HW 2

Math 4512
Differential Equations with Applications

Fall 2019
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December 20, 2019 Compiled on December 20, 2019 at 10:43am

Contents

1 Section 1.8, problem 8

A tank contains 300 gallons of water and 100 gallons of pollutant. Fresh water is pumped into the tank at rate 2 gal/min, and the well stirred mixture leaves at the same rate. How long does it take for the concentration of pollutants in the tank to decrease to $\frac{1}{10}$ of its original value?

Solution

Let $V(t)$ be the volume in gallons of the pollutant at time t. Hence

$$\frac{dV(t)}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is the rate in gallons per min that the pollutant is entering the tank and R_{out} is the rate in gallons per min that the pollutant is leaving the tank. In this problem

$$R_{in} = 0 \tag{1A}$$

Since no pollutant enters the tank. And $R_{out} = 2$ gal/min. But each gallon that leaves contains the ratio $\frac{V(t)}{400}$ of pollutant at any moment of time. This is because the volume of the tank is fixed at 400 gallons since same volume enters as it leaves. Hence

$$R_{out} = 2\frac{V(t)}{400} \qquad \text{gal/min} \tag{1B}$$

Using (1A,1B) in (1) gives

$$\begin{aligned} \frac{dV(t)}{dt} &= -\frac{2}{400}V(t) \\ \frac{dV(t)}{dt} + \frac{1}{200}V(t) &= 0 \end{aligned}$$

This is a linear ODE. The integration factor is $I = e^{\int \frac{1}{200}dt} = e^{\frac{t}{200}}$. Therefore the above can be written as

$$\begin{aligned} \frac{d}{dt}(V(t)I) &= 0 \\ \frac{d}{dt}\left(Ve^{\frac{t}{200}}\right) &= 0 \end{aligned}$$

Integrating gives the general solution as

$$Ve^{\frac{t}{200}} = C \tag{1}$$

Using initial conditions, at $t = 0$, $V = 100$ gallons. Substituting these in the above to solve for C gives

$$100 = C$$

Hence the solution (1) becomes

$$V(t) = 100e^{\frac{-t}{200}} \tag{2}$$

To find the time t when $V(t) = 10$ gallons (this is $\frac{1}{10}$ of the original volume of pollutant, which is 100 gallons), then the above becomes

$$10 = 100e^{\frac{-1}{200}t_0}$$

Solving for t_0 gives

$$\begin{aligned} \frac{1}{10} &= e^{\frac{-1}{200}t_0} \\ \ln\left(\frac{1}{10}\right) &= \frac{-1}{200}t_0 \\ t_0 &= -200\ln\left(\frac{1}{10}\right) \end{aligned}$$

Hence

$$\boxed{t_0 = 460.517 \text{ minutes}}$$

This is the time it takes for the pollutant volume to decrease to $\frac{1}{10}$ of its original value in the tank.

2 Section 1.8, problem 14

Find the orthogonal trajectory of the curve $y = c\sin x$

<u>Solution</u>

Let

$$F\left(x,y,c\right) = c\sin x - y \tag{1}$$

Then $F_x = c\cos x$ and $F_y = -1$. Hence the slope of the orhogonal projection is given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{F_y}{F_x}\\ &= \frac{-1}{c\cos x}\end{aligned}$$

From (1), we need to solve for c from $F\left(x,y,c\right) = 0$ which gives $c\sin x - y = 0$ or $c = \frac{y}{\sin x}$. Substituting this back into the above result gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{\left(\frac{y}{\sin x}\right)\cos x}\\ &= \frac{-\sin x}{y\cos x}\\ &= -\frac{1}{y}\tan x\end{aligned}$$

The above gives the ODE to sovle for the orthogonal trajectory curves. This is separable. Integrating gives

$$\int ydy = -\int \tan xdx$$

But $\int \tan xdx = -\ln|\cos(x)|$. Hence the above becomes

$$\begin{aligned}\frac{y^2}{2} &= \ln(|\cos(x)|) + C_1\\ y^2 &= 2\ln(|\cos x|) + C\end{aligned}$$

Where $C = 2C_1$. Solving for y gives two solutions

$$y(x) = \pm\sqrt{2\ln(|\cos x|) + C}$$

For illustration, the above was plotted for $C = 1,2,3,4,5$ in the following (shown in red color) against the function $\sin(x)$ (in blue color). It shows the projection curves all cross $\sin(x)$ at 90^0 everywhere as expected.

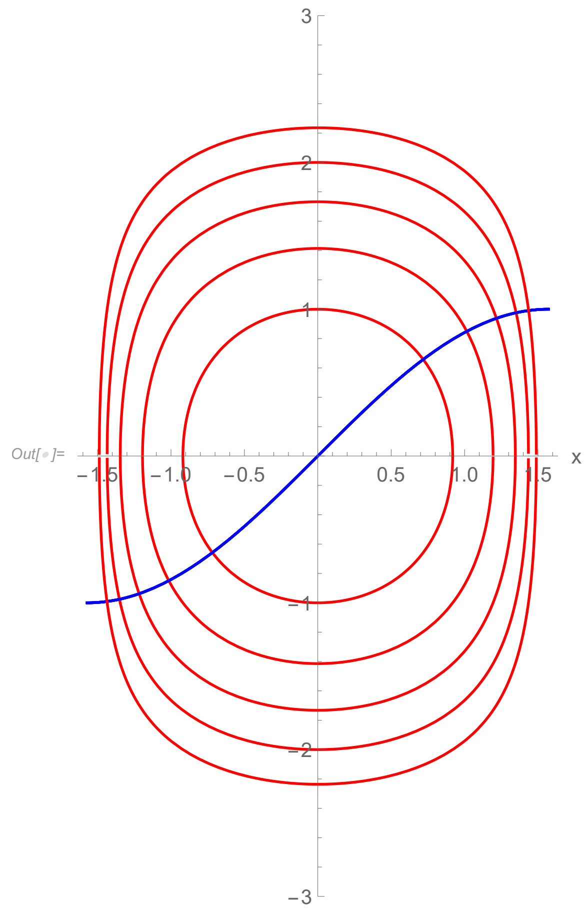

Figure 1: Orthogonal projections for different C values

```
In[ ]:= Show@Table[Plot[{Sin[x], Sqrt[2 Log[Abs[Cos[x]]] + c], -Sqrt[2 Log[Abs[Cos[x]]] + c]},
          {x, - Pi / 2, Pi / 2},
          PlotRange → {All, {-3, 3}},
          ImageSize → 300, AspectRatio → Automatic,
          PlotStyle → {Blue, Red, Red}, AxesLabel → {"x", None}, BaseStyle → 14], {c, 1, 5}]
```

Figure 2: code used for the above

The following plot is over a larger x range, from -2π to 2π

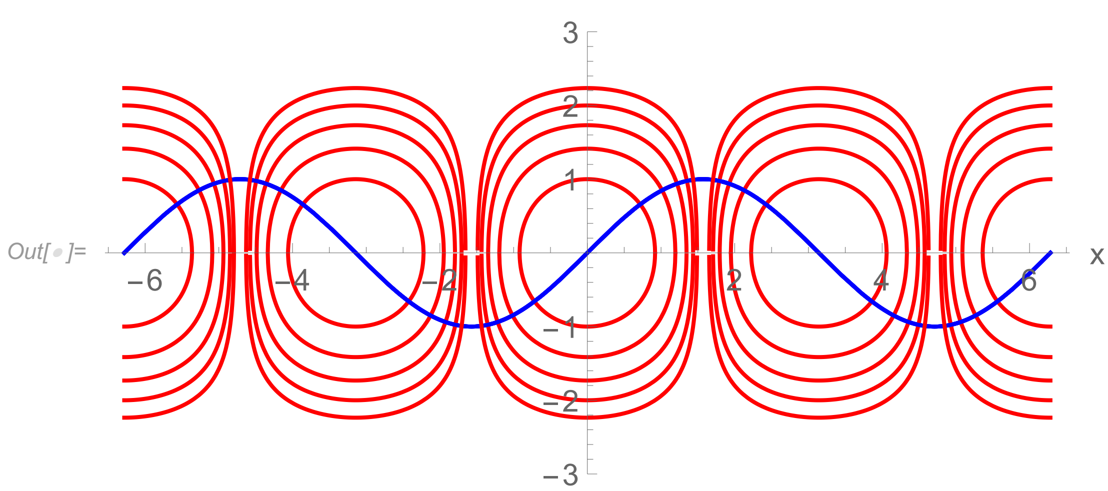

Figure 3: Orthogonal projections for different C values

3 section 1.10, problem 4

Show that the solution $y(t)$ of the given initial value problem exists on the specified interval.

$$y' = y^2 + \cos\left(t^2\right) \qquad y(0) = 0; \qquad 0 \le t \le \frac{1}{2}$$

Solution

Writing the ODE as

$$\begin{aligned} y' &= f\left(t, y\right) \\ &= y^2 + \cos\left(t^2\right) \end{aligned}$$

Let R be rectangle $0 \le t \le \frac{1}{2}, y_0 - b \le y \le y_0 + b$. But $y_0 = 0$ as given. Therefore

$$R = \left[0, \frac{1}{2}\right] \times [-b, b]$$

Now

$$\begin{aligned} M &= \max_{(t,y)\in R} \left|f\left(t, y\right)\right| \\ &= \max_{(t,y)\in R} \left|y^2 + \cos\left(t^2\right)\right| \\ &= b^2 + 1 \end{aligned}$$

Hence

$$\alpha = \min\left(a, \frac{b}{M}\right)$$

But $a = \frac{1}{2}, M = b^2 + 1$, therefore the above becomes

$$\alpha = \min\left(\frac{1}{2}, \frac{b}{b^2 + 1}\right)$$

The largest value α can obtain is when $g(b) = \frac{b}{b^2+1}$ is maximum.

$$\begin{aligned} g'(b) &= \frac{\left(b^2 + 1\right) - b(2b)}{\left(b^2 + 1\right)^2} \\ &= \frac{b^2 + 1 - 2b^2}{\left(b^2 + 1\right)^2} \\ &= \frac{1 - b^2}{\left(b^2 + 1\right)^2} \end{aligned}$$

Hence $g'(b) = 0$ gives $1 - b^2 = 0$ or $b = \pm 1$. Taking $b = 1$ gives $g_{\max}(b) = \frac{1}{1^2+1} = \frac{1}{2}$. Therefore

$$\begin{aligned} \alpha &= \min\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

This shows that the solution $y(t)$ exists on

$$t_0 \le t \le t_0 + \alpha$$

But $t_0 = 0, \alpha = \frac{1}{2}$, therefore

$$0 \le t \le \frac{1}{2}$$

Hence a unique solution exist inside rectangle

$$R = \left[0, \frac{1}{2}\right] \times [-1, 1]$$

4 Section 1.10, problem 17

Prove that $y(t) = -1$ is the only solution of the initial value problem

$$y' = t\left(1+y\right) \qquad y(0) = -1$$

Solution

The solution is found first to show it is $y(t) = -1$, then using the uniqueness theory, one can show it is unique. The above ODE is separable. Hence

$$\begin{aligned}
\int \frac{dy}{1+y} &= \int t dt \\
\ln\left(\left|1+y\right|\right) &= \frac{t^2}{2} + C \\
\left|1+y\right| &= e^{\frac{t^2}{2}+C} \\
1+y &= C_1 e^{\frac{t^2}{2}} \qquad (1)
\end{aligned}$$

Applying initial conditions gives

$$\begin{aligned}
1-1 &= C_1 \\
C_1 &= 0
\end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned}
1+y &= 0 \\
y(t) &= -1
\end{aligned}$$

To show the above is the only solution we need to show the uniqueness theorem applies to this ODE over all of $\Re$. Let

$$\begin{aligned}
y' &= f\left(t,y\right) \\
&= t\left(1+y\right)
\end{aligned}$$

The above shows that $f\left(t,y\right)$ is continuous in t over $-\infty < t < \infty$ and continuous in y over $-\infty < y < \infty$. Now

$$\frac{\partial f}{\partial y} = t$$

Hence $\frac{\partial f}{\partial y}$ is also continuous in y over $-\infty < y < \infty$. Therefore a solution exist and is unique in any region that includes the initial conditions. Hence the solution $y(t) = -1$ found above is the only solution.

5 Section 1.13, problem 2

Using Euler's method with step size $h = 0.1$, determine an approximate value of the solution at $t = 1$ for

$$y' = 2ty \qquad y(0) = 2$$

Which has analytical solution $y(t) = 2e^{t^2}$. Compute approximate value at $t = 1$ using just $h = 0.1$, and compare with $y(1)$.

Solution

Euler method is given by

$$\begin{aligned} y_1 &= y_0 + hf\left(t_0, y_0\right) \\ y_2 &= y_1 + hf\left(t_1, y_1\right) \\ &\vdots \\ y_{k+1} &= y_k + hf\left(t_k, y_k\right) \end{aligned}$$

Where $y_0 = 2$ in this problem, and $t_1 = t_0 + h, t_2 = t_1 + h$ and so on. Where $h = 0.1$. The following table shows the numerical value of $y(t)$ found at each t starting from $0, 0.1, 0.2, \cdots, 1.0$ and comparing it to the exact $y(t)$ and the error at each step using a small Mathematica program which implements the above method.

Out[•]=

t	appoximate y(t)	exact y(t)	error
0.	2	2.	0.
0.1	2.	2.0201	0.0201003
0.2	2.04	2.08162	0.0416215
0.3	2.1216	2.18835	0.0667486
0.4	2.2489	2.34702	0.0981257
0.5	2.42881	2.56805	0.139243
0.6	2.67169	2.86666	0.19497
0.7	2.99229	3.26463	0.272341
0.8	3.41121	3.79296	0.38175
0.9	3.95701	4.49582	0.53881
1.	4.66927	5.43656	0.767297

Figure 4: Table to compare Euler method with exact

```
f[t_, y_] := 2*t*y;
exacty[t_] := 2*Exp[t^2];
h = 1/10; t0 = 0; y0 = 2; N0 = 1/h;
y = Table[0, {N0 + 1}];
T = N@Table[t0 + i*h, {i, 0, N0}];
y[[1]] = y0;
data = Table[If[i == 1,
            {T[[1]], y0, exacty[T[[1]]], exacty[T[[1]]] - y0},
            {T[[i]],
              y[[i]] = y[[i - 1]] + h*f[T[[i - 1]], y[[i - 1]]], exacty[T[[i]]],
              exacty[T[[i]]] - y[[i]]}],
          {i, 1, n + 1}];
Grid[Prepend[data, {"t", "appoximate y(t)", "exact y(t)", "error"}],
   Frame → All, Alignment → Left]
```

Figure 5: Code for Euler method to generate the above table

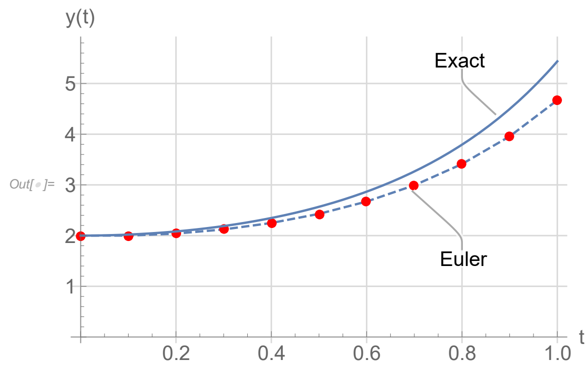

Figure 6: Plot of exact vs. Euler

```
p1 = ListLinePlot[
        Callout[Transpose@{data[[All, 1]], data[[All, 2]]}, "Euler", {0.8, 2}],
        Mesh → All, PlotStyle → Dashed, MeshStyle → Red];
p2 = Plot[Callout[2 * Exp[t^2], "Exact", {0.8, 5}], {t, 0, 1}];
Show[{p1, p2}, GridLines → Automatic, GridLinesStyle -> LightGray,
   PlotRange → All, AxesLabel → {"t", "y(t)"}, BaseStyle → 14]
```

Figure 7: Code to make plot