MATH 4512 – DIFFERENTIAL EQUATIONS WITH APPLICATIONS

HW1 - SOLUTIONS

1. (Section 1.2 - Exercise 8)

Find the solution of the given initial-value problem

$$\frac{dy}{dt} + \sqrt{1+t^2}\, y = 0, \qquad y(0) = \sqrt{5}.$$

The differential equation in this problem is of first-order and linear with $a(t) = \sqrt{1+t^2}$ and $b(t) = 0$. The integrating factor is

$$\mu(t) = \exp\left(\int a(t)dt\right) = \exp\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\mathrm{arsinh}\, t\right).$$

Then

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left(\int_0^t \mu(s)b(s)ds + \mu(0)y(0)\right) = \frac{\mu(0)y(0)}{\mu(t)} \\ &= \sqrt{5}\exp\left(-\frac{t}{2}\sqrt{1+t^2} - \frac{1}{2}\mathrm{arsinh}\, t\right), \end{aligned}$$

since $\mu(0) = 1$.

(Notice that this problem can also be considered as separable DE.)

2. (Section 1.2 - Exercise 17)

Find a continuous solution of the initial-value problem

$$\frac{dy}{dt} + y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

The differential equation is linear with $a(t) = 1$. The integrating factor is

$$\mu(t) = \mathrm{e}^{\int dt} = \mathrm{e}^{\mathrm{t}}$$

and the solution is

$$y(t) = \frac{1}{\mu(t)} \left(\int_0^t \mu(s) g(s) ds + \mu(0) y(0) \right) = \mathrm{e}^{-t} \int_0^t \mathrm{e}^s g(s) ds.$$

If $t \in [0, 1]$, then

$$y(t) = \mathrm{e}^{-t} \int_0^t 2 \cdot \mathrm{e}^s ds = 2\mathrm{e}^{-t} \mathrm{e}^s \Big|_0^t = 2\mathrm{e}^{-t}(\mathrm{e}^t - 1) = 2(1 - \mathrm{e}^{-t}).$$

If $t > 1$, then

$$y(t) = \mathrm{e}^{-t} \left(\int_0^1 2 \cdot \mathrm{e}^s ds + \int_1^t 0 \cdot \mathrm{e}^s ds \right) = 2\mathrm{e}^{-t} \mathrm{e}^s \Big|_0^1 = 2\mathrm{e}^{-t}(\mathrm{e} - 1).$$

Finally we get

$$y(t) = \begin{cases} 2(1 - \mathrm{e}^{-t}), & 0 \le t \le 1, \\ 2\mathrm{e}^{-t}(\mathrm{e} - 1), & t > 1. \end{cases}$$

The function y is continuous since

$$\lim_{t \to 1+} y(t) = \lim_{t \to 1+} 2\mathrm{e}^{-t}(\mathrm{e} - 1) = 2\mathrm{e}^{-1}(\mathrm{e} - 1) = 2(1 - \mathrm{e}^{-1}) = y(1).$$

3. (Section 1.4 - Exercise 10)

Solve initial-value problem

$$\cos y \, \frac{dy}{dt} = \frac{-t \sin y}{1+t^2}, \qquad y(1) = \frac{\pi}{2}$$

and determine the interval of existence of its solution.

First we will find a general solution to the separable DE:

$$\begin{aligned} \frac{\cos y}{\sin y} \frac{dy}{dt} &= -\frac{t}{1+t^2} \\ \int \cot y dy &= -\int \frac{t}{1+t^2} dt \\ \ln|\sin y| &= -\frac{1}{2} \ln|1+t^2| + c_1 \\ |\sin y| &= c_2(1+t^2)^{-1/2}, \qquad c_2 = \mathrm{e}_1^c \\ \sin y &= \frac{c}{\sqrt{1+t^2}} \\ y(t) &= \arcsin\left(\frac{c}{\sqrt{1+t^2}}\right). \end{aligned}$$

The initial condition

$$y(1) = \arcsin\left(\frac{c}{\sqrt{2}}\right) = \frac{\pi}{2}$$

implies $c/\sqrt{2} = 1$, i.e. $c = \sqrt{2}$. The final solution is

$$y(t) = \arcsin\left(\sqrt{\frac{2}{1+t^2}}\right)$$

and it is well defined if

$$\sqrt{\frac{2}{1+t^2}} \in [-1, 1].$$

Notice that for $t \geq 1$ or $t \leq -1$ we have that

$$0 < \frac{2}{1+t^2} \leq 1,$$

because

$$1 - \frac{2}{1+t^2} = \frac{t^2-1}{t^2+1} \geq 0.$$

Since we are looking for a continuous solution that contains $t_0 = 1$, we conclude that the interval of existence for the solution is $[1, \infty)$.

4. (Section 1.4 - Exercise 18)

Find the general solution for

$$\frac{dy}{dt} = \frac{t+y}{t-y}.$$

Following the hint, we will use the substitution

$$y(t) = tv(t), \qquad \frac{dy}{dt} = v(t) + t\frac{dv}{dt}.$$

Then the differential equation transforms into

$$\begin{aligned} v + t\frac{dv}{dt} &= \frac{t+tv}{t-tv} = \frac{1+v}{1-v} \\ t\frac{dv}{dt} &= \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v} \\ \frac{1-v}{1+v^2}\frac{dv}{dt} &= \frac{1}{t}. \end{aligned}$$

From

$$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{1+v^2}dv - \int \frac{v}{1+v^2}dv = \arctan v - \frac{1}{2}\ln(1+v^2)$$

it follows

$$\begin{aligned} \int \frac{1-v}{1+v^2}dv &= \int \frac{1}{t}dt \\ \arctan v - \frac{1}{2}\ln(1+v^2) &= \ln|t| + c \\ \arctan\frac{y}{t} - \frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right) &= \ln|t| + c. \end{aligned}$$

In the last step we again applied substitution in order to get the implicit form of the general solution.

5. (Section 1.5 - Exercise 4)

Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

The Malthusian law of population growth

$$\frac{dp}{dt} = a\,p(t), \qquad p(t_0) = p_0, \qquad a = const,$$

describes an exponential growth of a population. The solution to this problem is

$$p(t) = p_0\,\mathrm{e}^{a(t-t_0)}.$$

Suppose a population doubles its original size in 100 years. Then $p(t_0 + 100) = 2p_0$. Similarly, if this population triples its size in 200 years, then we should also have $p(t_0 + 200) = 3p_0$. From these two conditions we get

$$2p_0 = p(t_0 + 100) = p_0\,\mathrm{e}^{a(t_0+100-t_0)} = p_0\,\mathrm{e}^{100a} \qquad \longrightarrow \qquad a = \frac{\ln 2}{100} = 0.00693147\ldots$$

$$3p_0 = p(t_0 + 200) = p_0\,\mathrm{e}^{a(t_0+200-t_0)} = p_0\,\mathrm{e}^{200a} \qquad \longrightarrow \qquad a = \frac{\ln 3}{200} = 0.00549306\ldots,$$

which shows that the Malthusain law cannot be satisfied.

6. (Section 1.5 - Exercise 6(a))

A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it is (a) 10^8?

The population law

$$\frac{dp}{dt} = ap - bp^2, \qquad p(t_0) = p_0,$$

has the solution

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)\mathrm{e}^{-a(t-t_0)}}.$$

Here, the limiting population is $a/b = 5 \times 10^8$. Also, the population doubles every 40 minutes, i.e. $p(t_0 + 40) = 2p_0$.

(a) Let $p_0 = 10^8$. Then from $p(t_0 + 40) = 2p_0$ and $10^8 b = a/5$, we have that

$$\begin{aligned} 2 \cdot 10^8 &= \frac{10^8\, a}{10^8\, b + (a - 10^8\, b)\mathrm{e}^{-40a}} = \frac{10^8\, a}{\frac{a}{5} + (a - \frac{a}{5})\mathrm{e}^{-40a}} \\ &= \frac{10^8\, a}{\frac{a}{5} + \frac{4a}{5}\mathrm{e}^{-40a}} = \frac{5 \cdot 10^8}{1 + 4\mathrm{e}^{-40a}}. \end{aligned}$$

Finally, we can find the coefficients a and b:

$$\begin{aligned} 1 + 4\mathrm{e}^{-40a} &= \frac{5}{2} \\ \mathrm{e}^{-40a} &= \frac{3}{8} \\ a &= \frac{1}{40} \ln \frac{8}{3} = 0.0245207\ldots \\ b &= \frac{a}{5} 10^{-8} = 4.90415\ldots \times 10^{-11}. \end{aligned}$$

After two hours, the population will be

$$p(t_0 + 120) = \frac{10^8\, a}{10^8\, b + (a - 10^8\, b)\mathrm{e}^{-120a}} = 4.12903\ldots \times 10^8.$$