HW 4

Math 4512
Differential Equations with Applications

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Contents

1 Section 2.6, problem 4

A small object of mass 1 kg is attached to a spring with spring constant $k = 2$ N/m. This spring-mass system is immersed in a viscous medium with damping constant $c = 3$ N s/m. At time $t = 0$, the mass is lowered $\frac{1}{2}$ m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

Solution

The ODE is

$$my''(t) + cy' + ky = 0$$

Where $m = 1, c = 3, k = 2$. The above becomes

$$y''(t) + 3y' + 2y = 0$$

And initial conditions, using equilibrium position as $y = 0$ and hence below the equilibrium position y is taken as negative. Therefore

$$\begin{aligned} y(0) &= -\frac{1}{2} \\ y'(0) &= 0 \end{aligned}$$

The characteristic equation is

$$\begin{aligned} r^2 + 3r + 2 &= 0 \\ r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 4(2)} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 8} \\ &= \frac{-3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -2 \end{aligned}$$

Therefore the solution to the ODE is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \tag{1}$$

At $t = 0$ the above becomes

$$-\frac{1}{2} = c_1 + c_2 \tag{2}$$

Taking derivative of (1) gives

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

At $t = 0$ the above becomes

$$0 = -c_1 - 2c_2 \tag{3}$$

From (3) $c_1 = -2c_2$. Substituting into (2) gives

$$\begin{aligned} -\frac{1}{2} &= -2c_2 + c_2 \\ &= -c_2 \end{aligned}$$

Hence

$$c_2 = \frac{1}{2}$$

Therefore from (3)

$$\begin{aligned} 0 &= -c_1 - 2\left(\frac{1}{2}\right) \\ 0 &= -c_1 - 1 \\ c_1 &= -1 \end{aligned}$$

Hence the solution (1) becomes

$$y(t) = -e^{-t} + \frac{1}{2}e^{-2t}$$

We see now that as $t \to \infty$ the terms e^{-t}, e^{-2t} both go to zero. Therefore

$$\lim_{t\to\infty} y(t) = 0$$

Hence the mass will go back to equilibrium position $y = 0$ after long time.

The following is a plot of the solution above

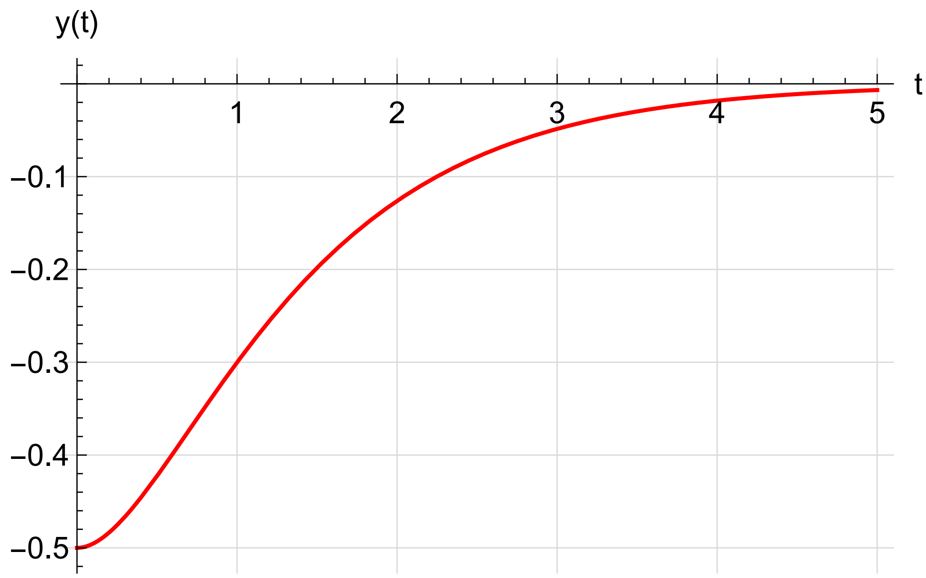

Figure 1: Plot showing solution in time

```
y[t_] := -Exp[-t] + 1/2 Exp[-2 t];
p = Plot[y[t], {t, 0, 5}, GridLines → Automatic, GridLinesStyle → LightGray,
        PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 2: Code used for the above plot

2 Section 2.9, problem 18

Find the Laplace transform of the solution of the following initial value problem.

$$\begin{aligned} y'' + y &= t^2 \sin t \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Solution

First we find the solution to the ODE then find its Laplace transform. The solution is given by

$$y(t) = y_h(t) + y_p(t)$$

Where $y_h(t)$ is the homogeneous solution to $y'' + y = 0$ and $y_p(t)$ is the particular solution to $y'' + y = t^2 \sin t$.

The characteristic equation is $r^2 + 1 = 0$. Hence $r^2 = -1$ or

$$r = \pm i$$

Therefore

$$y_h(t) = c_1 e^{it} + c_2 e^{-it} \tag{1}$$

To find the particular solution, we find the particular solution for $y'' + y = t^2 e^{it}$ instead, and then take the imaginary part. For this ODE, the RHS is $t^2 e^{it}$, therefore we start by guessing the particular solution to be

$$y_p = \left(At^2 + Bt + C\right) e^{it}$$

But from (1) we see that e^{it} is a fundamental solution to the homogeneous ode. Hence we adjust the above by multiplying by an extra t giving

$$y_p = \left(At^3 + Bt^2 + Ct\right) e^{it}$$

We now substitute the above back into $y'' + y = t^2 e^{it}$ in order to find A, B, C.

$$y_p' = \left(3At^2 + 2Bt + C\right) e^{it} + i\left(At^3 + Bt^2 + Ct\right) e^{it}$$

And

$$\begin{aligned} y_p'' &= (6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i^2\left(At^3 + Bt^2 + Ct\right) e^{it} \\ &= (6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} - \left(At^3 + Bt^2 + Ct\right) e^{it} \end{aligned}$$

Substituting the above in $y_p'' + y_p = t^2 e^{it}$ gives

$$\begin{aligned} &(6At + 2B) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} + i\left(3At^2 + 2Bt + C\right) e^{it} \\ &\qquad - \left(At^3 + Bt^2 + Ct\right) e^{it} + \left(At^3 + Bt^2 + Ct\right) e^{it} = t^2 e^{it} \end{aligned}$$

Canceling e^{it}

$$
\begin{aligned}
(6At+2B)+i\left(3At^2+2Bt+C\right)+i\left(3At^2+2Bt+C\right)-\left(At^3+Bt^2+Ct\right)+\left(At^3+Bt^2+Ct\right)&=t^2\\
(6At+2B)+i\left(3At^2+2Bt+C\right)+i\left(3At^2+2Bt+C\right)&=t^2\\
(6At+2B)+2i\left(3At^2+2Bt+C\right)&=t^2\\
(2B+2iC)+t\,(6A+4iB)+t^2\,(6iA)&=t^2
\end{aligned}
$$

Comparing coefficients gives

$$
\begin{aligned}
2B+2iC&=0\\
6A+4iB&=0\\
6Ai&=1
\end{aligned}
$$

Hence $A=\frac{1}{6i}=-\frac{i}{6}$. From the second equation

$$
\begin{aligned}
6\left(-\frac{i}{6}\right)+4iB&=0\\
-i+4iB&=0\\
B&=\frac{i}{4i}=\frac{1}{4}
\end{aligned}
$$

From the first equation

$$
\begin{aligned}
\frac{1}{2}+2iC&=0\\
2C&=-\frac{1}{2i}\\
C&=\frac{i}{4}
\end{aligned}
$$

Substituting the above back into $y_p=\left(At^3+Bt^2+Ct\right)e^{it}$ gives

$$
\begin{aligned}
y_p&=\left(-\frac{i}{6}t^3+\frac{1}{4}t^2+\frac{i}{4}t\right)e^{it}\\
&=\left(-\frac{i}{6}t^3+\frac{1}{4}t^2+\frac{i}{4}t\right)(\cos t+i\sin t)\\
&=-\frac{i}{6}t^3\cos t+\frac{1}{4}t^2\cos t+\frac{i}{4}t\cos t-\frac{i}{6}t^3\,(i\sin t)+\frac{1}{4}t^2\,(i\sin t)+\frac{i}{4}t\,(i\sin t)\\
&=-\frac{i}{6}t^3\cos t+\frac{1}{4}t^2\cos t+\frac{i}{4}t\cos t+\frac{1}{6}t^3\sin t+\frac{1}{4}it^2\sin t-\frac{1}{4}t\sin t\\
&=\left(\frac{1}{4}t^2\cos t+\frac{1}{6}t^3\sin t-\frac{1}{4}t\sin t\right)+i\left(-\frac{1}{6}t^3\cos t+\frac{1}{4}t\cos t+\frac{1}{4}t^2\sin t\right)
\end{aligned}
$$

The particular solution of the original ODE $y''+y=t^2\sin t$ is the imaginary part of the above which is

$$
y_p=-\frac{1}{6}t^3\cos t+\frac{1}{4}t\cos t+\frac{1}{4}t^2\sin t
$$

The homogeneous solution from (1) is $y_h(t)=c_1e^{it}+c_2e^{-it}$ which can be written using Euler relation as $y_h(t) = C_1 \cos t + C_2 \sin t$, therefore the general solution is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= C_1 \cos t + C_2 \sin t - \frac{1}{6} t^3 \cos t + \frac{1}{4} t \cos t + \frac{1}{4} t^2 \sin t \end{aligned} \tag{2}$$

What is left is to find C_1, C_2 from initial conditions. At $t = 0$ the above becomes

$$0 = C_1$$

Hence (2) becomes

$$y(t) = C_2 \sin t - \frac{1}{6} t^3 \cos t + \frac{1}{4} t \cos t + \frac{1}{4} t^2 \sin t \tag{3}$$

Taking derivative gives

$$y'(t) = C_2 \cos t - \frac{3}{6} t^2 \cos t + \frac{1}{6} t^3 \sin t + \frac{1}{4} \cos t - \frac{1}{4} t \sin t + \frac{2}{4} t \sin t + \frac{1}{4} t^2 \cos t$$

At $t = 0$ the above becomes

$$\begin{aligned} 0 &= C_2 + \frac{1}{4} \\ C_2 &= -\frac{1}{4} \end{aligned}$$

Hence (3) becomes the final solution

$$y(t) = -\frac{1}{4} \sin t - \frac{1}{6} t^3 \cos t + \frac{1}{4} t \cos t + \frac{1}{4} t^2 \sin t \tag{3A}$$

The following is a plot of the above solution. The solution blows up in time due to resonance.

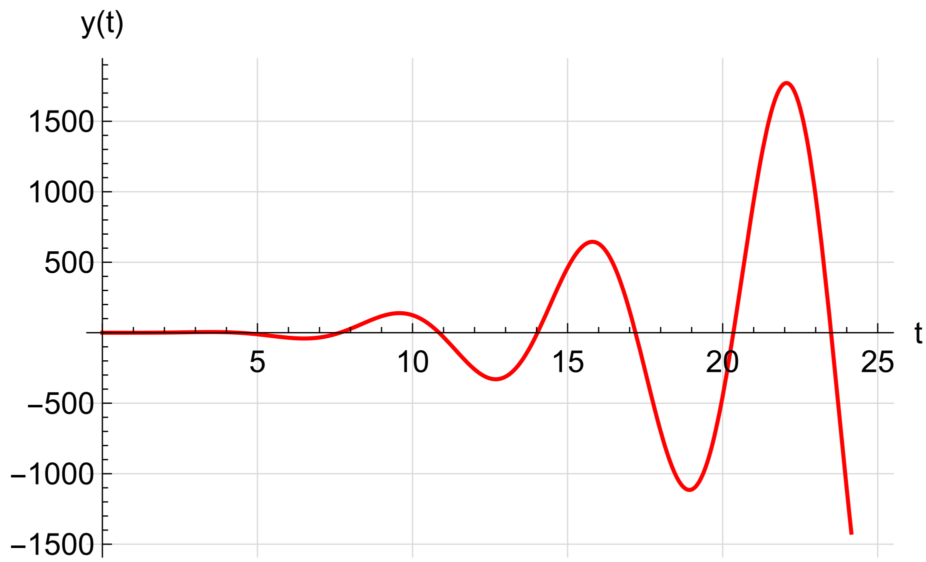

Figure 3: Plot showing solution in time

```
y[t_] := -1/4 Sin[t] - 1/6 t^3 Cos[t] + 1/4 t Cos[t] + 1/4 t^2 Sin[t];
p = Plot[y[t], {t, 0, 25}, GridLines → Automatic, GridLinesStyle → LightGray,
     PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 4: Code used for the above plot

The problem now asks to find the Laplace transform of the above. To obtain the Laplace Transform of the above, the following relations will be used (In the following, the notation ⇔ means the Laplace transform from left to right and the inverse Laplace transform from right to left).

$$\sin(at) \Leftrightarrow \frac{a}{a^2+s^2}$$
$$\cos(at) \Leftrightarrow \frac{s}{a^2+s^2}$$
$$t^n f(t) \Leftrightarrow (-1)^n \frac{d^n}{ds^n} F(s)$$

Hence

$$\sin(t) \Leftrightarrow \frac{1}{1+s^2} \tag{4}$$

$$\cos(t) \Leftrightarrow \frac{s}{1+s^2} \tag{5}$$

And

$$t\sin(t) \Leftrightarrow (-1)\frac{d}{ds}\mathscr{L}(\sin(t))$$

But

$$\begin{aligned}\frac{d}{ds}\mathscr{L}(\sin(t)) &= \frac{d}{ds}\left(\frac{1}{1+s^2}\right)\\ &= \frac{-2s}{\left(1+s^2\right)^2}\end{aligned}$$

Therefore

$$\begin{aligned}t\sin(t) &\Leftrightarrow (-1)\frac{-2s}{\left(1+s^2\right)^2}\\ &\Leftrightarrow \frac{2s}{\left(1+s^2\right)^2} \qquad (6)\end{aligned}$$

And

$$\mathscr{L}\left(t^2\sin(t)\right) = (-1)^2\frac{d^2}{ds^2}\mathscr{L}(\sin(t))$$

But

$$\begin{aligned}\frac{d^2}{ds^2}\mathscr{L}(\sin(t)) &= \frac{d}{ds}\left(\frac{-2s}{\left(1+s^2\right)^2}\right)\\ &= \frac{-2\left(1+s^2\right)^2 + 2s(2)\left(1+s^2\right)(2s)}{\left(1+s^2\right)^2}\\ &= \frac{-2\left(1+s^2\right)^2 + 8s^2\left(1+s^2\right)}{\left(1+s^2\right)^4}\\ &= \frac{-2\left(1+s^2\right) + 8s^2}{\left(1+s^2\right)^3}\\ &= \frac{-2+6s^2}{\left(1+s^2\right)^3}\end{aligned}$$

Therefore

$$\begin{aligned}\mathscr{L}\left(t^2 \sin(t)\right) &= (-1)^2 \frac{-2+6s^2}{\left(1+s^2\right)^3} \\ &= \frac{-2+6s^2}{\left(1+s^2\right)^3}\end{aligned} \tag{7}$$

And

$$\mathscr{L}(t\cos(t)) = (-1)\frac{d}{ds}\mathscr{L}(\cos(t))$$

But

$$\begin{aligned}\frac{d}{ds}\mathscr{L}(\cos(t)) &= \frac{d}{ds}\left(\frac{s}{1+s^2}\right) \\ &= \frac{\left(1+s^2\right) - s(2s)}{\left(1+s^2\right)^2} \\ &= \frac{1-s^2}{\left(1+s^2\right)^2}\end{aligned}$$

Therefore

$$\begin{aligned}\mathscr{L}(t\cos(t)) &= (-1)\frac{1-s^2}{\left(1+s^2\right)^2} \\ &= \frac{s^2-1}{\left(1+s^2\right)^2}\end{aligned} \tag{8}$$

And

$$\mathscr{L}\left(t^2\cos(t)\right) = (-1)^2 \frac{d^2}{ds^2}\mathscr{L}(\cos(t))$$

But

$$\frac{d^2}{ds^2}\mathscr{L}(\cos(t)) = \frac{d}{ds}\left(\frac{1-s^2}{\left(1+s^2\right)^2}\right)$$
$$= \frac{-2s\left(1+s^2\right)^2 - \left(1-s^2\right)(2)\left(1+s^2\right)(2s)}{\left(1+s^2\right)^4}$$
$$= \frac{-2s\left(1+s^2\right) - \left(1-s^2\right)(2)(2s)}{\left(1+s^2\right)^3}$$
$$= \frac{-2s\left(1+s^2\right) - 4s\left(1-s^2\right)}{\left(1+s^2\right)^3}$$
$$= \frac{-2s - 2s^3 - 4s + 4s^3}{\left(1+s^2\right)^3}$$
$$= \frac{-6s + 2s^3}{\left(1+s^2\right)^3}$$

Therefore

$$\mathscr{L}\left(t^2\cos(t)\right) = (-1)^2\left(\frac{-6s+2s^3}{\left(1+s^2\right)^3}\right)$$
$$= \frac{-6s+2s^3}{\left(1+s^2\right)^3} \tag{9}$$

And finally

$$\mathscr{L}\left(t^3\cos(t)\right) = (-1)^3\frac{d^3}{ds^3}\mathscr{L}(\cos(t))$$

But

$$\frac{d^3}{ds^3}\mathscr{L}(\cos(t)) = \frac{d}{ds}\left(\frac{-6s+2s^3}{\left(1+s^2\right)^3}\right)$$
$$= \frac{\left(-6+6s^2\right)\left(1+s^2\right)^3 - \left(-6s+2s^3\right)3\left(1+s^2\right)^2(2s)}{\left(1+s^2\right)^6}$$
$$= \frac{\left(-6+6s^2\right)\left(1+s^2\right) - \left(-6s+2s^3\right)3(2s)}{\left(1+s^2\right)^4}$$
$$= \frac{-6s^4+36s^2-6}{\left(1+s^2\right)^4}$$

Therefore

$$\begin{aligned}\mathscr{L}\left(t^3\cos(t)\right) &= (-1)^3\left(\frac{-6s^4+36s^2-6}{\left(1+s^2\right)^4}\right)\\ &= \frac{6s^4-36s^2+6}{\left(1+s^2\right)^4}\end{aligned} \tag{10}$$

Using (4,5,6,7,8,9,10) in (3A) gives

$$\begin{aligned}\mathscr{L}\left(y(t)\right) &= -\frac{1}{4}\mathscr{L}(\sin t) - \frac{1}{6}\mathscr{L}\left(t^3\cos t\right) + \frac{1}{4}\mathscr{L}(t\cos t) + \frac{1}{4}\mathscr{L}\left(t^2\sin t\right)\\ &= -\frac{1}{4}\frac{1}{1+s^2} - \frac{1}{6}\frac{6s^4-36s^2+6}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{s^2-1}{\left(1+s^2\right)^2} + \frac{1}{4}\frac{-2+6s^2}{\left(1+s^2\right)^3}\\ &= -\frac{1}{4}\frac{\left(1+s^2\right)^3}{\left(1+s^2\right)^4} - \frac{1}{6}\frac{6s^4-36s^2+6}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{\left(s^2-1\right)\left(1+s^2\right)^2}{\left(1+s^2\right)^4} + \frac{1}{4}\frac{\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\ &= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\ &= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\\ &= \frac{-\frac{1}{4}\left(1+s^2\right)^3 - \frac{1}{6}\left(6s^4-36s^2+6\right) + \frac{1}{4}\left(s^2-1\right)\left(1+s^2\right)^2 + \frac{1}{4}\left(-2+6s^2\right)\left(1+s^2\right)}{\left(1+s^2\right)^4}\end{aligned}$$

Which can be simplified to

$$\mathscr{L}\left(y(t)\right) = \frac{6s^2-2}{\left(1+s^2\right)^4}$$

3 section 2.10, problem 14

Find the inverse Laplace transform of each of the following functions

$$\frac{1}{s(s+4)^2}$$

Solution

Let

$$F(s)=\frac{1}{s(s+4)^2}$$

Using partial fractions

$$\frac{1}{s(s+4)^2}=\frac{A}{s}+\frac{B}{(s+4)}+\frac{C}{(s+4)^2}$$

Hence

$$\begin{aligned}\frac{1}{s(s+4)^2}&=\frac{A(s+4)^2+Bs(s+4)+Cs}{s(s+4)^2}\\&=\frac{A\left(s^2+16+8s\right)+Bs^2+4Bs+Cs}{s(s+4)^2}\\&=\frac{s^2(A+B)+s(8A+4B+C)+16A}{s(s+4)^2}\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}16A&=1\\8A+4B+C&=0\\A+B&=0\end{aligned}$$

From first equation $A=\frac{1}{16}$. From the third equation $B=-\frac{1}{16}$. From the second equation $8\left(\frac{1}{16}\right)+4\left(-\frac{1}{16}\right)+C=0$, hence $C=-\frac{1}{4}$, Therefore

$$\frac{1}{s(s+4)^2}=\frac{1}{16}\frac{1}{s}-\frac{1}{16}\frac{1}{(s+4)}-\frac{1}{4}\frac{1}{(s+4)^2}\tag{1}$$

Now we use the relation

$$\mathscr{L}^{-1}\left(\frac{1}{s}\right)=H_0(t)\tag{2}$$

For $\frac{1}{(s+4)}$ we will use the relation that

$$\mathscr{L}\left(e^{at}f(t)\right)=F(s-a)$$

Where here $\mathscr{L}\left(f(t)\right)=F(s)$. Therefore if we take $F(s)=\frac{1}{s}$ then we see that $\mathscr{L}\left(e^{-4t}f(t)\right)=\frac{1}{s+4}$. Therefore

$$\mathscr{L}^{-1}\left(\frac{1}{s+4}\right)=e^{-4t}\tag{3}$$

For $\frac{1}{(s+4)^2}$ we will use the relation that

$$\mathscr{L}\left(t^n f(t)\right) = (-1)^n F^{(n)}(s)$$

If we put $n = 1$ and $f(t) = e^{-4t}$ then

$$\begin{aligned}\mathscr{L}\left(te^{-4t}\right) &= (-1)\frac{d}{ds}\left(\frac{1}{s+4}\right)\\ &= (-1)\left(\frac{-1}{(s+4)^2}\right)\\ &= \frac{1}{(s+4)^2}\end{aligned}$$

Therefore we see that

$$\mathscr{L}^{-1}\left(\frac{1}{(s+4)^2}\right) = te^{-4t} \tag{4}$$

Substituting (2,3,4) back into (1) gives

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{1}{s(s+4)^2}\right) &= \frac{1}{16}\mathscr{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{16}\mathscr{L}^{-1}\left(\frac{1}{(s+4)}\right) - \frac{1}{4}\mathscr{L}^{-1}\left(\frac{1}{(s+4)^2}\right)\\ &= \frac{1}{16}H_0(t) - \frac{1}{16}e^{-4t} - \frac{1}{4}te^{-4t}\end{aligned}$$

Or, taking $t \geq 0$, then $H_0(t)$ can be replaced by 1 and above can be simplifies to

$$\mathscr{L}^{-1}\left(\frac{1}{s(s+4)^2}\right) = \frac{1}{16} - \frac{1}{16}e^{-4t} - \frac{1}{4}te^{-4t}$$

4 Section 2.10, problem 20

Solve the following initial-value problems by the method of Laplace transforms

$$y'' + y = t \sin t$$
$$y(0) = 1$$
$$y'(0) = 2$$

Solution

Taking the Laplace transform of the ODE gives

$$\mathscr{L}\left(y'' + y\right) = \mathscr{L}(t \sin t)$$
$$\mathscr{L} y'' + \mathscr{L} y = \mathscr{L}(t \sin t) \tag{1}$$

But from the above problem Section 2.9, problem 18 we have already found that

$$\mathscr{L}(t \sin t) = \frac{2s}{\left(1+s^2\right)^2}$$

And using

$$\mathscr{L} y'' = s^2 Y(s) - s y(0) - y'(0)$$

where $Y(s) = \mathscr{L}\left(y(t)\right)$, then (1) becomes

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{2s}{\left(1+s^2\right)^2}$$

Substituting the initial conditions into the above gives

$$s^2 Y(s) - s - 2 + Y(s) = \frac{2s}{\left(1+s^2\right)^2}$$
$$Y(s)\left(s^2+1\right) - s - 2 = \frac{2s}{\left(1+s^2\right)^2}$$
$$Y(s)\left(s^2+1\right) = \frac{2s}{\left(1+s^2\right)^2} + s + 2$$
$$Y(s) = 2\frac{s}{\left(1+s^2\right)^3} + \frac{s}{\left(s^2+1\right)} + 2\frac{1}{\left(s^2+1\right)} \tag{1A}$$

Now we ready to apply inverse Laplace transform using the relations

$$\mathscr{L} \cos t = \frac{s}{s^2+1} \tag{2A}$$
$$\mathscr{L} \sin t = \frac{1}{s^2+1} \tag{2B}$$

The only term left is $\frac{s}{\left(1+s^2\right)^3}$. But this is the same as $\frac{s}{\left(1+s^2\right)^2}\frac{1}{(1+s)}$ and we already found that $\frac{2s}{\left(1+s^2\right)^2} \Leftrightarrow t \sin t$ from above solving section 2.9, problem 18, and $\frac{1}{(1+s)} \Leftrightarrow \sin t$. Therefore we can use convolution as follows

$$\left(\frac{2s}{\left(1+s^2\right)^2}\right)\left(\frac{1}{(1+s)}\right) \Leftrightarrow \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Where we assume that $\frac{2s}{\left(1+s^2\right)^2} \Leftrightarrow f(t) = t\sin t$ and $\frac{1}{(1+s)} \Leftrightarrow g(t) = \sin t$. Hence the above becomes

$$\frac{2s}{\left(1+s^2\right)^3} \Leftrightarrow \int_0^t \tau \sin(\tau)\sin(t-\tau)\, d\tau \tag{2}$$

Let $A = \tau, B = t-\tau$ and using $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, then

$$\begin{aligned}\sin(\tau)\sin(t-\tau) &= \frac{1}{2}(\cos(\tau-(t-\tau)) - \cos(\tau+(t-\tau))) \\ &= \frac{1}{2}(\cos(2\tau-t) - \cos(t))\end{aligned}$$

Substituting the above in (2) gives

$$\begin{aligned}\frac{s}{\left(1+s^2\right)^3} &\Leftrightarrow \int_0^t \tau\left(\frac{1}{2}(\cos(2\tau-t) - \cos(t))\right) d\tau \\ &\Leftrightarrow \frac{1}{2}\int_0^t \tau\cos(2\tau-t)\, d\tau - \frac{1}{2}\int_0^t \tau\cos(t)\, d\tau \\ &\Leftrightarrow \frac{1}{2}\int_0^t \tau\cos(2\tau-t)\, d\tau - \frac{1}{2}\cos(t)\int_0^t \tau d\tau \end{aligned}\tag{3}$$

Using integration by parts on the first integral. Let $u = \tau, dv = \cos(2\tau-t), du = 1, v = \frac{\sin(2\tau-t)}{2}$, hence

$$\begin{aligned}\int_0^t \tau\cos(2\tau-t)\, d\tau &= \frac{1}{2}[\tau\sin(2\tau-t)]_0^t - \int_0^t \frac{\sin(2\tau-t)}{2} d\tau \\ &= \frac{1}{2}[t\sin(t)] - \frac{1}{2}\int_0^t \sin(2\tau-t)\, d\tau \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}[\cos(2\tau-t)]_0^t \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}[\cos(t) - \cos(-t)] \\ &= \frac{1}{2}t\sin(t) + \frac{1}{4}[\cos(t) - \cos(t)] \\ &= \frac{1}{2}t\sin(t)\end{aligned}$$

Substituting the above in (3) gives

$$\begin{aligned}\frac{s}{\left(1+s^2\right)^3} &\Leftrightarrow \frac{1}{2}\left(\frac{1}{2}t\sin(t)\right) - \frac{1}{4}t^2\cos(t) \\ &\Leftrightarrow \frac{1}{4}t\sin(t) - \frac{1}{4}t^2\cos(t)\end{aligned}\tag{2C}$$

We have found the inverse Laplace transform for all the terms. Substituting (2A,2B,2C) into (1A) gives

$$\mathscr{L}^{-1}Y(s) = \mathscr{L}^{-1}\frac{2s}{\left(1+s^2\right)^3} + \mathscr{L}^{-1}\frac{s}{\left(s^2+1\right)} + 2\mathscr{L}^{-1}\frac{1}{\left(s^2+1\right)}$$

$$y(t) = \left(\frac{1}{4}t\sin(t) - \frac{1}{4}t^2\cos(t)\right) + \cos t + 2\sin t$$

$$= -\frac{1}{4}t^2\cos t + \frac{1}{4}t\sin t + \cos t + 2\sin t$$

The following is a plot of the above solution. The solution blows up in time due to resonance.

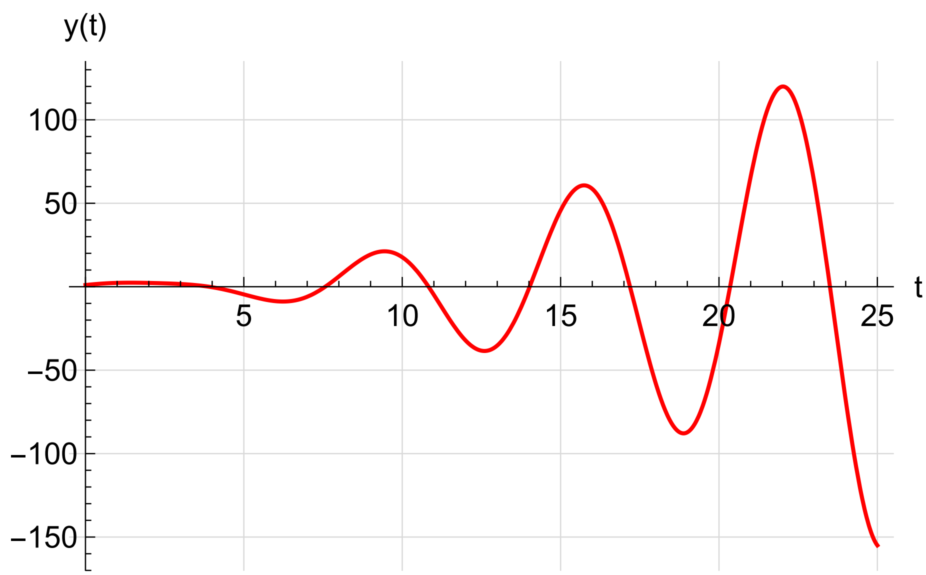

Figure 5: Plot showing solution in time

```
y[t_] := -1/4 t^2 Cos[t] + 1/4 t Sin[t] + Cos[t] + 2 Sin[t]
p = Plot[y[t], {t, 0, 25}, GridLines → Automatic, GridLinesStyle → LightGray,
        PlotStyle → Red, AxesLabel → {"t", "y(t)"}, BaseStyle → 12];
```

Figure 6: Code used for the above plot