MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS
HW3 - SOLUTIONS

1. (Section 2.1 - Exercise 11) Let $y_1 = t^2$ and $y_2(t) = t\,|t|$.
   (a) Show that y_1 and y_2 are linearly dependent on the interval $0 \le t \le 1$.
   (b) Show that y_1 and y_2 are linearly independent on the interval $-1 \le t \le 1$.
   (c) Show that $W[y_1, y_2](t)$ is identically zero.
   (d) Show that y_1 and y_2 can never be two solutions of

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

   on the interval $-1 < t < 1$, if both p and q are continuous in this interval.

(a) When $0 \le t \le 1$ we have that $|t| = t$ and $y_2(t) = t^2 = c\,y_1(t)$, with $c = 1$. Thus, functions y_1 and y_2 are linearly dependent.

(b) If we assume that y_1 and y_2 are linearly dependent on the interval $-1 \le t \le 1$, then there exists a constant c such that

$$y_2(t) = c \cdot y_1(t), \qquad -1 \le t \le 1.$$

From (a) we obtained $c = 1$ when $0 \le t \le 1$. But $c = -1$ if $-1 \le t \le 0$, because we then have $y_2(t) = -t^2 = -y_1(t)$.
Since the value for c should be unique on the whole interval $[-1, 1]$, we conclude that y_1 and y_2 are not linearly dependent, i.e. linearly independent on the interval $-1 \le t \le 1$.

(c) First let $-1 \le t \le 0$. Then the Wronskian for the functions $y_1(t) = t^2$ and $y_2(t) = -t^2$ is

$$W[y_1, y_2](t) = t^2 \cdot (-2t) - 2t \cdot (-t^2) = 0.$$

For $0 \le t \le 1$ and $y_1(t) = t^2 = y_2(t)$, we easily get

$$W[y_1, y_2](t) = t^2 \cdot 2t - 2t \cdot t^2 = 0.$$

(d) Functions y_1 and y_2 have zero Wronskian on $(-1, 1)$ and therefore are linearly dependent, reducing to only one solution (up to a constant) of the given problem.

(see also Theorem 4 and its Corollary in M. Braun's book)

2. (Subsection 2.2.1 - Exercise 6) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0, \qquad y(0) = 0, \qquad y'(0) = 2.$$

The characteristic equation

$$r^2 + 2r + 5 = 0$$

has two complex roots $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$. The general solution has the form

$$y(t) = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t),$$

with

$$y'(t) = (-c_1 + 2c_2)\mathrm{e}^{-t} \cos(2t) + (-2c_1 - c_2)\mathrm{e}^{-t} \sin(2t).$$

The initial conditions $y(0) = 0$, $y'(0) = 2$ give us two equations for finding constants c_1 and c_2:

$$0 = y(0) = c_1$$

$$2 = y'(0) = -c_1 + 2c_2.$$

The constants are $c_1 = 0$ and $c_2 = 1$ and the final solution is

$$y(t) = \mathrm{e}^{-t} \sin(2t).$$

3. (Section 2.2.2 - Exercise 6) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0, \qquad y(2) = 1, \qquad y'(2) = -1.$$

The characteristic equation

$$r^2 + 2r + 1 = 0$$

has one root $r = -1$. The general solution has the form

$$y(t) = (c_1 + c_2 t)\mathrm{e}^{-t}$$

with

$$y'(t) = -c_1\mathrm{e}^{-t} + c_2(1-t)\mathrm{e}^{-t}.$$

The initial conditions $y(2) = 1$, $y'(2) = -1$ give us two equations for finding constants c_1 and c_2:

$$1 = y(2) = c_1\mathrm{e}^{-2} + 2c_2\mathrm{e}^{-2}$$

$$-1 = y'(2) = -c_1\mathrm{e}^{-2} - c_2\mathrm{e}^{-2}.$$

The constants are $c_1 = \mathrm{e}^2$ and $c_2 = 0$ and the final solution is

$$y(t) = \mathrm{e}^{2-t}.$$

4. (Section 2.4 - Exercise 6) Solve the initial-value problem

$$y'' + 4y' + 4y = t^{5/2}\mathrm{e}^{-2t}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

First we solve the homogeneous problem. The characteristic equation

$$r^2 + 4r + 4 = 0$$

has one root $r = -2$. The functions

$$y_1(t) = \mathrm{e}^{-2t}, \qquad y_2(t) = t\,\mathrm{e}^{-2t},$$

form the fundamental set of solutions. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \mathrm{e}^{-4t}.$$

Now we find the particular solution ψ in the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

If $g(t) = t^{5/2}\mathrm{e}^{-2t}$, then

$$u_1(t) = -\int \frac{g(t)y_2(t)}{W[y_1, y_2](t)}dt = -\int t^{7/2}dt = -\frac{2}{9}t^{9/2}$$

$$u_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}dt = \int t^{5/2}dt = \frac{2}{7}t^{7/2}.$$

The particular solution is

$$\psi(t) = -\frac{2}{9}t^{9/2}\mathrm{e}^{-2t} + \frac{2}{7}t^{7/2}t\,\mathrm{e}^{-2t} = \frac{4}{63}t^{9/2}\mathrm{e}^{-2t}.$$

The general solution of the starting problem has the form

$$y(t) = (c_1 + c_2 t)\mathrm{e}^{-2t} + \psi(t)$$

with

$$y'(t) = -2c_1\mathrm{e}^{-2t} + c_2(1 - 2t)\mathrm{e}^{-2t} + \frac{4}{63}\left(\frac{9}{2}t^{7/2} - 2t^{9/2}\right)\mathrm{e}^{-2t}.$$

The initial conditions $y(0) = y'(0) = 0$ give us two equations for finding constants c_1 and c_2:

$$0 = y(0) = c_1$$

$$0 = y'(0) = -2c_1 + c_2.$$

The constants are $c_1 = c_2 = 0$ and the final solution is

$$y(t) = \psi(t) = \frac{4}{63}t^{9/2}\mathrm{e}^{-2t}.$$

5. (Section 2.5 - Exercise 14) Find a particular solution of

$$y'' + 2y' = 1 + t^2 + \mathrm{e}^{-2t}.$$

We will find the function ψ by splitting our problem into two parts:
First we will find a particular solution ψ_1 of the problem

$$y'' + 2y' = 1 + t^2.$$

Then we will find a particular solution ψ_2 of the problem

$$y'' + 2y' = \mathrm{e}^{-2t}.$$

Finally, $\psi = \psi_1 + \psi_2$. In both of those cases, we will use the guessing technique.

(1) We propose

$$\psi_1(t) = t(At^2 + Bt + C),$$

with unknown constants A, B, C. Using

$$\psi_1'(t) = 3At^2 + 2Bt + C$$
$$\psi_1''(t) = 6At + 2B$$

the differential equation $\psi_1'' + 2\psi_1' = 1 + t^2$ becomes

$$6At^2 + (6A + 4B)t + 2B + 2C = 1 + t^2.$$

This further implies $A = 1/6$, $B = -1/4$, $C = 3/4$, and

$$\psi_1(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{3t}{4}.$$

(2) Now we propose

$$\psi_2(t) = Dt\mathrm{e}^{-2t}.$$

with an unknown constant D. Using

$$\psi_2'(t) = D\mathrm{e}^{-2t} - 2Dt\mathrm{e}^{-2t}$$
$$\psi_2''(t) = -4D\mathrm{e}^{-2t} + 4Dt\mathrm{e}^{-2t}$$

the differential equation $\psi_2'' + 2\psi_2' = \mathrm{e}^{-2t}$ becomes

$$-4D\mathrm{e}^{-2t} + 4Dt\mathrm{e}^{-2t} + 2(D\mathrm{e}^{-2t} - 2Dt\mathrm{e}^{-2t}) = \mathrm{e}^{-2t}$$
$$-2D\mathrm{e}^{-2t} = \mathrm{e}^{-2t}.$$

This further implies $D = -1/2$, and

$$\psi_2(t) = -\frac{t}{2}\,\mathrm{e}^{-2t}.$$

The particular solution for the starting problems is

$$\psi(t) = \psi_1(t) + \psi_2(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{3t}{4} - \frac{t}{2}\,\mathrm{e}^{-2t}.$$