HW 3

Math 4512
Differential Equations with Applications

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Contents

1 Section 2.1, problem 11

Let $y_1(t) = t^2$ and $y_2(t) = t|t|$

1. Show that y_1, y_2 are linearly dependent (L.D.) on the interval $0 \le t \le 1$
2. Show that y_1, y_2 are linearly independent (L.I.) on the interval $-1 \le t \le 1$
3. Show that $W[y_1, y_2](t)$ is identically zero.
4. Show that y_1, y_2 can never be two solutions of (3) which is $y'' + p(t)y' + q(t)y = 0$, on the interval $-1 < t < 1$ if both p, q are continuous in this interval.

Solution

1.1 Part a

On the interval $0 \le t \le 1$, then $|t| = t$ since t is positive. Hence $y_2(t) = t^2$, which is the same as $y_1(t) = t^2$. Therefore they are linearly dependent (same solution). In other words, $y_1(t) = c_1 y_2(t)$ where $c_1 = 1$.

1.2 Part b

When $t \le 0$ now $y_2(t) = -t^2$. Hence we have $y_1 = y_2$ for $0 \le t \le 1$ and $y_1 = -y_2$ for $-1 \le t < 0$. Therefore it is not possible to find the same constant c such that $y_1 = cy_2$ which will work for all t regions. This implies that $y_1(t)$ and $y_2(t)$ are linearly independent on $-1 \le t \le 1$.

1.3 Part c

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

If $W(t) = 0$ is in some region or at some point, then it must be zero anywhere. Therefore let us pick the interval $0 \le t \le 1$ to calculate $W(t)$. This way we avoid having to deal with the $|t|$ when taking derivatives since on this interval, $y_1 = t^2$ and also $y_2 = t^2$. Now $W(t)$ becomes

$$\begin{aligned} W(t) &= t^2(2t) - t^2(2t) \\ &= 0 \end{aligned}$$

Therefore $W(t) = 0$ everywhere.

1.4 Part d

Since p, q are continuous on $-1 < t < 1$, then by uniqueness theorem, we know there are two fundamental solutions y_1, y_2, which must be linearly independent that their linear combination give the general solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

But from part(b) above we found that the given functions y_1, y_2 are not linearly independent on $-1 < t < 1$, hence these can never be the fundamental solutions to $y'' + p(t)y' + q(t)y = 0$.

2 Section 2.2.1, problem 6 (page 144, complex roots)

Solve $y'' + 2y' + 5y = 0$ with $y(0) = 0, y'(0) = 2$

Solution

Let $y = e^{\lambda t}$. Substituting in the above ODE gives

$$\begin{aligned} \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5e^{\lambda t} &= 0 \\ e^{\lambda t}\left(\lambda^2 + 2\lambda + 5\right) &= 0 \end{aligned}$$

Since $e^{\lambda t} \neq 0$, the above simplifies to $\lambda^2 + 2\lambda + 5 = 0$. The roots are $\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = \frac{-2}{2} \pm \frac{1}{2}\sqrt{4 - 4(5)}$ or $\lambda = -1 \pm \frac{1}{2}\sqrt{-16}$. Hence

$$\lambda = -1 \pm 2i$$

Therefore the general solution is linear combination of

$$\begin{aligned} y(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ &= c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \\ &= e^{-t}\left(c_1 e^{2it} + c_2 e^{-2it}\right) \end{aligned}$$

But $c_1 e^{2it} + c_2 e^{-2it}$ can be rewritten, using Euler relation, as $C_1 \cos 2t + C_2 \sin 2t$. The above solution becomes

$$y(t) = e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right) \tag{1}$$

C_1, C_2 are now found from initial conditions. At $t = 0$

$$0 = C_1$$

The solution (1) simplifies to

$$y(t) = C_2 e^{-t} \sin 2t \tag{2}$$

Taking time derivative gives

$$y'(t) = C_2\left(-e^{-t}\sin 2t + 2e^{-t}\cos 2t\right)$$

At $t = 0$ the above becomes

$$\begin{aligned} 2 &= 2C_2 \\ C_2 &= 1 \end{aligned}$$

Substituting the above in (2) gives the final general solution

$$y(t) = e^{-t}\sin 2t$$

3 section 2.2.2, problem 6 (page 149, equal roots)

Solve the following initial-value problems $y'' + 2y' + y = 0$ with $y(2) = 1, y'(2) = -1$

Solution

Let $y = e^{\lambda t}$. Substituting in the above ODE gives

$$\begin{aligned} \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} &= 0 \\ e^{\lambda t}\left(\lambda^2 + 2\lambda + 1\right) &= 0 \end{aligned}$$

Since $e^{\lambda t} \neq 0$, the above simplifies to $\lambda^2 + 2\lambda + 1 = 0$ or $(\lambda + 1)^2 = 0$. Hence there is a double root $\lambda = -1$. One fundamental solution is

$$y_1 = e^{-t}$$

To find the second solution, reduction of order is used. Let the second solution be

$$\begin{aligned} y_2(t) &= y_1(t)\,u(t) \\ &= e^{-t}u \qquad (1) \end{aligned}$$

Hence

$$\begin{aligned} y_2' &= -e^{-t}u + e^{-t}u' \qquad (2) \\ y_2'' &= e^{-t}u - e^{-t}u' - e^{-t}u' + e^{-t}u'' \qquad (3) \end{aligned}$$

Substituting (1,2,3) into the ODE gives (since y_2 is assumed to be a solution)

$$\begin{aligned} \left(e^{-t}u - e^{-t}u' - e^{-t}u' + e^{-t}u''\right) + 2\left(-e^{-t}u + e^{-t}u'\right) + \left(e^{-t}u\right) &= 0 \\ (u - u' - u' + u'') + 2(-u + u') + u &= 0 \\ u'' - 2u' + u - 2u + 2u' + u &= 0 \\ u'' &= 0 \end{aligned}$$

Hence the solution is $u = C_1 t + C_2$. Therefore from (1) the second solution is

$$\begin{aligned} y_2(t) &= y_1(t)\,u(t) \\ &= e^{-t}(C_1 t + C_2) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y(t) &= C_3 y_1 + C_4 y_2 \\ &= C_3 e^{-t} + C_4 e^{-t}(C_1 t + C_2) \end{aligned}$$

Combining constants gives

$$\begin{aligned} y(t) &= C_3 e^{-t} + e^{-t}(C_1 t + C_2) \\ &= (C_3 + C_2)e^{-t} + C_1 t e^{-t} \end{aligned}$$

Let $A = (C_3 + C_2), B = C_1$, then the final solution is

$$y(t) = Ae^{-t} + Bte^{-t} \qquad (4)$$

Now A, B are found from initial conditions $y(2) = 1, y'(2) = -1$. First initial condition gives from (4)

$$1 = Ae^{-2} + 2Be^{-2} \qquad (5)$$

Taking derivative of (4) gives

$$y'(t) = -Ae^{-t} + B\left(e^{-t} - te^{-t}\right)$$

Applying second initial condition on the above gives

$$\begin{aligned} -1 &= -Ae^{-2} + B\left(e^{-2} - 2e^{-2}\right) \\ &= -Ae^{-2} - Be^{-2} \qquad (6) \end{aligned}$$

Now we need to solve (5,6) for (A, B). Adding (5,6) gives

$$0 = Be^{-2}$$

Hence $B = 0$. Therefore from (5) we can now solve for A

$$\begin{aligned} 1 &= Ae^{-2} \\ A &= e^2 \end{aligned}$$

Hence (4) now becomes

$$
\begin{aligned}
y(t) &= e^2 e^{-t} \\
&= e^{2-t}
\end{aligned}
$$

4 Section 2.4, problem 6 (page 156, Variation of parameters)

Solve the following initial-value problems $y'' + 4y' + 4y = t^{\frac{5}{2}}e^{-2t}$ with $y(0) = 0, y'(0) = 0$

Solution

The first step is to solve the homogenous ODE $y''+4y'+4y = 0$. The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$
$$(\lambda + 2)(\lambda + 2) = 0$$

Hence a double root at $\lambda = -2$. The first solution is $y_1 = e^{=2t}$. Therefore the second solution is $y_2 = te^{-2t}$ (obtained using reduction of order as was done in the above problem with equal roots). Therefore the homogenous $y_h(t)$ is

$$y_h(t) = C_1 e^{-2t} + C_2 te^{-2t}$$

To find the particular solution $y_p(t)$, Variation of parameters will be used. Assuming the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) f(t)}{W(t)} dt \tag{1}$$

And

$$u_2(t) = \int \frac{y_1(t) f(t)}{W(t)} dt \tag{2}$$

Where in the above $f(t) = t^{\frac{5}{2}}e^{-2t}$ and $y_1 = e^{=2t}, y_2 = te^{-2t}$. We now need to find $W(t)$

$$\begin{aligned}
W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_2 y_1' \\
&= e^{=2t}\left(e^{-2t} - 2te^{-2t}\right) + 2te^{-2t}e^{-2t} \\
&= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\
&= e^{-4t}
\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
u_1(t) &= -\int \frac{te^{-2t}t^{\frac{5}{2}}e^{-2t}}{e^{-4t}} dt \\
&= -\int t^{\frac{5}{2}+1} dt \\
&= -\int t^{\frac{7}{2}} dt \\
&= -\frac{t^{\frac{9}{2}}}{\frac{9}{2}} \\
&= \frac{-2}{9} t^{\frac{9}{2}}
\end{aligned}$$

And (2) becomes

$$\begin{aligned}
u_2(t) &= \int \frac{e^{-2t}t^{\frac{5}{2}}e^{-2t}}{e^{-4t}} dt \\
&= \int t^{\frac{5}{2}} dt \\
&= \frac{t^{\frac{7}{2}}}{\frac{7}{2}} \\
&= \frac{2}{7} t^{\frac{7}{2}}
\end{aligned}$$

Since $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, then using the above results we obtain the particular solution

$$\begin{aligned}
y_p(t) &= \left(\frac{-2}{9}t^{\frac{9}{2}}\right)e^{-2t} + \left(\frac{2}{7}t^{\frac{7}{2}}\right)te^{-2t} \\
&= e^{-2t}\left(\frac{-2}{9}t^{\frac{9}{2}} + \frac{2}{7}t^{\frac{9}{2}}\right) \\
&= \frac{4}{63}e^{-2t}t^{\frac{9}{2}}
\end{aligned}$$

Since $y(t) = y_h(t) + y_p(t)$ then the final solution is

$$\begin{aligned}
y(t) &= \left(C_1e^{-2t} + C_2te^{-2t}\right) + \frac{4}{63}e^{-2t}t^{\frac{9}{2}} \\
&= e^{-2t}\left(C_1 + C_2t + \frac{4}{63}t^{\frac{9}{2}}\right) \qquad (3)
\end{aligned}$$

Now initial conditions are applied to find C_1, C_2. From $y(0) = 0$, then (3) becomes

$$0 = C_1$$

Hence the solution (3) simplifies to

$$y(t) = e^{-2t}\left(C_2t + \frac{4}{63}t^{\frac{9}{2}}\right) \qquad (4)$$

Taking derivatives

$$\begin{aligned}
y'(t) &= -2e^{-2t}\left(C_2t + \frac{4}{63}t^{\frac{9}{2}}\right) + e^{-2t}\left(C_2 + \left(\frac{4}{63}\right)\left(\frac{7}{2}\right)t^{\frac{7}{2}}\right) \\
&= -2e^{-2t}\left(C_2t + \frac{4}{63}t^{\frac{9}{2}}\right) + e^{-2t}\left(C_2 + \frac{2}{9}t^{\frac{7}{2}}\right)
\end{aligned}$$

Applying the second BC $y'(0) = 0$ to the above gives

$$0 = C_2$$

The solution (4) now reduces to

$$y(t) = \frac{4}{63}t^{\frac{9}{2}}e^{-2t}$$

Which is just the particular solution. This makes sense, since both initial conditions are zero, then the homogenous solution will be zero.

5 Section 2.5, problem 14 (page 164, Guessing method)

Find the particular solution for $y'' + 2y' = 1 + t^2 + e^{-2t}$

Solution

The first step is to solve the homogeneous solution $y_h(t)$ of the ODE $y'' + 2y' = 0$. Let $u = y'$. Then the ODE becomes

$$u' + 2u = 0$$

The integrating factor is $I = e^{\int 2dt} = e^{2t}$. The above becomes

$$\begin{aligned}\frac{d}{dt}\left(ue^{2t}\right) &= 0\\ ue^{2t} &= C_1\\ u &= C_1 e^{-2t}\end{aligned}$$

But $y' = u$. Integrating gives

$$\begin{aligned}y_h(t) &= \int C_1 e^{-2t} dt + C_2\\ &= \frac{-1}{2} C_1 e^{-2t} + C_2\\ &= C_3 e^{-2t} + C_2\end{aligned}$$

Hence the fundamental solutions are

$$\begin{aligned}y_1 &= e^{-2t}\\ y_2 &= 1\end{aligned}$$

We now go back to the original ODE and find the particular solution y_p. Since the RHS is $p(t) + e^{-2t}$ where $p(t) = 1 + t^2$, we can use linearity and find particular solution $y_{p_1}(t)$ associated with $p(t)$ only and then find $y_{p_2}(t)$ associated with e^{-2t} only and then add them together to obtain $y_p(t)$. In other words

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$

To find $y_{p_1}(t)$ associated with $1 + t^2$ we guess $y_{p_1}(t) = C_0 + C_1 t + C_2 t^2$. But because the ODE is missing the y term in it, then we have to multiply this guess by an extra t. Therefore it becomes

$$y_{p_1}(t) = t\left(C_0 + C_1 t + C_2 t^2\right)$$

To find $y_{p_2}(t)$ associated with e^{-2t} we guess $y_{p_2} = Ae^{-2t}$. But because e^{-2t} is also a fundamental solution of the homogenous solution found above, we have to again adjust this and multiply the guess by t. Hence it becomes

$$y_{p_2}(t) = Ate^{-2t}$$

Therefore the full guess for particular solution becomes

$$\begin{aligned}y_p(t) &= y_{p_1}(t) + y_{p_2}(t)\\ &= t\left(C_0 + C_1 t + C_2 t^2\right) + Ate^{-2t}\\ &= tC_0 + C_1 t^2 + C_2 t^3 + Ate^{-2t} \qquad \text{(1A)}\end{aligned}$$

Now

$$y_p'(t) = C_0 + 2C_1 t + 3C_2 t^2 + Ae^{-2t} - 2Ate^{-2t} \tag{1}$$

And

$$y_p''(t) = 2C_1 + 6C_2 t - 2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t} \tag{2}$$

Substituting (1,2) into LHS of $y'' + 2y' = 1 + t^2 + e^{-2t}$ gives

$$\begin{aligned}\left(2C_1 + 6C_2 t - 2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}\right) + 2\left(C_0 + 2C_1 t + 3C_2 t^2 + Ae^{-2t} - 2Ate^{-2t}\right) &= 1 + t^2 + e^{-2t}\\ 2C_0 + 2C_1 + 4tC_1 + 6tC_2 - 2Ae^{-2t} + 6t^2 C_2 &= 1 + t^2 + e^{-2t}\\ e^{-2t}(-2A) + t(4C_1 + 6C_2) + t^2(6C_2) + (2C_0 + 2C_1) &= 1 + t^2 + e^{-2t}\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} -2A &= 1 \\ 4C_1 + 6C_2 &= 0 \\ 6C_2 &= 1 \\ 2C_0 + 2C_1 &= 1 \end{aligned}$$

Solving gives $A = -\frac{1}{2}, C_0 = \frac{3}{4}, C_1 = -\frac{1}{4}, C_2 = \frac{1}{6}$. Substituting the above in (1A) gives the particular solution as

$$\begin{aligned} y_p(t) &= t\left(\frac{3}{4} - \frac{1}{4}t + \frac{1}{6}t^2\right) - \frac{1}{2}te^{-2t} \\ &= \frac{3}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 - \frac{1}{2}te^{-2t} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= C_3e^{-2t} + C_2 + \left(\frac{3}{4}t - \frac{1}{4}t^2 + \frac{1}{6}t^3 - \frac{1}{2}te^{-2t}\right) \end{aligned}$$