HW 1

Math 4512
Differential Equations with Applications

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Contents

1 Problem 8, section 1.2

Solve $\frac{dy}{dt} + \sqrt{1+t^2}y = 0, y(0) = \sqrt{5}$

<u>Solution</u>

This is separable first order ODE. Therefore

$$\int \frac{dy}{y} = -\int \sqrt{1+t^2}dt \tag{1}$$

The LHS becomes

$$\int \frac{dy}{y} = \ln|y| \tag{2}$$

For the RHS of (1), the integral $\int \sqrt{1+t^2}dt$ can be evaluated as follows. Let $t = \sinh(\theta)$. Hence $\frac{dt}{d\theta} = \cosh(\theta)$. Therefore

$$\int \sqrt{1+t^2}dt = \int \sqrt{1+\sinh^2(\theta)}\cosh(\theta)\,d\theta$$

$$= \int \cosh^2(\theta)\,d\theta$$

$$= \int \frac{1}{2}(1+\cosh(2\theta))\,d\theta$$

$$= \frac{1}{2}\left(\int d\theta + \int \cosh(2\theta)\,d\theta\right)$$

$$= \frac{1}{2}\left(\theta + \frac{\sinh(2\theta)}{2}\right)$$

$$= \frac{1}{2}\theta + \frac{\sinh(2\theta)}{4}$$

Since $\sinh(2\theta) = 2\sinh\theta\cosh\theta$, the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\theta + \frac{\sinh\theta\cosh\theta}{2}$$

Since $\cosh^2(\theta) - \sinh^2(\theta) = 1$ then $\cosh^2\theta = 1 + \sinh^2(\theta)$ and the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\left(\theta + \sinh\theta\sqrt{1+\sinh^2(\theta)}\right)$$

But $t = \sinh(\theta)$ and $\theta = \text{arcsinh}(t)$. Therefore the above becomes

$$\int \sqrt{1+t^2}dt = \frac{1}{2}\left(\text{arcsinh}(t) + t\sqrt{1+t^2}\right) \tag{3}$$

Using (2,3) in (1) gives

$$\ln|y| = -\frac{1}{2}\left(\text{arcsinh}(t) + t\sqrt{1+t^2}\right) + C \tag{4}$$

Where C is arbitrary constant of integration. Writing $\text{arcsinh}(t)$ using known identity as $\ln\left|t + \sqrt{1+t^2}\right|$. And since $\sqrt{1+t^2}$ is always larger than t, then the absolute sign is not needed.

Eq. (4) becomes

$$\ln|y| = -\frac{1}{2}\left(\ln\left(t + \sqrt{1 + t^2}\right) + t\sqrt{1 + t^2}\right) + C$$

$$|y| = e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}e^C$$

$$y = C_1 e^{-\frac{1}{2}\left(\ln\left(t+\sqrt{1+t^2}\right)+t\sqrt{1+t^2}\right)}$$

$$= C_1 e^{-\frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)}e^{t\sqrt{1+t^2}}$$

Therefore the general solution is

$$y(t) = C_1 \frac{e^{t\sqrt{1+t^2}}}{\left(t + \sqrt{1 + t^2}\right)^{\frac{1}{2}}}$$

Now initial conditions are used to determine C_1. From $y(0) = \sqrt{5}$ then the above gives

$$\sqrt{5} = C_1$$

Therefore the particular solution is

$$y(t) = \sqrt{5}\frac{e^{t\sqrt{1+t^2}}}{\left(t + \sqrt{1 + t^2}\right)^{\frac{1}{2}}}$$

2 Problem 17, section 1.2

Find a continuous solution of the IVP $y + y' = g(t), y(0) = 0$ where

$$g(t) = \begin{cases} 2 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

Solution

This is linear first order ODE. The integrating factor is $\mu = e^{\int dt} = e^t$. Hence the ODE becomes

$$\frac{d}{dt}\left(y\mu\right) = \mu g(t)$$

$$\frac{d}{dt}\left(ye^t\right) = e^t g(t)$$

Integrating gives

$$ye^t = \int e^t g(t)\, dt + C \tag{1}$$

Breaking the problem into two phases, and solving the above for $0 \leq t \leq 1$ gives

$$ye^t = \int 2e^t dt + C$$

$$= 2e^t + C$$

$$y(t) = 2 + Ce^{-t}$$

Applying initial conditions gives $0 = 2 + C$, or $C = -2$ and the above becomes

$$y(t) = 2 - 2e^{-t} \qquad 0 \leq t \leq 1 \tag{2}$$

The above solution is valid for $0 \leq t \leq 1$.

To solve for $t > 1$, initial conditions are first found for $t = 1$. At $t = 1$ the above gives

$$y(1) = 2 - \frac{2}{e}$$

Hence for $t > 1$, initial conditions are $y(1) = 2 - \frac{2}{e}$. Now the second phase is solved. From (1)

$$ye^t = \int e^t g(t)\, dt + C$$

But now $g(t) = 0$. The above simplifies to

$$ye^t = C \tag{3}$$

$$y = Ce^{-t}$$

But at $t = 1, y = 2 - \frac{2}{e}$. Therefore

$$2 - \frac{2}{e} = Ce^{-1}$$

$$C = 2e - 2$$

$$= 2(e - 1)$$

Substituting the above C into (3) gives

$$y = 2\left(e - 1\right)e^{-t} \qquad t > 1 \tag{4}$$

Using (2,4) the final solution is therefore

$$y\left(t\right) = \begin{cases} 2 - 2e^{-t} & 0 \le t \le 1 \\ 2\left(e - 1\right)e^{-t} & t > 1 \end{cases}$$
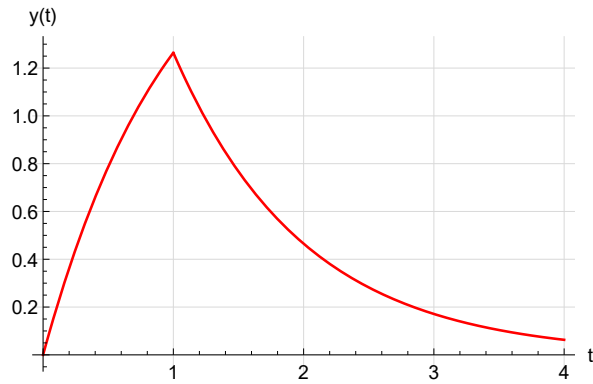

Figure 1: Plot of the solution $y(t)$

3   Problem 10, section 1.4

Solve $\cos y \frac{dy}{dt} = \frac{-t \sin y}{1+t^2}, y(1) = \frac{\pi}{2}$

Solution

This is separable first order ODE

$$\int \frac{\cos y}{\sin y} dy = -\int \frac{t}{1+t^2} dt$$

But $\int \frac{\cos y}{\sin y} dy = \int \frac{\frac{d}{dy} \sin y}{\sin y} dy = \ln|\sin(y)|$ and $\int \frac{t}{1+t^2} dt = \frac{1}{2} \ln|1+t^2| = \frac{1}{2} \ln(1+t^2)$ since $1+t^2$ is positive. Hence the above becomes

$$\ln|\sin(y)| = -\frac{1}{2}\ln(1+t^2) + C$$

Where C is the integration constant. Hence

$$|\sin(y)| = e^{-\frac{1}{2}\ln(1+t^2)+C}$$
$$= e^{-\frac{1}{2}\ln(1+t^2)} e^C$$

Therefore

$$\sin(y) = C_1 e^{-\frac{1}{2}\ln(1+t^2)}$$
$$= C_1 \frac{1}{\sqrt{1+t^2}} \tag{1}$$

From initial conditions $y(1) = \frac{\pi}{2}$ the above becomes

$$\sin\left(\frac{\pi}{2}\right) = C_1 \frac{1}{\sqrt{2}}$$
$$C_1 = \sqrt{2}$$

Hence (1) becomes

$$\sin(y) = \sqrt{2} \frac{1}{\sqrt{1+t^2}}$$
$$y(t) = \arcsin\left(\frac{\sqrt{2}}{\sqrt{1+t^2}}\right)$$

4 Problem 18, section 1.4

Solve $\frac{dy}{dt} = \frac{t+y}{t-y}$

<u>Solution</u>

Let $u = \frac{y}{t}$ or $y = ut$. Hence $\frac{dy}{dt} = u + t\frac{du}{dt}$. Therefore the ODE becomes

$$u + t\frac{du}{dt} = \frac{t + ut}{t - ut}$$
$$u + t\frac{du}{dt} = \frac{t(1 + u)}{t(1 - u)}$$
$$t\frac{du}{dt} = \frac{(1 + u)}{(1 - u)} - u$$
$$= -\frac{u^2 + 1}{u - 1}$$
$$= \frac{1 + u^2}{1 - u}$$

This is now separable ODE. Therefore

$$\frac{1 - u}{1 + u^2}\frac{du}{dt} = \frac{1}{t}$$
$$\int \frac{1 - u}{1 + u^2}du = \int \frac{1}{t}dt \tag{1}$$

But

$$\int \frac{1 - u}{1 + u^2}du = \int \frac{1}{1 + u^2}du - \int \frac{u}{1 + u^2}du$$
$$= \arctan(u) - \frac{1}{2}\ln|1 + u^2|$$

but $1 + u^2$ is positive. Hence

$$\int \frac{1 - u}{1 + u^2}du = \arctan(u) - \frac{1}{2}\ln\left(1 + u^2\right)$$

And $\int \frac{1}{t}dt = \ln|t|$. Hence (1) becomes

$$\arctan(u) - \frac{1}{2}\ln\left(1 + u^2\right) = \ln|t| + C$$

But $\frac{y}{t}$, and the above becomes

$$\arctan\left(\frac{y}{t}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{t}\right)^2\right) = \ln|t| + C$$

The above solution is implicit in $y(t)$.

5 Problem 4, section 1.5

Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

<u>Solution</u>

In Malthusian law of population growth, the rate at which population changes is fixed in the model. It is given by a below

$$\frac{dp}{dt} = ap(t)$$

Where a is constant. But the problem says the population is doubled in first 100 years. So if p_0 was initial population, then after 100 years the population now has become $2p_0$. There one will expect that after another 100 years the population will double again to become $4p_0$.

But the problem says that the population triples in 200 years, becoming $3p_0$ and not $4p_0$. This shows that the rate of growth is not constant. Hence this do not satisfy Malthusian law of population growth.

6 Problem 6(a), section 1.5

A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it was (a) 10^8 ?

Solution

In the logistic law, the population model is given by

$$\frac{dp}{dt} = ap - bp^2$$

Where $p(t)$ is population at time t and a is the growth rate (constant) and b is the competition rate (also constant). In this model

$$\lim_{t \to \infty} p(t) = \frac{a}{b}$$

Therefore

$$\frac{a}{b} = 5 \times 10^8 \tag{1}$$

The problem says that $a = 100\%$ (per 40 minute) or $a = 1$ (per 40 minute). Therefore $a = \frac{1}{40}$ per minute. And $p_0 = 10^8$. Using the solution of this model, given in the textbook at page 30 as

$$p(t) = \frac{ap_0}{bp_0 + \left(a - bp_0\right)e^{-a(t-t_0)}} \tag{3}$$

And using $t_0 = 0$, then the population size at t is now be calculated. From (1), $b = \frac{\frac{1}{40}}{5 \times 10^8} = \frac{1}{2} \times 10^{-10} = 5 \times 10^{-11}$. Eq. (3) now becomes

$$p(t) = \frac{\frac{1}{40}\left(10^8\right)}{\left(5 \times 10^{-11}\right)\left(10^8\right) + \left(\frac{1}{40} - \left(5 \times 10^{-11}\right)\left(10^8\right)\right)e^{-\frac{1}{40}t}}$$

$$= \frac{\frac{1}{40}\left(10^8\right)}{\left(5 \times \frac{1}{1000}\right) + \left(\frac{1}{40} - 5 \times \frac{1}{1000}\right)e^{-\frac{1}{40}t}}$$

For $t = 120$ (minutes) the above becomes

$$p(120) = \frac{\frac{1}{40}\left(10^8\right)}{\left(5 \times \frac{1}{1000}\right) + \left(\frac{1}{40} - 5 \times \frac{1}{1000}\right)e^{-\frac{1}{40}120}}$$

$$= \frac{\frac{1}{40}\left(10^8\right)}{\left(5 \times \frac{1}{1000}\right) + \left(\frac{1}{40} - 5 \times \frac{1}{1000}\right)e^{-3}}$$

$$= 4.1696 \times 10^8$$

Hence

$$p(120) = 4.1696 \times 10^8$$

The inflection point is

$$\frac{a}{2b} = \frac{\frac{1}{40}}{(2)\left(5 \times 10^{-11}\right)}$$
$$= 2.5 \times 10^8$$

The following plot was generated to compare the population $p(t)$ between case (a) and case (b). It shows that when starting with initial population of $p_0 = 10^8$ which is case (a) and when starting with $p_0 = 10^9$ which is case (b), both populations will eventually reach the limiting population of 5×10^8. The S curve shows up only when starting with population below the limiting population.
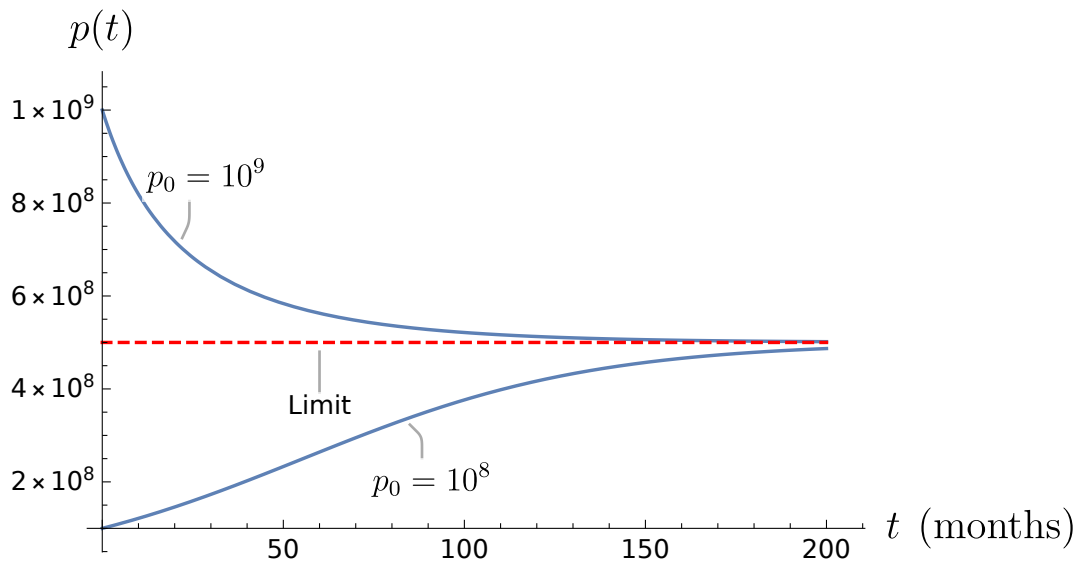

Figure 2: Population $p(t)$ change depends on p_0