MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW4 - SOLUTIONS

1. (Section 2.6 - Exercise 4) A small object of mass 1 kg is attached to a spring with spring constant 2 N/m. This spring-mass system is immersed in a viscous medium with damping constant 3 N·s/m. At time $t = 0$, the mass is lowered 1/2 m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

In this spring-mass system we have that $m = 1$, $c = 3$, $k = 2$, and zero external force $F(t)$. The corresponding initial-value problem is

$$y'' + 3y' + 2y = 0, \qquad y(0) = 0.5, \qquad y'(0) = 0.$$

The characteristic equation

$$r^2 + 3r + 2 = 0$$

has two real roots $r_1 = -1$ and $r_2 = -2$. The general solution is

$$y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t}$$

with its first derivative

$$y'(t) = -c_1 \mathrm{e}^{-t} - 2c_2 \mathrm{e}^{-2t}.$$

Initial conditions imply

$$0.5 = y(0) = c_1 + c_2, \qquad 0 = y'(0) = -c_1 - 2c_2.$$

Thus $c_1 = 1$, $c_2 = -0.5$ and

$$y(t) = \mathrm{e}^{-t} - 0.5\mathrm{e}^{-2t}.$$

Finally,

$$\lim_{t \to \infty} y(t) = 0.$$

2. (Section 2.9 - Exercise 18) Find the Laplace transform of the solution of the following initial value problem

$$y'' + y = t^2 \sin t, \qquad y(0) = y'(0) = 0.$$

The Laplace transform $Y(s)$ of the solution $y(t)$ can be obtained from the formula

$$Y(s) = \frac{1}{s^2+1} F(s),$$

where $F(s) = \mathcal{L}\{t^2 \sin t\}$. Next we find $F(s)$:

$$F(s) = -\mathcal{L}\{-t \cdot t \sin t\} = -\frac{d}{ds}\mathcal{L}\{t \sin t\}$$

$$= \frac{d}{ds}\mathcal{L}\{-t \sin t\} = \frac{d}{ds}\left(\frac{d}{ds}\mathcal{L}\{\sin t\}\right) = \frac{d^2}{ds^2}\mathcal{L}\{\sin t\}$$

$$= \frac{d^2}{ds^2}\frac{1}{s^2+1} = \frac{2(3s^2-1)}{(s^2+1)^3}.$$

Finally, the Laplace transform of the solution of the initial value problem is

$$Y(s) = \frac{2(3s^2-1)}{(s^2+1)^4}.$$

3. (Section 2.10 - Exercise 14) Find the inverse Laplace transform of the following function

$$\frac{1}{s(s+4)^2}.$$

Let

$$F(s) = \frac{1}{s(s+4)^2} = \frac{1}{s} \cdot \frac{1}{(s+4)^2}.$$

Notice that

$$\frac{1}{s} = \mathcal{L}\{1\}$$

and

$$\frac{1}{(s+4)^2} = -\frac{d}{ds}\frac{1}{s+4} = -\frac{d}{ds}\mathcal{L}\{\mathrm{e}^{-4t}\} = \mathcal{L}\{t\,\mathrm{e}^{-4t}\}.$$

Now

$$F(s) = \mathcal{L}\{1\} \cdot \mathcal{L}\{t\,\mathrm{e}^{-4t}\} = \mathcal{L}\{1 * t\,\mathrm{e}^{-4t}\}.$$

Thus, the inverse Laplace transform of $F(s)$ is

$$\begin{aligned} 1 * t\,\mathrm{e}^{-4t} &= \int_0^t u\,\mathrm{e}^{-4u}\,du = -\frac{1}{4}u\,\mathrm{e}^{-4u}\Big|_0^t + \frac{1}{4}\int_0^t \mathrm{e}^{-4u}\,du \\ &= -\frac{1}{4}t\,\mathrm{e}^{-4t} - \frac{1}{16}\mathrm{e}^{-4u}\Big|_0^t = -\frac{1}{4}t\,\mathrm{e}^{-4t} - \frac{1}{16}\mathrm{e}^{-4t} + \frac{1}{16}. \end{aligned}$$

4. (Section 2.10 - Exercise 20) Solve the following initial-value problem by the method of Laplace transforms:

$$y'' + y = t\sin t, \qquad y(0) = 1, \qquad y'(0) = 2.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{t\sin t\}$. Then

$$\begin{aligned} Y(s) &= \frac{1}{s^2+1}\left(s + 2 + F(s)\right) = \frac{s}{s^2+1} + \frac{2}{s^2+1} + \frac{1}{s^2+1}F(s) \\ &= \mathcal{L}\{\cos t\} + 2\mathcal{L}\{\sin t\} + \mathcal{L}\{\sin t\}\cdot\mathcal{L}\{t\sin t\} \\ &= \mathcal{L}\{\cos t + 2\sin t + \sin t * t\sin t\}. \end{aligned}$$

The solution of the starting initial-value problem is $y(t) = \cos t + 2\sin t + \sin t * t\sin t$. It remains to calculate the convolution between $\sin t$ and $t\sin t$. We proceed as follows:

$$\begin{aligned} \sin t * t\sin t &= \int_0^t \sin(t-u)u\sin u\,du = \int_0^t (\sin t\cos u - \cos t\sin u)u\sin u\,du \\ &= \sin t\int_0^t u\sin u\cos u\,du - \cos t\int_0^t u\sin^2 u\,du \\ &= \frac{1}{2}\sin t\int_0^t u\sin 2u\,du - \frac{1}{2}\cos t\int_0^t u(1-\cos 2u)\,du. \end{aligned}$$

The first integral is

$$\begin{aligned} \int_0^t u\sin 2u\,du &= -\frac{1}{2}u\cos 2u\Big|_0^t + \frac{1}{2}\int_0^t \cos 2u\,du \\ &= -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t, \end{aligned}$$

while the second integral is

$$\begin{aligned} \int_0^t u(1-\cos 2u)\,du &= \frac{t^2}{2} - \frac{1}{2}u\sin 2u\Big|_0^t + \frac{1}{2}\int_0^t \sin 2u\,du \\ &= \frac{t^2}{2} - \frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}. \end{aligned}$$

Then

$$
\begin{aligned}
\sin t * t\sin t &= \frac{1}{2}\sin t\left(-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right) \\
&\qquad - \frac{1}{2}\cos t\left(\frac{t^2}{2} - \frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}\right) \\
&= -\frac{t^2}{4}\cos t + \frac{t}{4}\left(\sin 2t\cos t - \cos 2t\sin t\right) \\
&\qquad + \frac{1}{8}\left(\cos 2t\cos t + \sin 2t\sin t\right) - \frac{1}{8}\cos t \\
&= -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t + \frac{1}{8}\cos t - \frac{1}{8}\cos t = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t.
\end{aligned}
$$

Finally

$$
y(t) = \cos t + 2\sin t - \frac{t^2}{4}\cos t + \frac{t}{4}\sin t = \left(1 - \frac{t^2}{4}\right)\cos t + \left(2 + \frac{t}{4}\right)\sin t.
$$