MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW2 - SOLUTIONS

1. (Section 1.8 - Exercise 8) A tank contains 300 gallons of water and 100 gallons of pollutants. Fresh water is pumped into the tank at the rate of 2 gal/min, and the well-stirred mixture leaves at the same rate. How long does it take for the concentration of pollutants in the tank to decrease to 1/10 of its original value?

Initially there are $V_0 = 300$ gal of water and $S_0 = 100$ gal of pollutants. Inflow and outflow rates are $r_i = r_o = 2$ gal/min, while the inflow concentration of pollutants is 0, since only pure water is pumped into the tank. If $S(t)$ denotes the amount of pollutants in the tank at time t, then IVP for this mixture problem is

$$\frac{dS}{dt} = 0 - 2 \cdot \frac{S(t)}{400}, \qquad S(0) = 100.$$

Its solution is $S(t) = 100\,\mathrm{e}^{-t/200}$. Thus the concentration $c(t)$ of pollutants in the tank at time t is

$$c(t) = \frac{S(t)}{400} = \frac{1}{4}\,\mathrm{e}^{-t/200}.$$

In order to find how long does it take for the concentration of pollutants in the tank to decrease to 1/10 of its original value, we need to solve for t the problem

$$c(t) = \frac{1}{10}\,c(0).$$

Then we get

$$\begin{aligned} \frac{1}{4}\,\mathrm{e}^{-t/200} &= \frac{1}{40} \\ \mathrm{e}^{-t/200} &= \frac{1}{10} \\ -\frac{t}{200} &= \ln\frac{1}{10} \\ t &= 200\ln 10 = 460.517\ldots \text{min} \approx 7\text{h}\,40\text{min}. \end{aligned}$$

2. (Section 1.8 - Exercise 14)

Find the orthogonal trajectories of the given family of curves

$$y = c \sin x.$$

Here we can take $F(x, y, c) = y - c \sin x$. Then from

$$F_x = -c \cos x, \qquad F_y = 1, \qquad c = \frac{y}{\sin x},$$

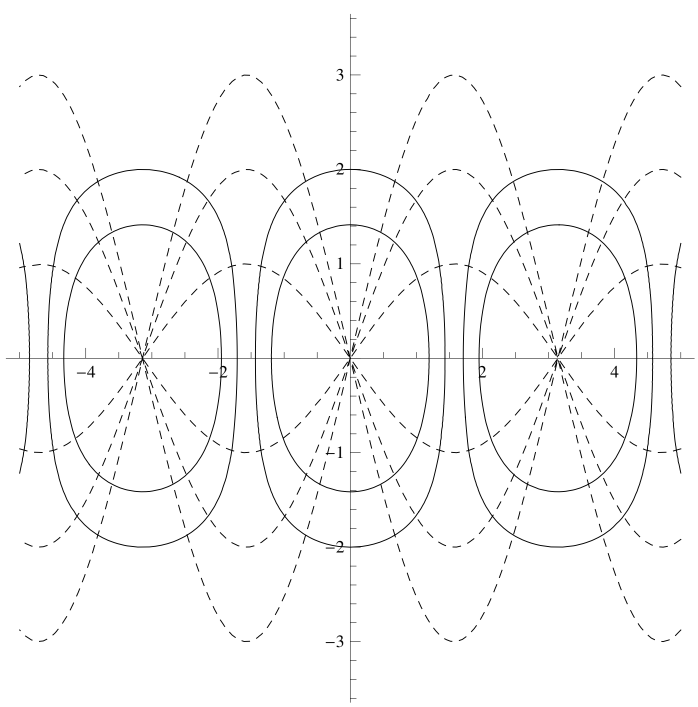
the orthogonal trajectories of the given family are the solution curves of the equation

$$\frac{dy}{dx} = \frac{F_y}{F_x} = -\frac{\tan x}{y}.$$

This is a separable differential equation and we solve it as follows:

$$\int y\, dy = -\int \tan x dx$$

$$\frac{y^2}{2} = \ln|\cos x| + c.$$



Curves $y = c \sin x$ (dashed) and $\frac{y^2}{2} = \ln|\cos x| + c$ (solid).

3. (Section 1.10 - Exercise 4)

Show that the solution y of the initial-value problem

$$\frac{dy}{dt} = y^2 + \cos t^2, \qquad y(0) = 0,$$

exists on the interval $0 \leq t \leq \frac{1}{2}$.

Let $f(t, y) = y^2 + \cos t^2$. The functions f and $f_y = 2y$ are continuous on a rectangle

$$R = [t_0, t_0 + a] \times [y_0 - b, y_0 + b] = [0, a] \times [-b, b],$$

for arbitrary constants $a > 0$ and $b > 0$. Then there exists a unique solution of the IVP on the interval $[0, \alpha]$, with

$$\alpha = \min\{a, \frac{b}{M}\}, \qquad M = \max_{(t,y)\in R} |y^2 + \cos t^2|.$$

Since

$$M = \max_{(t,y)\in R} |y^2 + \cos t^2| = b^2 + 1,$$

then

$$\alpha = \min\{a, \frac{b}{M}\} = \min\{a, \frac{b}{b^2 + 1}\}.$$

Let

$$g(b) = \frac{b}{b^2 + 1}.$$

For $b > 0$, the function g is positive and

$$g'(b) = \frac{1 - b^2}{(b^2 + 1)^2} = \frac{(1 - b)(1 + b)}{(b^2 + 1)^2}.$$

The point $b = 1$ is the local maximum of g on $(0, \infty)$ since

$$b \in (0, 1) \Rightarrow g'(b) > 0 \Rightarrow g \text{ is increasing},$$

$$b \in (1, \infty) \Rightarrow g'(b) < 0 \Rightarrow g \text{ is decreasing}.$$

Therefore

$$\frac{b}{b^2 + 1} = g(b) \leq g(1) = \frac{1}{2}.$$

Consequently, the largest possible value for α is $1/2$ (obtained for $b = 1$ and any $a \geq 1/2$), that concludes the proof.

4. (Section 1.10 - Exercise 17)

Prove that $y(t) = -1$ is the only solution of the initial-value problem

$$\frac{dy}{dt} = t(1+y), \qquad y(0) = -1.$$

First notice that the constant function $y(t) = -1$ is the solution of the given IVP (its derivative is zero, $1 + y = 0$ and $y(0) = -1$). In order to prove that this is the only solution, we need to analyze the function $f(t, y) = t(1 + y)$ and its partial derivative $f_y = t$. On a rectangle

$$R = [0, a] \times [-1 - b, -1 + b],$$

both f and f_y are continuous functions, for arbitrary positive constants a, b. Let

$$M = \max_{(t,y)\in R} |f(t, y)| = \max_{(t,y)\in R} |t(1 + y)| = a\, b,$$

and

$$\alpha = \min\{a, \frac{b}{M}\} = \min\{a, \frac{1}{a}\} = 1.$$

(Remark: Conclusion $\alpha = 1$ can be deduced from assuming first that $\min\{a, 1/a\} = a$. Then $a \leq 1/a$ and

$$a - \frac{1}{a} \leq 0$$

$$\frac{a^2 - 1}{a} \leq 0$$

$$\frac{(a-1)(a+1)}{a} \leq 0 \quad \longrightarrow \quad a \leq 1 \quad \longrightarrow \quad \min\{a, \frac{1}{a}\} = a \leq 1.$$

Similarly, assuming $\min\{a, 1/a\} = 1/a$ we obtain $1/a \leq a$ and

$$a - \frac{1}{a} \geq 0$$

$$\frac{(a-1)(a+1)}{a} \geq 0 \quad \longrightarrow \quad a \geq 1 \quad \longrightarrow \quad \min\{a, \frac{1}{a}\} = \frac{1}{a} \leq 1.)$$

From the existence-uniqueness theorem, we conclude that the solution $y(t) = -1$ of the IVP is unique in the interval $t_0 \leq t \leq t_0 + \alpha$, i.e. when $0 \leq t \leq 1$.

5. (Section 1.13 - Exercise 2 with $h = 0.1$)
Using Euler's method with step size $h = 0.1$, determine an approximate value of the solution at $t = 1$ for the initial-vale problem

$$\frac{dy}{dt} = 2ty, \qquad y(0) = 2,$$

and compare the results with the exact solution $y(t) = 2\mathrm{e}^{t^2}$.

Let $t_0 = 0$, $y_0 = 2$ and $f(t, y) = 2ty$. Using equidistant points

$$t_{k+1} = t_k + h, \qquad k = 0, 1, \ldots, 9, \qquad h = 0.1,$$

Euler's method

$$y_{k+1} = y_k + h\, f(t_k, y_k), \qquad k = 0, 1, \ldots, 9, \qquad y_0 = y(t_0),$$

will generate the following data

k	t_k	y_k
0	0	2
1	0.1	2
2	0.2	2.04
3	0.3	2.1216
4	0.4	2.2489
5	0.5	2.42881
6	0.6	2.67169
7	0.7	2.99229
8	0.8	3.41121
9	0.9	3.95701
10	1	4.66927

From this table we read $y_{10} = 4.66927$ is the approximation to $y(1) = 2\,\mathrm{e} = 5.43656$. Absolute error is

$$|y(1) - y_{10}| = 0.767297.$$