HW 1

Math 4512
Differential Equations with Applications

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Contents

1 Problem 8, section 1.2

Solve $\frac{dy}{dt}+\sqrt{1+t^2}y=0, y(0)=\sqrt{5}$

Solution

This is separable first order ODE. Therefore

$$\int\frac{dy}{y}=-\int\sqrt{1+t^2}dt \tag{1}$$

The LHS becomes

$$\int\frac{dy}{y}=\ln|y| \tag{2}$$

For the RHS of (1), the integral $\int\sqrt{1+t^2}dt$ can be evaluated as follows. Let $t=\sinh(\theta)$. Hence $\frac{dt}{d\theta}=\cosh(\theta)$. Therefore

$$\begin{aligned}\int\sqrt{1+t^2}dt&=\int\sqrt{1+\sinh^2(\theta)}\cosh(\theta)\,d\theta\\&=\int\cosh^2(\theta)\,d\theta\\&=\int\frac{1}{2}(1+\cosh(2\theta))\,d\theta\\&=\frac{1}{2}\left(\int d\theta+\int\cosh(2\theta)\,d\theta\right)\\&=\frac{1}{2}\left(\theta+\frac{\sinh(2\theta)}{2}\right)\\&=\frac{1}{2}\theta+\frac{\sinh(2\theta)}{4}\end{aligned}$$

Since $\sinh(2\theta)=2\sinh\theta\cosh\theta$, the above becomes

$$\int\sqrt{1+t^2}dt=\frac{1}{2}\theta+\frac{\sinh\theta\cosh\theta}{2}$$

Since $\cosh^2(\theta)-\sinh^2(\theta)=1$ then $\cosh^2\theta=1+\sinh^2(\theta)$ and the above becomes

$$\int\sqrt{1+t^2}dt=\frac{1}{2}\left(\theta+\sinh\theta\sqrt{1+\sinh^2(\theta)}\right)$$

But $t=\sinh(\theta)$ and $\theta=\operatorname{arcsinh}(t)$. Therefore the above becomes

$$\int\sqrt{1+t^2}dt=\frac{1}{2}\left(\operatorname{arcsinh}(t)+t\sqrt{1+t^2}\right) \tag{3}$$

Using (2,3) in (1) gives

$$\ln|y|=-\frac{1}{2}\left(\operatorname{arcsinh}(t)+t\sqrt{1+t^2}\right)+C \tag{4}$$

Where C is arbitrary constant of integration. Writing $\operatorname{arcsinh}(t)$ using known identity as $\ln\left|t+\sqrt{1+t^2}\right|$. And since $\sqrt{1+t^2}$ is always larger than t, then the absolute sign is not needed. Eq. (4) becomes

$$\begin{aligned}\ln|y|&=-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)+C\\|y|&=e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}e^C\\y&=C_1e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}\\&=C_1e^{-\frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)}e^{t\sqrt{1+t^2}}\end{aligned}$$

Therefore the general solution is

$$y(t)=C_1\frac{e^{t\sqrt{1+t^2}}}{\left(t+\sqrt{1+t^2}\right)^{\frac{1}{2}}}$$

Now initial conditions are used to determine C_1. From $y(0)=\sqrt{5}$ then the above gives

$$\sqrt{5}=C_1$$

Therefore the particular solution is

$$y(t) = \sqrt{5}\frac{e^{t\sqrt{1+t^2}}}{\left(t + \sqrt{1+t^2}\right)^{\frac{1}{2}}}$$

2 Problem 17, section 1.2

Find a continuous solution of the IVP $y + y' = g(t), y(0) = 0$ where

$$g(t) = \begin{cases} 2 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

Solution

This is linear first order ODE. The integrating factor is $\mu = e^{\int dt} = e^t$. Hence the ODE becomes

$$\begin{aligned} \frac{d}{dt}(y\mu) &= \mu g(t) \\ \frac{d}{dt}(ye^t) &= e^t g(t) \end{aligned}$$

Integrating gives

$$ye^t = \int e^t g(t)\,dt + C \tag{1}$$

Breaking the problem into two phases, and solving the above for $0 \le t \le 1$ gives

$$\begin{aligned} ye^t &= \int 2e^t dt + C \\ &= 2e^t + C \\ y(t) &= 2 + Ce^{-t} \end{aligned}$$

Applying initial conditions gives $0 = 2 + C$, or $C = -2$ and the above becomes

$$y(t) = 2 - 2e^{-t} \qquad 0 \le t \le 1 \tag{2}$$

The above solution is valid for $0 \le t \le 1$.

To solve for $t > 1$, initial conditions are first found for $t = 1$. At $t = 1$ the above gives

$$y(1) = 2 - \frac{2}{e}$$

Hence for $t > 1$, initial conditions are $y(1) = 2 - \frac{2}{e}$. Now the second phase is solved. From (1)

$$ye^t = \int e^t g(t)\,dt + C$$

But now $g(t) = 0$. The above simplifies to

$$\begin{aligned} ye^t &= C \\ y &= Ce^{-t} \end{aligned} \tag{3}$$

But at $t = 1, y = 2 - \frac{2}{e}$. Therefore

$$\begin{aligned} 2 - \frac{2}{e} &= Ce^{-1} \\ C &= 2e - 2 \\ &= 2(e-1) \end{aligned}$$

Substituting the above C into (3) gives

$$y = 2(e-1)e^{-t} \qquad t > 1 \tag{4}$$

Using (2,4) the final solution is therefore

$$y(t) = \begin{cases} 2 - 2e^{-t} & 0 \le t \le 1 \\ 2(e-1)e^{-t} & t > 1 \end{cases}$$

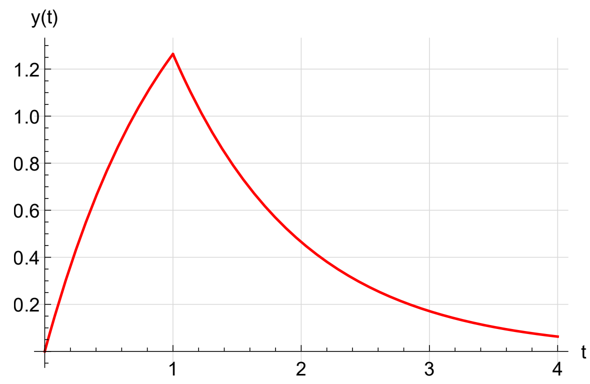

Figure 1: Plot of the solution $y(t)$

3 Problem 10, section 1.4

Solve $\cos y \frac{dy}{dt} = \frac{-t\sin y}{1+t^2}, y(1) = \frac{\pi}{2}$

<u>Solution</u>

This is separable first order ODE

$$\int \frac{\cos y}{\sin y} dy = -\int \frac{t}{1+t^2} dt$$

But $\int \frac{\cos y}{\sin y} dy = \int \frac{\frac{d}{dy}\sin y}{\sin y} dy = \ln \left|\sin\left(y\right)\right|$ and $\int \frac{t}{1+t^2} dt = \frac{1}{2}\ln\left|1+t^2\right| = \frac{1}{2}\ln\left(1+t^2\right)$ since $1+t^2$ is positive. Hence the above becomes

$$\ln\left|\sin\left(y\right)\right| = -\frac{1}{2}\ln\left(1+t^2\right) + C$$

Where C is the integration constant. Hence

$$\begin{aligned}\left|\sin\left(y\right)\right| &= e^{-\frac{1}{2}\ln\left(1+t^2\right)+C} \\ &= e^{-\frac{1}{2}\ln\left(1+t^2\right)} e^{C}\end{aligned}$$

Therefore

$$\begin{aligned}\sin\left(y\right) &= C_1 e^{-\frac{1}{2}\ln\left(1+t^2\right)} \\ &= C_1 \frac{1}{\sqrt{1+t^2}} \qquad (1)\end{aligned}$$

From initial conditions $y(1) = \frac{\pi}{2}$ the above becomes

$$\begin{aligned}\sin\left(\frac{\pi}{2}\right) &= C_1 \frac{1}{\sqrt{2}} \\ C_1 &= \sqrt{2}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\sin\left(y\right) &= \sqrt{2}\frac{1}{\sqrt{1+t^2}} \\ y(t) &= \arcsin\left(\frac{\sqrt{2}}{\sqrt{1+t^2}}\right)\end{aligned}$$

4 Problem 18, section 1.4

Solve $\frac{dy}{dt} = \frac{t+y}{t-y}$

Solution

Let $u = \frac{y}{t}$ or $y = ut$. Hence $\frac{dy}{dt} = u + t\frac{du}{dt}$. Therefore the ODE becomes

$$\begin{aligned}
u + t\frac{du}{dt} &= \frac{t + ut}{t - ut} \\
u + t\frac{du}{dt} &= \frac{t(1+u)}{t(1-u)} \\
t\frac{du}{dt} &= \frac{(1+u)}{(1-u)} - u \\
&= -\frac{u^2+1}{u-1} \\
&= \frac{1+u^2}{1-u}
\end{aligned}$$

This is now separable ODE. Therefore

$$\begin{aligned}
\frac{1-u}{1+u^2}\frac{du}{dt} &= \frac{1}{t} \\
\int \frac{1-u}{1+u^2} du &= \int \frac{1}{t} dt
\end{aligned} \tag{1}$$

But

$$\begin{aligned}
\int \frac{1-u}{1+u^2} du &= \int \frac{1}{1+u^2} du - \int \frac{u}{1+u^2} du \\
&= \arctan(u) - \frac{1}{2}\ln\left|1+u^2\right|
\end{aligned}$$

but $1 + u^2$ is positive. Hence

$$\int \frac{1-u}{1+u^2} du = \arctan(u) - \frac{1}{2}\ln\left(1+u^2\right)$$

And $\int \frac{1}{t} dt = \ln|t|$. Hence (1) becomes

$$\arctan(u) - \frac{1}{2}\ln\left(1+u^2\right) = \ln|t| + C$$

But $\frac{y}{t}$, and the above becomes

$$\arctan\left(\frac{y}{t}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{t}\right)^2\right) = \ln|t| + C$$

The above solution is implicit in $y(t)$.

5 Problem 4, section 1.5

Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

Solution

In Malthusian law of population growth, the rate at which population changes is fixed in the model. It is given by a below

$$\frac{dp}{dt} = ap(t)$$

Where a is constant. But the problem says the population is doubled in first 100 years. So if p_0 was initial population, then after 100 years the population now has become $2p_0$. There one will expect that after another 100 years the population will double again to become $4p_0$.

But the problem says that the population triples in 200 years, becoming $3p_0$ and not $4p_0$. This shows that the rate of growth is not constant. Hence this do not satisfy Malthusian law of population growth.

6 Problem 6(a), section 1.5

A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it was (a) 10^8 ?

Solution

In the logistic law, the population model is given by

$$\frac{dp}{dt}=ap-bp^2$$

Where $p(t)$ is population at time t and a is the growth rate (constant) and b is the competition rate (also constant). In this model

$$\lim_{t\to\infty}p(t)=\frac{a}{b}$$

Therefore

$$\frac{a}{b}=5\times10^8 \tag{1}$$

The problem says that $a=100\%$ (per 40 minute) or $a=1$ (per 40 minute). Therefore $a=\frac{1}{40}$ per minute. And $p_0=10^8$. Using the solution of this model, given in the textbook at page 30 as

$$p(t)=\frac{ap_0}{bp_0+\left(a-bp_0\right)e^{-a(t-t_0)}} \tag{3}$$

And using $t_0=0$, then the population size at t is now be calculated. From (1), $b=\frac{\frac{1}{40}}{5\times10^8}=\frac{1}{2}\times10^{-10}=5\times10^{-11}$. Eq. (3) now becomes

$$\begin{aligned}p(t)&=\frac{\frac{1}{40}\left(10^8\right)}{\left(5\times10^{-11}\right)\left(10^8\right)+\left(\frac{1}{40}-\left(5\times10^{-11}\right)\left(10^8\right)\right)e^{-\frac{1}{40}t}}\\&=\frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right)+\left(\frac{1}{40}-5\times\frac{1}{1000}\right)e^{-\frac{1}{40}t}}\end{aligned}$$

For $t=120$ (minutes) the above becomes

$$\begin{aligned}p(120)&=\frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right)+\left(\frac{1}{40}-5\times\frac{1}{1000}\right)e^{-\frac{1}{40}120}}\\&=\frac{\frac{1}{40}\left(10^8\right)}{\left(5\times\frac{1}{1000}\right)+\left(\frac{1}{40}-5\times\frac{1}{1000}\right)e^{-3}}\\&=4.1696\times10^8\end{aligned}$$

Hence

$$\boxed{p(120)=4.1696\times10^8}$$

The inflection point is

$$\begin{aligned}\frac{a}{2b}&=\frac{\frac{1}{40}}{(2)\left(5\times10^{-11}\right)}\\&=2.5\times10^8\end{aligned}$$

The following plot was generated to compare the population $p(t)$ between case (a) and case (b). It shows that when starting with initial population of $p_0=10^8$ which is case (a) and when starting with $p_0=10^9$ which is case (b), both populations will eventually reach the limiting population of 5×10^8. The S curve shows up only when starting with population below the limiting population.

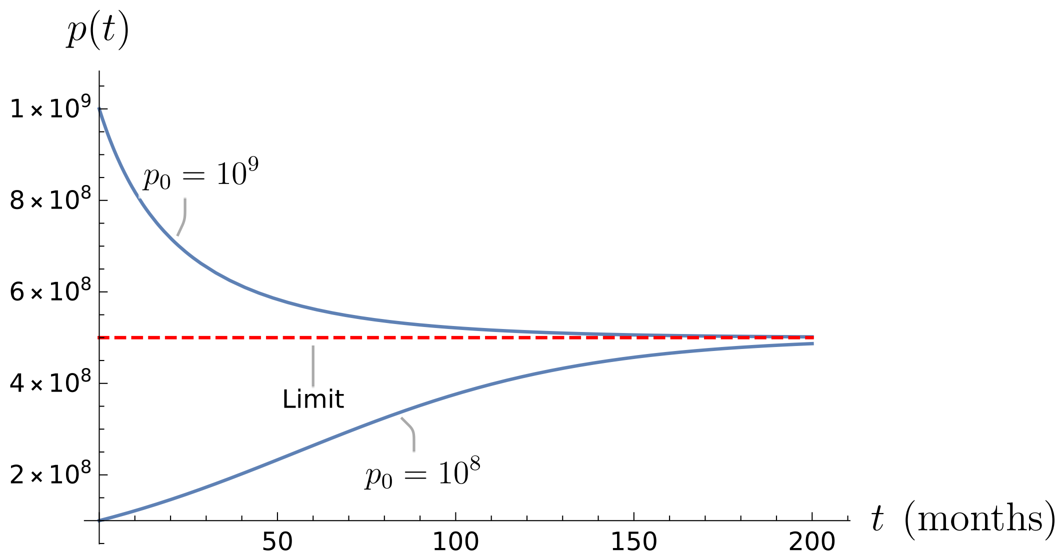

Figure 2: Population $p(t)$ change depends on p_0