HW 5

Math 4512
Differential Equations with Applications

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Nasser M. Abbasi

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Contents

1 Section 3.1, problem 4

Convert the pair of second-order equations

$$y''(t) + 3z'(t) + 2y(t) = 0$$
$$z''(t) + 3y'(t) + 2z(t) = 0$$

into a system of 4 first-order equations for the variables $x_1 = y, x_2 = y', x_3 = z, x_4 = z'$

<u>Solution</u>

$$x_1 = y, x_2 = y', x_3 = z, x_4 = z'$$

Taking derivative gives

$$\dot{x}_1 = y', \dot{x}_2 = y'', \dot{x}_3 = z', \dot{x}_4 = z''$$
$$\dot{x}_1 = x_2, \dot{x}_2 = -\left(3z' + 2y\right), \dot{x}_3 = x_4, \dot{x}_4 = -\left(3y' + 2z\right)$$

Hence

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_4 - 2x_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -3x_2 - 2x_3$$

Or In Matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

2 Section 3.2, problem 4

Determine whether the given set of elements $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1 + x_2 + x_3 = 1$ form a vector space under the properties of vector addition and scalar multiplication defined in Section 3.1.

Solution

We need to check that using vector addition + and scalar multiplication c the following is true. For any x, y in V then $(x + y)$ is still in V. And for x in V then cx is still in V.

Checking for addition This fails. Here is an example. Let $x = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, y = \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}$. Both x, y are in V, but vector addition gives

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

We see that the resulting vector is not in V, because sum of its elements $1 + \frac{1}{3} + \frac{2}{3} = 2 \neq 1$. Hence not in V.

So it does not form a vector space.

3 Section 3.3, problem 16

Find basis for $\Re^3$ which includes the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\3\\4\end{pmatrix}$

Solution

We need to find 3rd vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ such that it is linearly independent to above two vectors. If we take the cross product of the above two vectors, then we get a vector that is perpendicular to the plane that the two given vectors span. This will give us the third vector we need

$$\begin{aligned}\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\3\\4\end{pmatrix} &= \begin{vmatrix}i & j & k\\1 & 1 & 0\\1 & 3 & 4\end{vmatrix}\\ &= i(4) - j(4) + k(3-1)\\ &= 4i - 4j + 2k\end{aligned}$$

Hence the third vector is $\begin{pmatrix}4\\-4\\2\end{pmatrix}$.

To verify this result, we now check that $c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\3\\4\end{pmatrix} + c_3\begin{pmatrix}4\\-4\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ implies $c_1 = 0, c_2 = 0, c_3 = 0$ as only solution. Writing the above as

$$\begin{pmatrix}1 & 1 & 4\\1 & 3 & -4\\0 & 4 & 2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence augmented matrix is

$$\begin{pmatrix}1 & 1 & 4 & 0\\1 & 3 & -4 & 0\\0 & 4 & 2 & 0\end{pmatrix}$$

Replacing row 2 by row 2 minus row 1

$$\begin{pmatrix}1 & 1 & 4 & 0\\0 & 2 & -8 & 0\\0 & 4 & 2 & 0\end{pmatrix}$$

Replacing row 3 by row 3 minus twice row 2

$$\begin{pmatrix}1 & 1 & 4 & 0\\0 & 2 & -8 & 0\\0 & 0 & 16 & 0\end{pmatrix}$$

This implies that Gaussian elimination gives

$$\begin{pmatrix}1 & 1 & 4\\0 & 2 & -8\\0 & 0 & 16\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Back substituting gives $c_3 = 0$. From second row we obtain $2c_2 - 8c_3 = 0$, hence $c_2 = 0$ and from first row $c_1 + c_2 + 4c_3 = 0$ hence $c_1 = 0$. This shows that

$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\3\\4\end{pmatrix}, \begin{pmatrix}4\\-4\\2\end{pmatrix}$$

Are linearly independent. Hence they span $\Re^3$ and form a basis.

4 Section 3.4, problem 6

Determine whether the given solutions are a basis for the set of all solutions

$$\dot{x} = \begin{pmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{pmatrix} x$$

$$x^1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}, x^2(t) = \begin{pmatrix} 0 \\ e^{4t} \\ e^{4t} \end{pmatrix}, x^3(t) = \begin{pmatrix} e^{6t} \\ 0 \\ e^{6t} \end{pmatrix}$$

Solution

We pick $t = 0$ to check linear independence (we can choose any t value, but $t = 0$ is the simplest). At $t = 0$ the given solutions become

$$x^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, x^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, x^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

we now check that $c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $c_1 = 0, c_2 = 0, c_3 = 0$ as only solution.
Writing the above as

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence augmented matrix is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Replacing row 2 by row 2 minus row 1 gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Replacing row 3 by row 3 minus row 2 gives

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

This implies that Gaussian elimination gives

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Back substituting gives $2c_3 = 0$ or $c_3 = 0$. From second row $c_2 - c_3 = 0$. Hence $c_2 = 0$ and from first row $c_1 + c_3 = 0$, hence $c_1 = 0$. This shows that

$$x^1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}, x^2(t) = \begin{pmatrix} 0 \\ e^{4t} \\ e^{4t} \end{pmatrix}, x^3(t) = \begin{pmatrix} e^{6t} \\ 0 \\ e^{6t} \end{pmatrix}$$

Are linearly independent. Hence they form basis for the set of all solutions for the system given above.

5 Section 3.5, problem 6

Compute the determinant of

$$\begin{pmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} + 6\begin{vmatrix} 1 & 0 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 1 \end{vmatrix} \tag{1}$$

But

$$\begin{aligned} \begin{vmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} &= 0 - \begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ &= -2 - 3 \\ &= -5 \end{aligned} \tag{2}$$

And

$$\begin{aligned} \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} &= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= -2 + 2 - 1 \\ &= -1 \end{aligned} \tag{3}$$

And

$$\begin{aligned} \begin{vmatrix} 1 & 0 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & 0 \end{vmatrix} &= \begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix} + 0 - \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} \\ &= 2 - (-1 - 3) \\ &= 6 \end{aligned} \tag{4}$$

And

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} + 0 + \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} \\ &= 3 + (-1 - 3) \\ &= -1 \end{aligned} \tag{5}$$

Substituting (2,3,4,5) into (1) gives

$$\begin{aligned} \begin{vmatrix} 2 & -1 & 6 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{vmatrix} &= 2(-5) + (-1) + 6(6) - 3(-1) \\ &= 28 \end{aligned}$$

6 Section 3.6, problem 10

Find the inverse if it exist of

$$A = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

<u>Solution</u>

$$A^{-1} = \frac{1}{|A|} adj(A)^T \tag{1}$$

But

$$\begin{aligned} |A| &= \cos\theta \begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} - 0 - \sin\theta \begin{vmatrix} 0 & 1 \\ \sin\theta & 0 \end{vmatrix} \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned} \tag{2}$$

And

$$\begin{aligned} adj(A) &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \end{aligned}$$

Hence

$$adj(A)^T = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3}$$

Substituting (2,3) into (1) gives

$$A^{-1} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$