MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW5 - SOLUTIONS

1. (Section 3.1 - Exercise 4) Convert the pair of second-order equations

$$\frac{d^2y}{dt^2} + 3\frac{dz}{dt} + 2y = 0, \qquad \frac{d^2z}{dt^2} + 3\frac{dy}{dt} + 2z = 0$$

into a system of 4 first-order equations for the variables

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = z, \qquad \text{and} \qquad x_4 = z'.$$

Using new variables, differential equations can be expressed as

$$\frac{dx_2}{dt} + 3x_4 + 2x_1 = 0, \qquad \frac{dx_4}{dt} + 3x_2 + 2x_3 = 0.$$

The system of differential equations with unknown functions x_1, x_2, x_3, x_4 is

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -2x_1 - 3x_4 \\ \frac{dx_3}{dt} &= x_4 \\ \frac{dx_4}{dt} &= -3x_2 - 2x_3. \end{aligned}$$

The matrix form of this system is

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 \end{bmatrix} x(t), \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}.$$

2. (Section 3.2 - Exercise 4) Determine whether the set of all elements $x = [x_1, x_2, x_3]^\top$ where $x_1 + x_2 + x_3 = 1$ forms a vector space under the properties of vector addition and scalar multiplication.

Let V denote the given set of vectors, i.e.

$$V = \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1\}.$$

Consider vectors $x, y \in V$ where

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$x + y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin V,$$

we conclude that V is not a vector space.

3. (Section 3.3 - Exercise 16) Find a basis for $\mathbb{R}^3$ which includes the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

We only need to find a vector $x \in \mathbb{R}^3$ that is independent to given vectors. The choice for x is not unique, and each student can get a different answer. For example, let

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

A zero linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

implies

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 3c_2 \\ 4c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then $c_2 = 0$, $c_1 = 0$ and $c_3 = 0$. Thus, these three vectors are linearly independent and form a basis for $\mathbb{R}^3$.

4. (Section 3.4 - Exercise 6) For the differential equation

$$\dot{x} = \begin{bmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{bmatrix} x$$

determine whether the given solutions

$$x^1(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ 0 \end{bmatrix}, \qquad x^2(t) = \begin{bmatrix} 0 \\ e^{4t} \\ e^{4t} \end{bmatrix}, \qquad x^3(t) = \begin{bmatrix} e^{6t} \\ 0 \\ e^{6t} \end{bmatrix},$$

are a basis for the set of all solutions.

In order to show linear independence of the solutions $x^1(t), x^2(t), x^3(t)$, it is sufficient to prove that the vectors

$$x^1(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad x^2(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad x^3(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

are linearly independent. Their zero linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

implies

$$\begin{bmatrix} c_1 + c_3 \\ c_1 + c_2 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Substituting $c_1 = -c_3$ and $c_2 = -c_3$ into $c_1 + c_2 = 0$, we obtain $-2c_3 = 0$. Thus $c_1 = c_2 = c_3 = 0$ and vectors $x^1(0), x^2(0), x^3(0)$ are linearly independent.

5. (Section 3.5 - Exercise 6) Compute the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

One of the ways to find the determinant of the given matrix is:

$$\begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{vmatrix} \overset{-R_2+R_4}{=\!=\!=\!=} \begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$\overset{4^{th}\ row\ exp.}{=\!=\!=\!=\!=} (-1)(-1)^{4+2} \begin{vmatrix} 2 & 6 & 3 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} + 1(-1)^{4+4} \begin{vmatrix} 2 & -1 & 6 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix}$$

$$= -(4-6-3-12) + (-1+18-6) = 17 + 11 = 28.$$

6. (Section 3.6 - Exercise 10) Find the inverse, it is exists, of the given matrix

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Then

$$\det A = \begin{vmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0,$$

and A^{-1} exists. The cofactor matrix C for A is

$$C = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\theta & \cos\theta \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ \sin\theta & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{vmatrix} & +\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} & -\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & -\sin\theta \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = A.$$

Thus $\text{adj}A = C^\top = A^\top$ and

$$A^{-1} = \frac{1}{\det A}\,\text{adj}A = A^\top = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$