(2+y^2) = \frac{x^2}{8} + c_1$$
$$\ln(2+y^2) = \frac{x^2}{4} + c_2$$
$$y^2 = c\,\mathrm{e}^{x^2/4} - 2.$$

The only equilibrium solution is $(0,0)$. Thus, the orbits of the given system are

- equilibrium point $(0,0)$,
- curves $y^2 = c\,\mathrm{e}^{x^2/4} - 2$, $c \neq 2$,
- four curves

  (1) $y = \sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x > 0$,

  (2) $y = \sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x < 0$,

  (3) $y = -\sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x > 0$,

  (4) $y = -\sqrt{2\,\mathrm{e}^{x^2/4} - 2}$, $x < 0$.

(b) The nonlinear system in the matrix form is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ xy^2 \end{bmatrix}.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 8.$$

The eigenvalues of A are $\lambda_1 = -\sqrt{8}$ and $\lambda_2 = \sqrt{8}$. Since one eigenvalue has positive real part, the equilibrium solution $(0,0)$ is unstable.

(c) In (b) we already derived the matrix form

$$\dot{z} = Az + g(z), \qquad z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}.$$

Eigenvalues of A are $\lambda_1 = -\sqrt{8}$ and $\lambda_2 = \sqrt{8}$, and the equilibrium solution $(0,0)$ is saddle. In order to draw the phase portrait for $\dot{z} = Az$, we will determine eigenvectors corresponding to λ_1, λ_2.

From

$$(A - \lambda_1 I)v = \begin{bmatrix} \sqrt{8} & 4 \\ 2 & \sqrt{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we obtain

$$\sqrt{8}\, v_1 + 4\, v_2 = 0, \qquad v_1 = -\frac{4}{\sqrt{8}}\, v_2 = -\sqrt{2}\, v_2,$$

and we can choose

$$v^1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.$$

From

$$(A - \lambda_2 I)v = \begin{bmatrix} -\sqrt{8} & 4 \\ 2 & -\sqrt{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

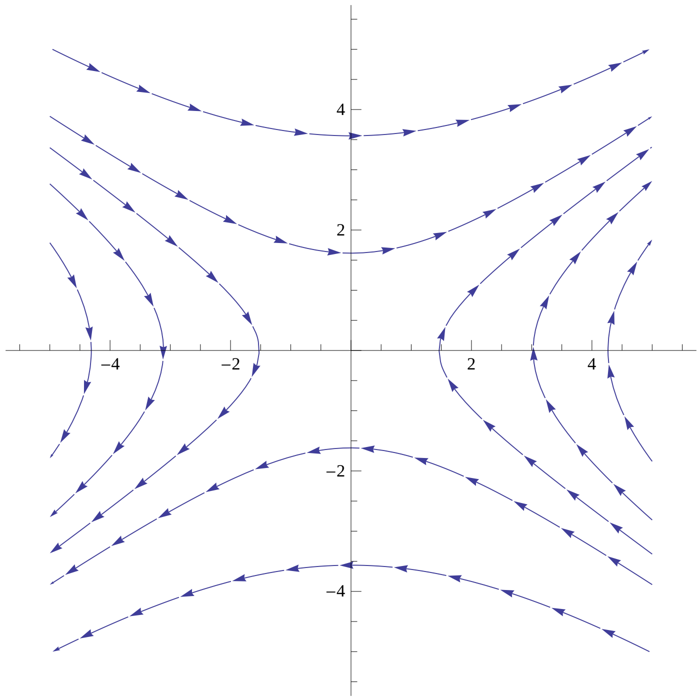
we obtain

$$-\sqrt{8}\, v_1 + 4\, v_2 = 0, \qquad v_1 = \frac{4}{\sqrt{8}}\, v_2 = \sqrt{2}\, v_2,$$

and we can choose

$$v^2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$$

The phase portrait of $\dot{z} = Az$:



5. (12 points) Determine whether the following statements are true or false. Explain your answer.

(a) (4 points) Initial-value problem

$$\frac{dy}{dt} = \mathrm{e}^t\,(y+1)^{2/3}, \qquad y(0) = -1,$$

has a unique solution $y(t) = -1$. True/False

(b) (4 points) Families of curves

$$y = c\,\tan x, \qquad y^2 + \sin^2 x = c,$$

are orthogonal. True/False

(c) (4 points) Vector-valued functions

$$x(t) = \begin{bmatrix} 3\mathrm{e}^t \\ -\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix}, \qquad y(t) = \begin{bmatrix} \sin t \\ \cos t \\ -\cos t \end{bmatrix}, \qquad z(t) = \begin{bmatrix} -\mathrm{e}^{-2t} \\ 1-\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}-1 \end{bmatrix},$$

are linearly independent. True/False

(a) False.

Notice that a constant function $y(t) = -1$ is one solution to the IVP.
Let $f(t,y) = \mathrm{e}^t\,(y+1)^{2/3}$ and $y_0 = -1$. Though function f is continuous for all $t \in \mathbb{R}$ and all $y \in \mathbb{R}$, its partial derivative

$$\frac{\partial f}{\partial y} = \mathrm{e}^t\,\frac{2}{3}(y+1)^{-1/3}$$

is not continuous in any neighborhood of y_0. Thus this IVP has more than one solution.

(b) True.

Starting from

$$F(x,y,c) = c\tan x - y, \qquad F_y = -1, \qquad F_x = \frac{c}{\cos^2 x} = \frac{y}{\tan x}\frac{1}{\cos^2 x} = \frac{y}{\sin x \cos x},$$

we can derive

$$\frac{dy}{dx} = \frac{F_y}{F_x} = \frac{-\sin x\cos x}{y}$$

$$\int y\,dy = -\int \sin x\cos x\,dx$$

$$\frac{y^2}{2} = -\frac{\sin^2 x}{2} + c_1$$

$$y^2 + \sin^2 x = c.$$

(c) False.

Choose $t = 0$ and consider a zero linear combination $c_1x(0) + c_2y(0) + c_3z(0) = 0$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 - c_3 \\ -c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}.$$

We can choose, for example, the following nonzero constants

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = 3,$$

and obtain $c_1x(0) + c_2y(0) + c_3z(0) = 0$. This implies that the vector-valued functions $x(t), y(t), z(t)$ are linearly dependent.