MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW7 - SOLUTIONS

1. (Section 4.1 - Exercise 6) Find all equilibrium values of the given system of differential equations

$$\frac{dx}{dt} = \cos y$$
$$\frac{dy}{dt} = \sin x - 1.$$

Equilibrium values are solutions to the system of nonlinear equations

$$\cos y = 0$$
$$\sin x - 1 = 0.$$

Solutions of the first equation $\cos y = 0$ are the points

$$y_l = \frac{\pi}{2} + l\,\pi, \qquad l \in \mathbb{Z},$$

while solutions of the second equation $\sin x - 1 = 0$ are

$$x_k = \frac{\pi}{2} + 2\,k\,\pi, \qquad k \in \mathbb{Z}.$$

Equilibrium points of this system are

$$(x_k, y_l), \qquad k, l \in \mathbb{Z}.$$

2. (Section 4.1 - Exercise 8) Find all equilibrium values of the given system of differential equations

$$\frac{dx}{dt} = x - y^2$$
$$\frac{dy}{dt} = x^2 - y$$
$$\frac{dz}{dt} = \mathrm{e}^z - x.$$

Equilibrium values are solutions to the system of nonlinear equations

$$x - y^2 = 0$$
$$x^2 - y = 0$$
$$\mathrm{e}^z - x = 0.$$

From $x = y^2$ we obtain $y^4 - y = 0$. Real solutions of this equation are

$$y_1 = 0, \qquad y_2 = 1.$$

Then $x_1 = y_1^2 = 0$ and $x_2 = y_2^2 = 1$. Notice that there is no value z_1 such that $\mathrm{e}^{z_1} = x_1 = 0$, while $z_2 = 0$, where $\mathrm{e}^{z_2} = x_2 = 1$. Therefore, the only equilibrium point is

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

3. (Section 4.2 - Exercise 9) Determine the stability or instability of all solutions of the following system of differential equations

$$\dot{x} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -2 & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda^3 - 4\lambda) - 2(-2\lambda^2 - 8) = \lambda^2(\lambda^2 + 4) + 4(\lambda^2 + 4) = (\lambda^2 + 4)^2.$$

For finding this determinant we used first-row element expansion.

The eigenvalues of the matrix A are $\lambda_1 = 2i$, $\lambda_2 = -2i$, both with multiplicity 2. It remains to check the number of linearly independent eigenvectors for each λ_1 and λ_2.

First consider the system $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first equation we obtain $-2iv_1 + 2v_2 = 0$, and $v_2 = iv_1$, while from the third $-2iv_3 + 2v_4 = 0$ it follows $v_4 = iv_3$. Thus every eigenvector v has the form

$$v = \begin{bmatrix} v_1 \\ iv_1 \\ v_3 \\ iv_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}.$$

Notice that

$$\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_1 = 2i$ that generate all other eigenvectors. Since the multiplicity of λ_1 is the same as the number of linearly independent eigenvectors, we proceed with analysis of the second eigenvalue.
Consider the system $(A - \lambda_2 I)v = 0$, i.e.

$$\begin{bmatrix} 2i & 2 & 0 & 0 \\ -2 & 2i & 0 & 0 \\ 0 & 0 & 2i & 2 \\ 0 & 0 & -2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the second equation $-2v_1 + 2iv_2 = 0$ we obtain $v_1 = iv_2$, while from the last equation $-2v_3 + 2iv_4 = 0$ it follows $v_3 = iv_4$. Thus every eigenvector v has the form

$$v = \begin{bmatrix} iv_2 \\ v_2 \\ iv_4 \\ v_4 \end{bmatrix} = v_2 \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$

Similarly to previous case, vectors

$$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_2 = -2i$ that generate all other eigenvectors.

Since the multiplicity of each λ_1 and λ_2 is the same as the number of corresponding linearly independent eigenvectors, we conclude that every solution of the starting system of DEs is stable.

4. (Section 4.2 - Exercise 10) Determine the stability or instability of all solutions of the following system of differential equations

$$\dot{x} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} x.$$

The characteristic polynomial of the system matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 1 & 0 \\ -2 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} + (-2)(-1)^{2+1} \begin{vmatrix} 2 & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda^3 - 4\lambda) + 2(2\lambda^2 + 8) = \lambda^2(\lambda^2 + 4) + 4(\lambda^2 + 4) = (\lambda^2 + 4)^2.$$

For finding this determinant we used first-column element expansion.

Eigenvectors for $\lambda_1 = 2i$ solve $(A - \lambda_1 I)v = 0$, i.e.

$$\begin{bmatrix} -2i & 2 & 1 & 0 \\ -2 & -2i & 0 & 1 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The third equation $-2iv_3 + 2v_4 = 0$ implies $v_4 = iv_3$. The second equation can be written as

$$0 = -2v_1 - 2iv_2 + v_4 = -2v_1 - 2iv_2 + iv_3 = -i(-2iv_1 + 2v_2 - v_3).$$

Combining the last relation with the first equation $-2iv_1 + 2v_2 + v_3 = 0$, we obtain $v_3 = 0$. Consequently $v_4 = 0$ and $v_2 = iv_1$. Every eigenvector v corresponding to $\lambda_1 = 2i$ can be represented as

$$v = \begin{bmatrix} v_1 \\ iv_1 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}.$$

Since the number of linearly independent eigenvectors is smaller than the multiplicity 2 of λ_1, we conclude that every solution of the starting system of DEs is unstable.

5. (Section 4.3 - Exercise 8) Verify that the origin is an equilibrium point of the following system of equations

$$\dot{x} = y + \cos y - 1$$
$$\dot{y} = -\sin x + x^3$$

and determine (if possible) whether it is stable or unstable.

Vector $[0,0]^\top$ is obviously an equilibrium point of this system.

First approach.
From the expansions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$$

we can write down $\dot{x} = y + \cos y - 1$, and $\dot{y} = x^3 - \sin x$, as

$$\dot{x} = y - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots = y + g_1(y)$$
$$\dot{y} = x^3 - x + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots = -x + g_2(x).$$

Then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1(y) \\ g_2(x) \end{bmatrix}.$$

The characteristic polynomial of the previous matrix is

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

Since its roots $\lambda_1 = i$, $\lambda_2 = -i$, both have zero real part, we cannot determine whether the vector $[0,0]^\top$ is stable or not.
(At this point of the course, we can only apply the theory from Sections 4.1-4.3).

Second approach.
Let $f_1(x,y) = y + \cos y - 1$ and $f_2(x,y) = -\sin x + x^3$. The Jacobian matrix for nonlinear vector-valued function

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

evaluated at the equilibrium point $[0,0]^\top$ is

$$A = \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(0,0) & \dfrac{\partial f_1}{\partial y}(0,0) \\ \dfrac{\partial f_2}{\partial x}(0,0) & \dfrac{\partial f_2}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 - \sin y \\ -\cos x + 3x^2 & 0 \end{bmatrix}_{x=0,y=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We obtained the same matrix and we can proceed as in the first approach.

6. (Section 4.3 - Exercise 10) Verify that the origin is an equilibrium point of the following system of equations

$$\dot{x} = \ln(1 + x + y^2)$$
$$\dot{y} = -y + x^3$$

and determine (if possible) whether it is stable or unstable.

Again the vector $[0, 0]^\top$ is obviously an equilibrium point of this system.

First approach.
Here we will use expansion

$$\ln(1 + x + y^2) = x + y^2 - \frac{(x + y^2)^2}{2} + \frac{(x + y^2)^3}{3} - \cdots .$$

Then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g(x, y) \\ x^3 \end{bmatrix},$$

where

$$g(x, y) = y^2 - \frac{(x + y^2)^2}{2} + \frac{(x + y^2)^3}{3} - \cdots .$$

The characteristic polynomial of the previous matrix is

$$p(\lambda) = \det \begin{bmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{bmatrix} = -(1 - \lambda)(1 + \lambda).$$

Since one eigenvalue of A has positive real part, the equilibrium value $[0, 0]^\top$ for this system is unstable.

Second approach.
Let $f_1(x, y) = \ln(1+x+y^2)$ and $f_2(x, y) = -y+x^3$. The Jacobian matrix for nonlinear vector-valued function

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

evaluated at the equilibrium point $[0, 0]^\top$ is

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x}(0,0) & \frac{\partial f_1}{\partial y}(0,0) \\ \frac{\partial f_2}{\partial x}(0,0) & \frac{\partial f_2}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} \frac{1}{1 + x + y^2} & \frac{2y}{1 + x + y^2} \\ 3x^2 & -1 \end{bmatrix}_{x=0, y=0}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We obtained the same matrix and we can proceed as in the first approach.