HW 7

Math 4512
Differential Equations with Applications

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Nasser M. Abbasi

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Contents

1 Section 4.1, problem 6 (page 377)

find all equilibrium values of the given system of differential equations.

$$\frac{dx}{dt} = \cos y$$
$$\frac{dy}{dt} = \sin x - 1$$

solution

The system can be written as $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$, where $\boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\boldsymbol{f} = \begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix}$. Equilibrium points are solution to $\begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This gives two equations to solve

$$\cos y = 0$$
$$\sin x - 1 = 0$$

The first equation has solution $y = \frac{\pi}{2} \pm 2n\pi$ for all integer n values. And the second equation is $\sin x = 1$ which has solution $x = \frac{\pi}{2} + 2n\pi$ for all integer n values. Since we want both components of $\boldsymbol{f}$ to be zero for equilibrium, then the common values that makes both zero at the same values is given by

$$\{x, y\} = \frac{\pi}{2} + 2n\pi$$

For all integer n. Here is partial list of values $\{x, y\} = \left\{\cdots, -\frac{7}{2}\pi, -\frac{3}{2}\pi, \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \cdots\right\}$. At any one of such values $\boldsymbol{f} = \begin{pmatrix} \cos y \\ \sin x - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2 Section 4.1, problem 8

find all equilibrium values of the given system of differential equations.

$$\frac{dx}{dt} = x - y^2$$
$$\frac{dy}{dt} = x^2 - y$$
$$\frac{dz}{dt} = e^z - x$$

solution

We need to find values of x, y, z which solves

$$x - y^2 = 0$$
$$x^2 - y = 0$$
$$e^z - x = 0$$

From the first equation $x = y^2$. From the third equation $e^z = x$ or $z = \ln x$. A solution that satisfies all these is

$$x = 1$$
$$y = 1$$
$$z = 0$$

At the above values the system is in equilibrium. No other real solutions exist.

3 Section 4.2, problem 9

Determine the stability or instability of all solutions of the following systems of differential equations

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \boldsymbol{x}$$

solution

The stability is determined from the eigenvalues. Therefore we need to find the eigenvalues of A first.

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
\begin{vmatrix} -\lambda & 2 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix} &= 0 \\
-\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} &= 0 \\
-\lambda \left(-\lambda \left(\lambda^2 + 4\right)\right) - 2 \left(-2 \left(\lambda^2 + 4\right)\right) &= 0 \\
\lambda^2 \left(\lambda^2 + 4\right) + 4 \left(\lambda^2 + 4\right) &= 0 \\
\left(\lambda^2 + 4\right) \left(\lambda^2 + 4\right) &= 0
\end{aligned}
$$

Hence roots are

$$
\begin{aligned}
\lambda_{1,2} &= 2i \qquad \text{multiplicity 2} \\
\lambda_{3,4} &= -2i \qquad \text{multiplicity 2}
\end{aligned}
$$

The real part is zero for all the above 4 eigenvalues. Since the real part is zero, then to check if it is stable, we need to check if the eigenvalue $2i$ and $-2i$ are defective or not. A defective eigenvalue is one which generates n linearly independent vectors where n is less than the multiplicity of the eigenvalue. So basically we need to find the eigenvectors associated with $\lambda = 2i$ and see if we obtain 2 linearly independent eigenvectors or not. If we obtain only one eigenvector, then the system is not stable. Same for $\lambda = -2i$.

Case $\lambda = 2i$

$$\begin{pmatrix}-\lambda & 2 & 0 & 0\\ -2 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 2\\ 0 & 0 & -2 & -\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-2i & 2 & 0 & 0\\ -2 & -2i & 0 & 0\\ 0 & 0 & -2i & 2\\ 0 & 0 & -2 & -2i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

The first two rows give same information, which is $-2v_1-2iv_2=0$. Or $v_1=-iv_2$. Row 3 and 4 also give same information, which is $-2v_3-2iv_4=0$ or $v_3=-iv_4$. Hence the eigenvector is

$$\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}-iv_2\\ v_2\\ -iv_4\\ v_4\end{pmatrix}=v_2\begin{pmatrix}-i\\1\\0\\0\end{pmatrix}+v_4\begin{pmatrix}0\\0\\-i\\1\end{pmatrix}$$

Hence we found two linearly independent eigenvector associated with $\lambda=2i$ which is the same number as the multiplicity which is 2. Hence this eigenvalue is not defective. Therefore stable eigenvalue. Now we check for the other eigenvalue $\lambda=-2i$ using same method.

Case $\lambda=-2i$

$$\begin{pmatrix}-\lambda & 2 & 0 & 0\\ -2 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 2\\ 0 & 0 & -2 & -\lambda\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2i & 2 & 0 & 0\\ -2 & 2i & 0 & 0\\ 0 & 0 & 2i & 2\\ 0 & 0 & -2 & 2i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

The first two rows give same information, which is $-2v_1+2iv_2=0$. Or $v_1=iv_2$. Row 3 and 4 also give same information, which is $-2v_3+2iv_4=0$ or $v_3=iv_4$. Hence the eigenvector is

$$\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}iv_2\\ v_2\\ iv_4\\ v_4\end{pmatrix}=v_2\begin{pmatrix}i\\1\\0\\0\end{pmatrix}+v_4\begin{pmatrix}0\\0\\i\\1\end{pmatrix}$$

Hence we found two linearly independent eigenvector associated with $\lambda=-2i$ which is the same number as the multiplicity which is 2. Hence this eigenvalue is not defective. Therefore stable eigenvalue.

In summary, the eigenvalues are $\{2i, 2i, -2i, -2i\}$ and the associated eigenvectors are

$$\overbrace{\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix}}^{\lambda=2i}, \overbrace{\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix}}^{\lambda=-2i}$$

Therefore the system is stable.

4 Section 4.2, problem 10

Determine the stability or instability of all solutions of the following systems of differential equations

$$\dot{\boldsymbol{x}}=\begin{pmatrix}0&2&1&0\\-2&0&0&1\\0&0&0&2\\0&0&-2&0\end{pmatrix}\boldsymbol{x}$$

solution

The stability is determined from the eigenvalues. Therefore we need to find the eigenvalues of A first.

$$
\begin{aligned}
|A-\lambda I| &= 0\\
\begin{vmatrix}-\lambda&2&1&0\\-2&-\lambda&0&1\\0&0&-\lambda&2\\0&0&-2&-\lambda\end{vmatrix} &= 0\\
-\lambda\begin{vmatrix}-\lambda&0&1\\0&-\lambda&2\\0&-2&-\lambda\end{vmatrix}-2\begin{vmatrix}-2&0&1\\0&-\lambda&2\\0&-2&-\lambda\end{vmatrix}+\begin{vmatrix}-2&-\lambda&1\\0&0&2\\0&0&-\lambda\end{vmatrix} &= 0\\
-\lambda\left(-\lambda\begin{vmatrix}-\lambda&2\\-2&-\lambda\end{vmatrix}+\begin{vmatrix}0&-\lambda\\0&-2\end{vmatrix}\right)-2\left(-2\begin{vmatrix}-\lambda&2\\-2&-\lambda\end{vmatrix}+\begin{vmatrix}0&-\lambda\\0&-2\end{vmatrix}\right)-2\,(0) &= 0\\
-\lambda\left(-\lambda\left(\lambda^2+4\right)+0\right)-2\left(-2\left(\lambda^2+4\right)+0\right) &= 0\\
\lambda^2\left(\lambda^2+4\right)+4\left(\lambda^2+4\right) &= 0\\
\left(\lambda^2+4\right)\left(\lambda^2+4\right) &= 0
\end{aligned}
$$

Hence the eigenvalues are the same as last problem.

$$
\begin{aligned}
\lambda_{1,2} &= 2i \qquad \text{multiplicity } 2\\
\lambda_{3,4} &= -2i \qquad \text{multiplicity } 2
\end{aligned}
$$

The real part is zero for all the above 4 eigenvalues. Since the real part is zero, then to check if it is stable, we need to check if the eigenvalue $2i$ and $-2i$ are defective or not.

A defective eigenvalue is one which generates n linearly independent vectors where n is less than the multiplicity of the eigenvalue. So basically we need to find the eigenvectors associated with $\lambda=2i$ and see if we obtain 2 linearly independent eigenvectors or not. If we obtain only one eigenvector, then the system is not stable. Same for $\lambda=-2i$.

Case $\lambda = 2i$

$$\begin{pmatrix} -\lambda & 2 & 1 & 0 \\ -2 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & 2 & 1 & 0 \\ -2 & -2i & 0 & 1 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} -2iv_1 + 2v_1 + v_3 &= 0 \\ -2v_1 - 2iv_2 + v_4 &= 0 \\ -2iv_3 + 2v_4 &= 0 \\ -2v_3 - 2iv_4 &= 0 \end{aligned}$$

Third and fourth equations gives same information which is $-2v_3 = 2iv_4$ or $v_3 = -iv_4$. Substituting this into first two equations gives

$$\begin{aligned} -2iv_1 + 2v_1 - iv_4 &= 0 \\ -2v_1 - 2iv_2 + v_4 &= 0 \end{aligned}$$

Multiplying second equation by $-i$ and adding the two equations gives $-2iv_4 = 0$. Hence $v_4 = 0$, which implies $v_3 = 0$. Therefore the above reduces to

$$\begin{aligned} -2iv_1 + 2v_1 &= 0 \\ -2v_1 - 2iv_2 &= 0 \end{aligned}$$

These two equations give the same information which is $-2v_1 = 2iv_2$ or $v_1 = -iv_2$. Therefore the eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} -iv_2 \\ v_2 \\ 0 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So only one eigenvector was found. But the multiplicity of the eigenvalue is two. Hence this eigenvalue is defective. Therefore the system is unstable. No need to check the second eigenvalue because if one eigenvalue with zero real part is defective then that is enough to make the system unstable.

5 Section 4.3, problem 8

Verify that the origin is an equilibrium point of each of the following systems of equations and determine, if possible, whether it is stable or unstable.

$$\begin{aligned} \dot{x} &= y + \cos y - 1 \\ \dot{y} &= -\sin x + x^3 \end{aligned}$$

solution

At $x = 0, y = 0$ the above becomes

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \end{aligned}$$

Hence origin $(0,0)$ is equilibrium point. To check if it is stable equilibrium or not, we find the Jacobian matrix, evaluate it at the origin and check the eigenvalues that results. The Jacobian is

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 - \sin y \\ -\cos x + 2x^2 & 0 \end{pmatrix}$$

At $x = 0, y = 0$ the above becomes

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence

$$p(\lambda) = |J - \lambda I| = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

Therefore $\lambda = \pm i$. The real part is zero. Since this is a nonlinear system, then we are not able to determine the stability of equilibrium at the origin.

6 Section 4.3, problem 10

Verify that the origin is an equilibrium point of each of the following systems of equations and determine, if possible, whether it is stable or unstable.

$$
\begin{aligned}
\dot{x} &= \ln\left(1 + x + y^2\right) \\
\dot{y} &= -y + x^3
\end{aligned}
$$

Solution

At $x = 0, y = 0$ the above becomes

$$
\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0
\end{aligned}
$$

Hence origin $(0, 0)$ is equilibrium point. To check if it is stable equilibrium or not, we find the Jacobian matrix, evaluate it at the origin and check the eigenvalues that results. The Jacobian is

$$
J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+x+y^2} & \frac{2y}{1+x+y^2} \\ 3x^2 & -1 \end{pmatrix}
$$

At $x = 0, y = 0$ the above becomes

$$
J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

Hence

$$
\begin{aligned}
p(\lambda) &= |J - \lambda I| \\
&= \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} \\
&= (1-\lambda)(-1-\lambda) \\
&= -1 - \lambda + \lambda + \lambda^2 \\
&= \lambda^2 - 1
\end{aligned}
$$

Hence eigenvalues are $\lambda = \pm 1$. Since one of the eigenvalues is positive, the origin is not a stable equilibrium point.