$\frac{d}{dx}\ln x = \frac{1}{x}$ | $\frac{d}{dx} e^{ax} = ae^{ax}$ | $\int \ln(x) = -x + x\ln(x)$ | $\frac{d}{dx}\sin x = \cos x$ | $\frac{d}{dx}\cos x = -\sin x$

$ax^2+bx+c=0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$d/dx \ln\frac{1}{x} = -\frac{1}{x}$

$e^{x\ln(a)} = (e^{\ln(a)})^x = a^x$

$\log_a x = \frac{\ln(x)}{\ln(a)}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\sin x = x - \frac{x^3}{3!} + \cdots$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

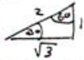
$\cos x = 1 - \frac{x^2}{2!} + \cdots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$

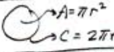
$\log x^y = y\log x$

$\int e^{at} = \frac{1}{a}e^{at}$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$(uv)' = uv' + vu'$



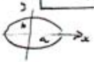


$\sin^2\phi = \frac{1-\cos 2\phi}{2}$

$\frac{A_1}{A} = \frac{\phi^\circ}{360}$ circle area

$\frac{ds}{\phi^\circ} = \frac{2\pi r}{360^\circ}$

$ds = r\phi \rightarrow$ in radian



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Incomplete $\Gamma(n,x) = \frac{x^{n-1}}{e^x}\left(1 + \frac{n-1}{x} + \frac{(n-1)(n-2)}{x^2} + \cdots\right)$

put $n = \frac{1}{2}$, $x = y^2$ to get $\operatorname{erf}(x) = 1 - \frac{\Gamma(\frac{1}{2}, x^2)}{\sqrt{\pi}}$

$(1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3$

$(1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 -$

$\log(xy) = \log x + \log y$

$\log x^n = n\log x$

$\log\frac{x}{y} = \log x - \log y$

$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

$\int \cos 2x\, dx = \cos x \sin x$

$\int \sin 2x = -\frac{1}{2}\cos(2x)$

Legendre DE $(1-x^2)y'' - 2xy' + l(l+1)y = 0$

$\int_{-1}^{1} P_l P_m = 0$

inner product $a_n = \boxed{\frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\frac{2}{2n+1}}}$

$\tilde{f}(x) = a_0 P_0 + a_1 P_1 + \cdots$

$\int_{-1}^{1} f(x) f(x) = \hat{N} \leftarrow$ norm so normalized ~~vector~~ function $= \frac{f(x)}{\sqrt{N^2}}$

$\int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha}e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$

$\frac{d}{dx}\ln(f(x)\ g(x)) = \frac{g(x) f'(x) + f(x) g'(x)}{f(x)\ g(x)}$

$\int_0^L ax\cos(bx) = \frac{a(-1+\cos(bL)+bL\sin(bL))}{b^2}$

$\lambda = \int g(x)$

$\downarrow$

$\frac{d}{dx}\lambda = g(x)$

$\int_0^L x \sin x$

$= -L\cos L + \sin L$

$\nabla^2 u = 0$ $\begin{cases} \sin kx\, e^{ky} \\ \sin kx\, e^{-ky} \\ \cos kx\, e^{ky} \\ \cos kx\, e^{-ky} \end{cases}$

$\boxed{\Gamma(n) = (n-1)!}$

$u = f(x) = \sum b_n \sin\frac{n\pi}{L}x$

$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ $\begin{cases} e^{-\alpha^2 k^2 t}\sin kx \\ e^{-\alpha^2 k^2 t}\cos kx \end{cases}$

$\Rightarrow \boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x}$

$$\int_0^\infty x^n e^{-ax} = \frac{n!}{a^{n+1}} \Rightarrow \int_0^\infty x^n e^{-x} = n! \xrightarrow{n \text{ noninteger}} \int_0^\infty x^{p-1} e^{-x} = \Gamma(p),\ p>0$$

$\Gamma(n) = (n-1)!$ | ex: $\Gamma(5)=4!$ $\Gamma(n+1) = n!$ | $\Gamma(p+1) = p\Gamma(p)$ | $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$, $\Gamma(-0.5) = \frac{1}{-0.5}\Gamma(0.5)$

$\Gamma(p) = \infty$ For $p = -1, -2, -3, \dots$ $\Gamma(p) = \infty$ at 0 | $\Gamma(\frac{1}{2}) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x} = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} = \sqrt{\pi}$ special: $\int_0^\infty e^{-t^2} = \frac{1}{2}\Gamma(\frac{1}{2})$ | $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$

$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ $\Leftarrow \int_0^1 \frac{x^4}{\sqrt{1-x^2}}$ $p>0,\ q>0$ | $B(p,q) = B(q,p)$ | $B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta$

$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ | $B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$ or $\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$ $\Leftarrow$ used to solve $\int_0^\infty \frac{x^3}{(1+x)^5}$ type integrals

$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ $\Leftarrow$ factor causes erf to be 1 | $P(-\infty, x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)$

$P(0,x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2} = \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)$ | $\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2} = 1 - \text{erf}(x)$

This area $= \frac{\text{erf}(x)\cdot\sqrt{\pi}}{2}$

This area $= 1 - \text{erf}(x)$

$1-p$ 0 1 p

$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ | p $p+1$ $p\Gamma(p) = \Gamma(p+1)$ | $\text{erf}(\infty) = 1$ since

$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} \dots\right)$ $|x| \ll 1$

sterling approx $n! = n^n e^{-n}\sqrt{2\pi n}$

elliptic integrals:

Legendre form: $F(k,\phi) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$, $E(k,\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}$ | $k = \sin\theta$ [redacted]

Complete: $F(k,\pi/2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$, $E(k,\pi/2) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi$

$P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, $P_3 = \frac{1}{2}(5x^3 - 3x)$

$P_\ell = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

Jacobi form: $F(k,\pi/2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, $E = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$

$x = \sin\phi$ in Legendre forms | $F(k,x) = \int_0^x$ | $E(k,x) = \int_0^x$

$\Phi(x,h) = (1-2xh+h^2)^{-1/2}$ expand in power series. $\Phi(x,h) \to P_0 + hP_1 + h^2P_2 + \dots$

$f(x), g(x)$ are orthogonal on (a,b) if $\int f^*(x)g(x)\,dx = 0$

$\int_{-\pi}^{\pi} \sin nx \sin mx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases}$

$\int_{-\pi}^{\pi} \sin nx \cos mx = 0$ for any n, m

$\int_{-1}^1 P_\ell P_m = 0$ $m \neq n$

$\int_{-1}^1 P_\ell = 0$ $\int_{-1}^1 P_\ell x^m = 0$ $m < \ell$

Associated legendre: $P_\ell^m = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_\ell$

$\text{erf}(x) = 2P(0, x\sqrt{2})$ | $\int_{-1}^1 P_\ell P_\ell = \frac{2}{2\ell+1}$

$\int_0^\infty e^{-ax}\sin(bx) = a(-b\cos(bL) + \dots \sin(bL)) \dots / b^2$

Leibniz Rule: example $\frac{d^q}{dx^q}(x\sin x) = x\frac{d^q}{dx^q}\sin x + q\frac{d}{dx}\frac{d^{q-1}}{dx^{q-1}}\sin x$

$\frac{d^n}{dx^n}fg = f\frac{d^n g}{dx^n} + nf'\frac{d^{n-1}}{dx^{n-1}}g + \frac{n(n-1)}{2!}f''\frac{d^{n-2}}{dx^{n-2}}g + \dots$ $+ \frac{q(q-1)}{2!}\frac{d^2}{dx^2}x\,\frac{d^{q-2}}{dx^{q-2}}\sin x$

$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots + b^n$ | to prove $B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}$ sub $x = \frac{y}{1+y}$

binomial coeff $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

For integer $(a+b)^n = a^n\binom{n}{0}b^0 + a^{n-1}\binom{n}{1}b^1 + a^{n-2}\binom{n}{2}b^2 + \dots + \binom{n}{n}a^0 b^n$

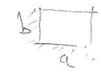
HW: Single card drawn from deck. What is prob. it is either ace or Red or both?
$P = P(\text{Ace}) + P(\text{Red}) - P(\text{Ace and Red}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

HW: player makes ball in 3 tries out of 4. How many tries need to have chance > .99 of at least one basket? prob. not scoring after n tries $= (1-P)^n = (\frac{1}{4})^n$.
Solve $(\frac{1}{4})^n = 0.01$ $\quad n \log \frac{1}{4} = \log(0.01) \Rightarrow n = 4$

What is prob that a number is $1 \le n \le 99$ is divisible by both 6 and 10? by either 6 or 10? or both? A = event divisible by 6 $\Rightarrow P(A) = \frac{16}{99}$
B = event divisible by 10 $\Rightarrow P(B) = \frac{9}{99}$
$P(AB) = P(A)\,P_A(B)$. $P_A(B) = \frac{3}{16} \Rightarrow P(AB) = \frac{1}{33}$
$P(A+B) = P(A) + P(B) - P(AB) = \frac{2}{9}$

Sterling approx $n! = n^n e^{-n}\sqrt{2\pi n}$

$$\int_0^\infty x^n e^{-ax} = \frac{n!}{a^{n+1}} \quad \int_0^\infty e^{-x} x^n = n! \quad \Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \quad \Gamma(n) = (n-1)!,\ \Gamma(n+1) = n! \quad \Gamma(p+1) = p\Gamma(p),\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p} \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$$

HW: Separate wave eq. in 2D rect. $z = (t, x, y)$.
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \Rightarrow z = XYT$$

$\Rightarrow YTX'' + XTY'' = \frac{1}{v^2} XYT''$
$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{v^2}\frac{T''}{T} \Rightarrow$ each is a function of one variable and not in others
$\Rightarrow \frac{X''}{X} = -k_x^2,\ \frac{Y''}{Y} = -k_y^2,\ \frac{T''}{T} = -(k_x^2 + k_y^2)v^2$
$\Rightarrow X = \begin{cases} \sin k_x x \\ \cos k_x x \end{cases},\ Y = \begin{cases} \sin k_y y \\ \cos k_y y \end{cases},\ T = \begin{cases} \sin(vt\sqrt{\ }) \\ \cos(vt\sqrt{\ }) \end{cases}$

Since attached to supports. $X = 0$ at $x = 0$ and $x = a$
$Y = 0$ at $y = 0,\ y = b$
so $X = \sin k_x x$, and at $x = a \Rightarrow \sin k_x a = 0$
$\Rightarrow k_x = \frac{n\pi}{a} \Rightarrow X = \sin\frac{n\pi x}{a}$. Similarly
$Y = \sin\frac{m\pi y}{b}$. $\Rightarrow T = \begin{cases} \sin\left(vt\pi\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}\right) \\ \cos\ \ '' \end{cases}$
so $f = \frac{v}{2}\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}$

Hermite Pols: $H_0 = 1,\ H_1 = 2x$, $H_2 = 4x^2 - 2,\ H_3 = -12x + 8x^3$

PDE	Solutions
$\nabla^2 u = 0$	$u(x,y) = \begin{cases} e^{ky}\cos kx \\ e^{ky}\sin kx \\ e^{-ky}\cos kx \\ e^{-ky}\sin kx \end{cases}$
$\nabla^2 = f$	$u(x,y,z) = -\frac{1}{4\pi}\iiint \frac{f(x',y',z')}{ds}\,dx'dy'dz'$
$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ Diffusion	$u(t,x) = \begin{cases} e^{-k^2\alpha^2 t}\cos kx \\ e^{-k^2\alpha^2 t}\sin kx \end{cases}$ For bar (one dimension)
$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$	$Y = XT$. $X = \begin{cases}\cos kx \\ \sin kx\end{cases}$ $T(t) = \begin{cases}\cos \omega t \\ \sin \omega t\end{cases}$. $Y = \begin{cases} \cos kx \cos \omega t \\ \cos kx \sin \omega t \\ \sin kx \cos \omega t \\ \sin kx \sin \omega t \end{cases}$ For 1D like string $\omega = kv$. $k = \frac{2\pi}{\lambda}$.
	$Z = XYT$ for 2D wave. $X = \begin{cases} \cos k_x x \\ \sin k_x x \end{cases}$ $Y = \begin{cases} \cos k_y y \\ \sin k_y y \end{cases}$ $T = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$

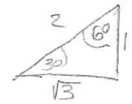
$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

HW: Separate Schrodinger eq. $\nabla^2\psi + (\epsilon - bV)\psi = 0$ in spherical.
write in spherical $= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + (\epsilon - bV(r))\psi = 0$.
$\psi = R(r)\Theta(\theta)\Phi(\phi) \Rightarrow$ sub in above and multiply by $\frac{r^2}{R\Theta\Phi}$
$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (\epsilon - bV(r))r^2 + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$
$= K$
$\Rightarrow K + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0 \Rightarrow$ multiply by $\sin^2\theta$ and let $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ in result $= -m^2$. $\Rightarrow \Phi = \begin{cases}\sin m\phi \\ \cos m\phi\end{cases}$. so we now have
$K + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0 \Rightarrow$ multiply by Θ
$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + K\Theta = 0 \Rightarrow$ solution $\Theta = P_\ell^m \cos\theta$
so $\psi = P_\ell^m \cos\theta \begin{cases}\sin m\phi \\ \cos m\phi\end{cases}$

HW: show that gravitational energy $V = -\frac{Gm}{r}$ satisfy Laplace:
$V = \frac{-Gm}{\sqrt{x^2+y^2+z^2}}$. Find $\frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}$. and add $\frac{\partial^2 V}{\partial y^2}, \frac{\partial^2 V}{\partial z^2} = 0$

Expansions of $f(x)$:
$f(x) = a_0P_0 + a_1P_1 + a_2P_2$
$a_i = \frac{\int_{-1}^1 f(x)P_i(x)}{\int_{-1}^1 P_i(x)P_i(x)} = \frac{\sim}{\frac{2}{2\ell+1}}$

Laplace eq in cylindrical:
$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$
$u = R(r)\Theta(\theta)Z(z)$. $Z = \begin{cases} e^{kz} \\ e^{-kz}\end{cases}$, $\Theta(\theta) = \begin{cases}\sin\theta \\ \cos\theta\end{cases}$ $\quad R(r) = J_n(kr)$
$\Rightarrow u = \begin{cases} J_n(kr)\sin n\theta\, e^{-kz} \\ J_n(kr)\cos n\theta\, e^{-kz}\end{cases} \xrightarrow{\text{symmetry}} u = J_0(K_m r)e^{-K_m z}$
K is zero of J_n.

HW: Find solution for steady state temp distribution in solid semi-infinite cylinder if $u = 0$ at $r = 1$, $u = y = r\sin\theta$ at $z = 0$.
$u = \begin{cases} J_n(kr)\sin(n\theta)e^{-kz} \\ J_n(kr)\cos(n\theta)e^{-kz}\end{cases}$ since $u = r\sin\theta$ when $z = 0$, keep and set $n = 1$
$\Rightarrow u = J_n(kr)\sin(n\theta)e^{-kz} \Rightarrow u = \sum_{m=1}^{\infty} C_m J_1(K_m r)\sin\theta\, e^{-K_m z}$

$z = 0$. B.C. to find C_m. $u = r\sin\theta = \sum C_m J_1(K_m r)\sin\theta$.
take inner product w.r.t. $rJ_1(K_u r)$ from $r = 0$ to 1.
$$\int_0^1 r\sin\theta\, rJ_1(K_u r)\,dr = \sum C_m \int_0^1 J_1(K_m r)\sin\theta\, rJ_1(K_u r)\,dr$$
$$\Rightarrow \int_0^1 r^2 J_1(K_u r) = \sum C_m \int_0^1 J_1(K_m r)\, rJ_1(K_u r)\,dr$$
From orthogonality of Bessel: $\int_0^1 J_p(K_m r)\, r\, J_p(K_u r)\,dr = 0 \quad m \ne u$.
$$\Rightarrow \int_0^1 r^2 J_1(K_m r)\,dr = C_m\int_0^1 rJ_1(K_m r)J_1(K_m r)\,dr$$
$$C_m = \frac{\int_0^1 r^2 J_1(K_m r)\,dr}{\int_0^1 rJ_1(K_m r)J_1(K_m r)\,dr} = \frac{\int\ ''}{\frac{1}{2}[J_2(K_m)]^2}$$ but $\int_0^1 r^2 J_1(K_m r) = \frac{1}{K_m}J_2(K_m)$
$$\Rightarrow C_m = \frac{\frac{1}{K_m}J_2(K_m)}{\frac{1}{2}[J_2(K_m)]^2} = \frac{2}{K_m J_2(K_m)} \Rightarrow u = \sum C_m J_1(K_m r)\sin\theta\, e^{-K_m z} = \sum \frac{2}{K_m J_2(K_m)} J_1(K_m r)\sin\theta\, e^{-K_m z}$$

Laplace spherical: $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$
$u = R(r)\Theta(\theta)\Phi(\phi)$, $\Phi = \begin{cases}\sin m\phi \\ \cos m\phi\end{cases}$ $\quad \Theta = \{P_\ell^m\cos\theta$, $R(r) = \begin{cases} r^\ell \\ r^{-\ell-1}\end{cases}$

$u = r^\ell P_\ell^m\cos\theta \begin{cases}\sin m\phi \\ \cos m\phi\end{cases}$. For symm. $m = 0 \Rightarrow u = r^\ell P_\ell \cos\theta$

HW: Find steady state temp dist. inside sphere; $r = 1$ when surface temp is $u = \cos\theta - (\cos\theta)^3$. $u = r^\ell P_\ell^m(\cos\theta)\begin{cases}\sin m\phi \\ \cos m\phi\end{cases} \Rightarrow$ set $m = 0$ since symm
$u = r^\ell P_\ell(\cos\theta) \Rightarrow u_g = \sum c_\ell r^\ell P_\ell(\cos\theta)$. when $r = 1$, $u = \cos\theta - (\cos\theta)^3$
$\Rightarrow \cos\theta - (\cos\theta)^3 = \sum c_\ell P_\ell(\cos\theta)$. Let $x = \cos\theta$. We see that
$x - x^3 = \frac{2}{5}P_1(x) - \frac{2}{5}P_3 \Rightarrow c_1 = \frac{2}{5}, c_3 = -\frac{2}{5} \Rightarrow u = \frac{2}{5}r^1 P_1(\cos\theta) - \frac{2}{5}r^3 P_3(\cos\theta)$

Fuch Theorem: write ODE as $y'' + f(x)y' + g(x)y = 0$. if xf and $x^2 g$ are expandable in convergent power series $\sum_0 a_n x^n$, we say ODE is regular at origin. if one solution found, second $y_2 = y_1 \ln x + \text{Series Frob.}$ example: $4x^2y'' + y = 0$ has one solution $y_1 = \sqrt{x}$. so second is $y_2 = \sqrt{x}\ln x + \sum_0 b_n x^{n+s}$. sub into DE
$y' = \frac{1}{2}\frac{1}{\sqrt{x}}\ln x + \frac{\sqrt{x}}{x} + \sum b_n(n+s)x^{n+s-1}$, $y'' = -\frac{1}{4}\frac{1}{x^{3/2}}\ln x + \frac{1}{2}x^{-3/2} - \frac{1}{2}x^{-3/2} + \sum b_n(n+s)(n+s-1)x^{n+s-2}$
sub into ODE, set $n = 0 \Rightarrow$ indicial eq $\Rightarrow s = \frac{1}{2}$.

Bessel: $x^2y'' + xy' + (x^2 - p^2)y = 0$. $y_1 = J_p(x) = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}$
$y_2 = N_p = \frac{\cos(\pi p)J_p(x) - J_{-p}}{\sin \pi p}$
orthogonality: $\int_0^1 x J_p(ax) J_p(bx)\,dx = \begin{cases} 0 & a \ne b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J'^2_p(a) & a = b\end{cases}$
recursive: $\frac{d}{dx}(x^p J_p) = x^p J_{p-1}$, $J_{p-1} - J_{p+1} = 2J_p'$, $\frac{d}{dx}\left(\frac{1}{x^p}J_p\right) = -\frac{1}{x^p}J_{p+1}$
$J_p' = -\frac{p}{x}J_p + J_{p-1} = \frac{p}{x}J_p - J_{p+1}$, $j_n = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

Legendre: $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$, $P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$. normalization: $\int_{-1}^1 (P_\ell(x))^2 = \frac{2}{2\ell+1}$
$y_1 = P_\ell(x)$. orthogonality: $\int_{-1}^1 P_\ell(x)P_m(x) = 0 \quad m \ne \ell$ also $\int_{-1}^1 P_\ell(x) \times \text{any poly deg} < \ell = 0$

$P_r^n = \frac{n!}{(n-r)!}$ order not imp

Leibniz Rule: example: $\frac{d^9}{dx^9}(x\sin x) = x\frac{d^9}{dx^9}\sin x + 9\frac{d}{dx}x\frac{d^8}{dx^8}\sin x + \frac{9 \times 8}{2}\frac{d^2}{dx^2}x\frac{d^7}{dx^7}\sin x + \cdots$ $\binom{n}{1}$ $\binom{n}{2}$

Example: how many ways can 10 people be seated on bench with 4 seats? $\binom{10}{4}$.

$\int_0^L \sin^2\frac{n\pi}{L}x\,dx = \frac{L}{2}$

HW: box A has 5 red ball, 6 black. Box B has 5 red ball, 8 white. What is prob of picking red ball? $= P(\text{Pick A}) \times P(\text{Pick red from A}) + P(\text{Pick B}) \times (P\text{ pick red from B})$

it is known that 1% of population have cancer. it is also known that a test of this cancer is positive in 99% of people who have it but is also positive in 2% of people who don't have it. What is prob. that a person who test positive has cancer of this type? Let A = event person has cancer, Let B = Event test is positive. Find $P_B(A) = \frac{P(BA)}{P(B)} = \frac{P(A)\,P_A(B)}{P(B)}$. $P(A)$ given as 1% (0.01), $P(B) = 99\% \times 1\% + 2\% \times 99\% = 0.99\times0.01 + 0.02\times0.99 = 0.0297$. $P_A(B) = 0.99$ given. then $P_B(A) = \frac{0.01\times0.99}{0.0297} = 0.333$ so 33% chance person who test positive has cancer.

HW 5 Cards are dealt from Deck. what is P() they are all same suite? $P = $ # ways to select suite × # way to select 5 Cards out of same suite $= \frac{4\times\binom{13}{5}}{\binom{52}{5}} = 1.98\times10^{-3}$. what is Prob they are all diamonds? $= \frac{\binom{13}{5}}{\binom{52}{5}} = 4.95\times10^{-4}$ (# ways to select 5 Cards from 52), what is Prob they are all face card? $\frac{\binom{12}{5}}{\binom{52}{5}}$

HW What is prob they are from same suite and in series? $= 4\times\frac{\binom{13}{5}\times\frac{9}{\binom{13}{5}}}{\binom{52}{5}} = 1.39\times10^{-5}$

In family of 5 children, what is Prob. there are 2 boys and 2 girls? $= \frac{\binom{5}{3}}{2^5} = \frac{5}{16}$

what is prob. that 2 oldest are boys and the others are girls? Let A = event first 2 born are boys, B = Event Last 3 born are Girls. Then $P(AB) = P(A)\,P_A(B) = \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{1}{32}$

what is prob. that 2 and 3 of clubs are next to each others in Deck of card? $= \left(\frac{1}{52}\times\frac{1}{51} + \frac{1}{52}\times\frac{1}{51} + \frac{50}{52}\times\frac{2}{51}\right) = \frac{1}{26}$

Birthday formula: $\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$

generalized Power Series $y = a_0x^s + a_1x^{s+1} + \ldots$

Maxwell-Blotzman: Balls are different. Can have more the one ball in Box.

Fermi-Dirace: Balls are same, each Box can have one Ball only, just at most.

Bose-Einstein: Balls are same, can have more than one ball in same Box

MB $= n^b$ (b = # Boxes, n = # balls), FD: number of ways to put b balls in n boxes $= \binom{n}{b}$

BE: $\binom{n-1+b}{b}$

$\sin 2x = 2\sin x\cos x$, $\cos 2x = 2\cos^2 x - 1$ | $\int e^{at} = \frac{1}{a}e^{at}$ | $e^{x\ln a} = a^x$

A weighted coin with p prop. for head is tossed 3 times, x = number of heads minus tail. find μ, σ.

	3h	2h,1t	2t,1h	3t
$x_i =$	3	1	−1	−3
$P_i =$	p^3	$\binom{3}{1}p^2(1-p)$	$\binom{3}{1}p(1-p)^2$	$(1-p)^3$

$\frac{d}{dx}\ln x = \frac{1}{x}$, $\int \ln x = -x + x\ln x$

$\mu = \sum x_iP_i = 3p^3 + 3p^2(1-p) - 3p(1-p)^2 - 3(1-p)^3 = 3(2p-1)$

$Var = \sum P_i(x_i-\mu)^2 = 12p(1-p)$ $\sigma = \sqrt{Var} = 2\sqrt{3p(1-p)}$ or use $Var = \left(\sum x_i^2P_i\right) - \mu^2$

For binomial distribution $\boxed{\bar{x} = np,\ \sigma = \sqrt{npq}}$, $Var = npq$. Var can also be written $\overline{x^2} - \bar{x}^2$

Trick $\lim_{N\to\infty}\left(1-\frac{1}{N}\right)^N = \frac{1}{e}$. to proof: $\left(1-\frac{1}{N}\right)^N = e^{N\log\left(1-\frac{1}{N}\right)} = e^{N\left(1-\frac{1}{N}-\cdots\right)} = e^{-1}$

recursive Hermite: $H_{n+1}(x) = 2x\,H_n - 2n\,H_{n-1}$

Gaussian probability density for a random variable with μ, σ is $P = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, Cummulative $= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt$

Example: Find p of exactly 52 heads in 100 toss: $\mu = np = 100\cdot\frac{1}{2} = 50$

$\sigma = \sqrt{npq} = \sqrt{100\cdot\frac{1}{2}\cdot\frac{1}{2}} = 5 \Rightarrow t = \frac{x-\mu}{\sigma} = \frac{52-50}{5} = 0.4$. Then

$P(x) = \frac{1}{\sigma}\phi(t) = \frac{1}{5}\phi(0.4)$. but $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2} \Rightarrow \phi(t) = 0.36 \Rightarrow P = 0.07$

HW ~~Find~~ using Normal approx (Gaussian) to binomial Find prop. of getting exactly 120 aces in 720 tosses of die: $\sigma = \sqrt{npq} = \sqrt{720\cdot\frac{1}{6}\cdot\frac{5}{6}} = 10$

$\mu = np = 720\times\frac{1}{6} = 120 \Rightarrow p = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ where $t = \frac{x-\mu}{\sigma} \Rightarrow P = 0.0398$

HW: Using Gaussian distribution, Find prob of getting between 499,000 and 501,000 heads in 10^6 tosses: $\frac{1}{\sigma\sqrt{2\pi}}\int_{t_1}^{t_2}\sigma e^{-\frac{t^2}{2}}dt$. $t_1 = \frac{x_1-\mu}{\sigma}$, $t_2 = \frac{x_2-\mu}{\sigma}$

$\mu = np = 10^6\times\frac{1}{2} = 500{,}000$

$\sigma = \sqrt{npq} = \sqrt{10^6\times\frac{1}{2}\times\frac{1}{2}} = 500 \Rightarrow t_1 = \frac{499{,}000-500{,}000}{500}$, $t_2 = \frac{501{,}000-500{,}000}{500}$. $t_1 = -2, t_2 = 2$

So want $P(-2,2) = P(-2,0) + P(0,2) = 2P(0,2)$ → need table to find. This is $\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-t^2/2}dt$ error function

$= 2\times\frac{1}{2}\operatorname{erf}\left(\frac{2}{\sqrt{2}}\right)$ ← From Tables $= 0.954$

Summary Gaussian: Find t_1, t_2. Then μ, σ. Then write $P(-x,x) = P(-x,0) + P(0,x) = 2P(0,x) = 2\times\frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$ (to find prob. between t_1, t_2).

Sample space for 2 die sum $x = 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12$

$P_i = 1\ 2\ 3\ 4\ 5\ 6\ 5\ 4\ 3\ 2\ 1 \quad \div 36$

$\mu = 7$, $Var = \frac{35}{6}$

2 Cards drawn from Deck. what is prob. they are both ace? $= \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{1}{221}$ or $\frac{4}{52}\times\frac{3}{51}$; if you know one is Ace, what is prob. both are ace? $P = \frac{\binom{4}{2}}{(4\times48)+\binom{4}{2}} = \frac{6}{(4\times48)+6} = \frac{1}{33}$.

if one is ace of spade, what is prob both are aces? $= \frac{3}{48+3} = \frac{1}{17}$

make sample space of possible sets of 2 cards.

if a coin is tossed 5 times, what is prob. of exactly 3 heads? $P = \binom{5}{3}\left(\frac{1}{2}\right)^5$. P = head, q tail. we set $\binom{5}{3}p^3q^2$

For weighted coin

binomial Probability density: $f(x) = \binom{n}{x}p^xq^{n-x}$. This is prob. of exactly x success out of n trials. p = prob. of success

Prob. density function answer question: what is prob. x is ...

distribution (or cummulative) answers: what is prob of at most x.

If biased coin tossed n times, what is chance of K heads? $= \binom{N}{K}p^Kq^{N-K}$ where p = chance of head

operator $x\frac{d}{dx}: x^n \to nx^n$

Gaussion distro is good approx to poisson when μ is large

Poisson prob. distribution: $P_n = \frac{\mu^n}{n!}e^{-\mu}$. where μ = averge Count per unit time.

Poisson is good approx. to binomial for small n, small np.

Normal is good " " " " Large n, Large np

Poisson approximation: $\binom{n}{x}p^xq^{n-x} \sim \frac{(np)^x e^{-np}}{x!}$ For small P Large n

Example: using binomial dist: if 1000 each select number between 1 and 500 what is prob. that 3 people selected 29? using binomial: $\binom{1000}{3}\left(\frac{1}{500}\right)^3\left(\frac{499}{500}\right)^{997}$

using poisson: $np = 2 \Rightarrow P = \frac{2^3e^{-2}}{3!}$

HW poisson. In an alpha particle counting, number of alpha particles is recorded each minute for 50 hrs. total particles is 6000. in how many 1 minute intervals you expect no particles? Exactly n=1?

here $\mu = \frac{6000}{50\times60} = 2$ paticles per min $\Rightarrow P = \frac{\mu^n}{n!}e^{-\mu}$.

For n=0 ⟹ $P = e^{-2} = 0.135$. so 13% of time. i.e 406 1 minute intervals.

n=1 ⟹ $P = \frac{2^1}{1}e^{-2} = 812$ 1-minute intervals.

HW poisson. Suppose you have 5 exams during 5 days. Find prob on a given day you have no exam?, one exam? $P = \frac{\mu^n}{n!}e^{-\mu}$. $\mu = 1$ exam per day. $\Rightarrow P_0 = e^{-1}$ etc..

HW: if there are 100 misprints in a 40 pages. on how many pages you expect to find no misprints? 2 misprint? $\mu = \frac{100}{40} = 2.5$. so Prob. of no misprint on given page $= P_0 = \frac{\mu^0 e^{-\mu}}{0!} = 0.082$

⟹ expected pages with no misprints $= 40\times P_0 = 3.3$

⟹ " " " 2 misprints $= 40P_2 = 40\times\frac{2.5^2e^{-2.5}}{2!} = 10.3$ etc..

HW: Coin is tossed repeatedly. x = number of toss in which head first appear. Find $P_i, \bar{x}, Var$. $x_i = 1\ 2\ 3 \cdots$, $P_i = \frac{1}{2}\ \left(\frac{1}{2}\right)^2\ \left(\frac{1}{2}\right)^3 \cdots \Rightarrow \mu = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$.

$\sum x^n = \frac{x}{1-x} \Rightarrow \frac{d}{dx} \Rightarrow \sum nx^{n-1} = \frac{x}{(1-x)^2} + \frac{1}{1-x}$ ⟹ multiply by x ⟹ $\sum nx^n = \frac{x^2}{(1-x)^2} + \frac{x}{1-x}$ ⟹ sub $x = \frac{1}{2}$ ⟹ $\sum n\left(\frac{1}{2}\right)^n = 2 = \mu$

$Var = \left(\sum x_i^2P_i\right) - \mu^2 = \sum n^2\left(\frac{1}{2}\right)^n - 4$. to find $\sum n^2\left(\frac{1}{2}\right)^n \Rightarrow \frac{d}{dx}\left(\frac{x^2}{(1-x)^2} + \frac{x}{1-x}\right) = \frac{2x^2}{(1-x)^3} + \frac{2x}{1-x^2} + \frac{x}{(1-x)^2} + \frac{1}{1-x}$ ⟹ multiply by x. and sub $x = \frac{1}{2}$ ⟹ $Var = 6 - 4 = 2$