HW 9

MATH 121B

Spring 2004
UC BERKELEY

Nasser M. Abbasi

Spring 2004

Contents

1 Chapter 13, problem 6.1 Mary Boas. Second edition

chapter 13

6.1

will use $\boxed{z = J_n (K_{mn} r) \cos n\theta \cos K_{mn} \upsilon t}$ to
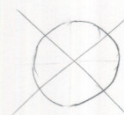
Complete Figure 6.1.

K_{mn} is the mth zero of J_n.

Circle is divided into as many sectors as $2n$. for example, when $n=1$, we will set
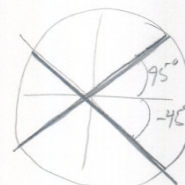
when $n=2$, we will set

this is because we want a solution

$$\cos n\theta = \pm \frac{\pi}{2}$$

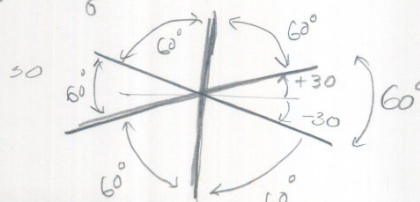which means $\theta = \pm \frac{\pi}{2n}$

so for $n=1$, $\theta = +\frac{\pi}{2}, -\frac{\pi}{2}$

$n=2$ $2\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{4}$

for $n=3$ $3\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{6}$

so

etc...

now, for changing of the m.

from $z = J_n(k_{mn} r) \cos n\theta \cos k_{mn} v t$

we want $J_n(k_{mn} r)$ to zero.

as m increases, $k_{m_2 n} > k_{m_1 n}$ when $m_2 > m_1$

so r becomes smaller for each m increasing.

if original radius of drum is 1, then

r for m=1 ⟹ 1

r for m=2 ⟹ $\frac{k_{10}}{k_{20}} < 1$

r for m=3 → $\frac{k_{10}}{k_{30}} < \frac{k_{10}}{k_{20}}$

So each time m increases by 1 we add a smaller radius inside. so now I show the complete figure for $n = 0, 1, 2$, $m = 1, 2, 3$.
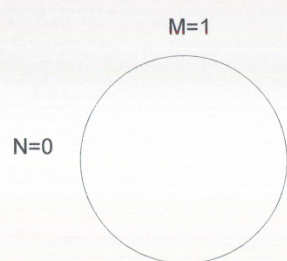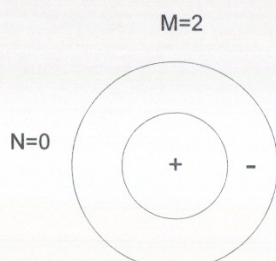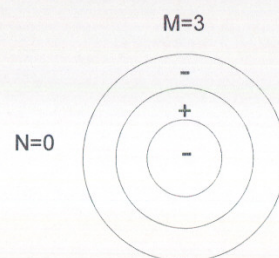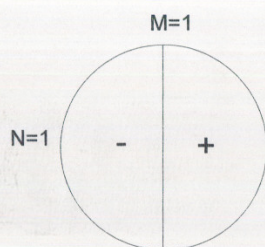
$\longrightarrow$

M=1 N=0
$$z = J_0(k_{10}r)\cos(k_{10}vt)$$

M=2 N=0
$$z = J_0(k_{20}r)\cos(k_{20}vt)$$

M=3 N=0
$$z = J_0(k_{30}r)\cos(k_{30}vt)$$

M=1 N=1
$$z = J_1(k_{11}r)\cos\theta\cos(k_{11}vt)$$

M=2 N=1
$$z = J_1(k_{21}r)\cos\theta\cos(k_{21}vt)$$

M=3 N=1
$$z = J_1(k_{31}r)\cos\theta\cos(k_{31}vt)$$

M=1 N=2
$$z = J_2(k_{12}r)\cos 2\theta\cos(k_{12}vt)$$

M=2 N=2
$$z = J_2(k_{22}r)\cos 2\theta\cos(k_{22}vt)$$
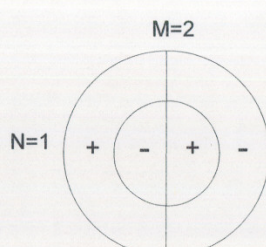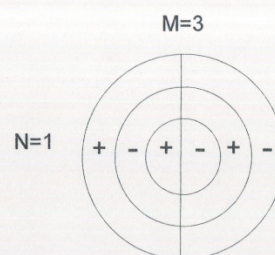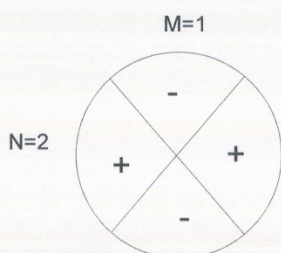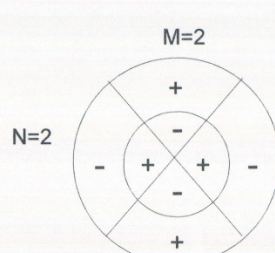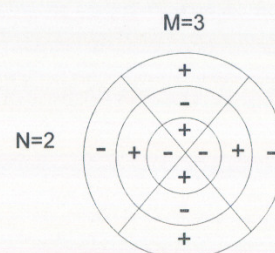
M=3 N=2
$$z = J_2(k_{32}r)\cos 2\theta\cos(k_{32}vt)$$

chapter 13, 6.2

---

look up first 3 zeros of K_{mn} of each Bessel function J_0, J_1, J_2, J_3. find the first 6 frequencies of a vibrating circular membrane as multiples of fundamental frequency.

Solution

frequency is given by $W_{mn} = K_{mn} v$

Fundamental frequency is $W_{10} v$.

First Zero of → J_0

So ratio of frequency W_{mn} to Fundamental is

$$\frac{W_{mn}}{W_{10}} = \frac{K_{mn}}{K_{10}}$$

First Zero of J_0.

So $$W_{mn} = W_{10} \frac{K_{mn}}{K_{10}}$$

using this I can find. frequencies as multiple of Fundamental freq.

→

First 3 Zeros of Bessel functions. Using Handbook.

J_0

		zero
m=1	$\longrightarrow$	2.4
m=2	$\longrightarrow$	5.65
m=3	$\longrightarrow$	8.55

J_1

m=1	$\longrightarrow$	3.75
m=2	$\longrightarrow$	7.25
m=3	$\longrightarrow$	10.05

J_2

m=1	$\longrightarrow$	5.05
m=2	$\longrightarrow$	8.45
m=3	$\longrightarrow$	11.55

J_3

m=1	$\longrightarrow$	6.3
m=2	$\longrightarrow$	13.1
m=3	$\longrightarrow$	16.3

So now I can find the first 6 frequencies.

First $= \omega_{10}$ This is Fundamental Frequencies.

Second $= \omega_{11} = \omega_{10}\dfrac{K_{11}}{K_{10}} = \omega_0\dfrac{3.75}{2.4} = \boxed{1.56\,\omega_0}$

Third $= \omega_{12} = \omega_{10}\dfrac{K_{12}}{K_{10}} = \omega_{10}\dfrac{5.05}{2.4} = \boxed{2.1\,\omega_0}$

Fourth $= \omega_{20} = \omega_{10}\dfrac{K_{20}}{K_{10}} = \omega_{10}\dfrac{5.65}{2.4} = \boxed{2.35\,\omega_0}$

Fifth $= \omega_{21} = \omega_{10}\dfrac{K_{21}}{K_{10}} = \omega_{10}\dfrac{7.25}{2.4} = \boxed{3.01\,\omega_0}$

Sixth $= \omega_{22} = \omega_{10}\dfrac{K_{22}}{K_{10}} = \omega_{10}\dfrac{8.45}{2.4} = \boxed{3.5\,\omega_0}$

2 chapter 13, problem 4.1. Mary Boas, second edition

Complete the plucked string problem to get equation 4.0

Solution

Here we start with the solution given in 4.8

$$y_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad (1)$$

Where $f(x)$ represents the initial position (shape) of the string.
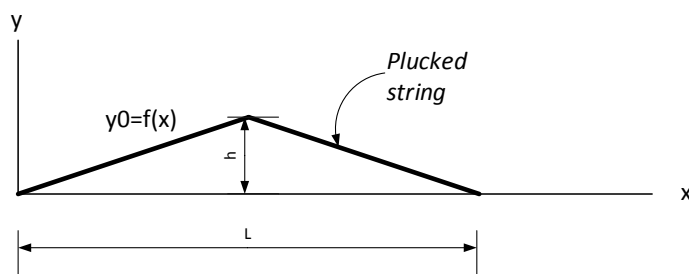
Now need to find b_n

First need to define $f(x)$, from diagram we see that from $x = 0$ to $x = L/2$ the slope is $\frac{h}{L/2} = \frac{2h}{L}$ hence from equation of line we get $y = \frac{2h}{L}x$

From $x = L/2$ to $x = L$, slope is $-\frac{2h}{L}$, so $y = h - \frac{2h}{L}\left(x - \frac{L}{2}\right) = h - \frac{2h}{L}x + h = 2h - \frac{2h}{L}x = 2\left(h - \frac{hx}{L}\right)$

so we have

$$f(x) = \begin{cases} \frac{2h}{L}x & 0 \le x \le \frac{L}{2} \\ \\ 2\left(h - \frac{hx}{L}\right) & \frac{L}{2} < x \le L \end{cases}$$

so from (1) we get, after applying inner product w.r.t. $\sin\left(\frac{n\pi x}{L}\right)$

$$b_n \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) = \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n \frac{L}{2} = \int_0^{\frac{L}{2}} f(x)\sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{L}{2}}^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n \frac{L}{2} = \int_0^{\frac{L}{2}} \frac{2h}{L}x \, \sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{L}{2}}^L 2\left(h - \frac{hx}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n \frac{L}{2} = \frac{2h}{L}\int_0^{\frac{L}{2}} x \, \sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{L}{2}}^L 2h \, \sin\left(\frac{n\pi x}{L}\right) dx - \int_{\frac{L}{2}}^L \frac{2hx}{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n \frac{L}{2} = \frac{2h}{L}\int_0^{\frac{L}{2}} x \, \sin\left(\frac{n\pi x}{L}\right) dx + 2h \int_{\frac{L}{2}}^L \sin\left(\frac{n\pi x}{L}\right) dx - \frac{2h}{L} \int_{\frac{L}{2}}^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n \frac{L}{2} = \frac{16\,h\,L\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}$$

$$b_n = \frac{32\,h\,\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}$$

so

$$b_n = \frac{32\,h\,L\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}$$

Looking at few values of n to see the pattern

$$b_n = \frac{32\,h\,\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)^3}{1^2\pi^2}, \frac{32\,h\,\cos\left(\frac{2\pi}{4}\right)\sin\left(\frac{2\pi}{4}\right)^3}{2^2\pi^2}, \frac{32\,h\,\cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)^3}{3^2\pi^2}, \dots$$

$$= \frac{8h}{\pi^2}, 0, -\frac{8h}{9\,\pi^2}, 0, \frac{8h}{25\,\pi^2}, \dots$$

$$= \frac{8h}{\pi^2}\left(1, 0, -\frac{1}{9}, 0, \frac{1}{25}, \dots\right)$$

Notice that we have terms for only odd n.

Now, substituting the above in the general solution given in equation 4.7 in book, which is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

Gives

$$y = \frac{8h}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) + 0 + -\frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi vt}{L}\right) + 0 + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi vt}{L}\right) + \dots\right)$$
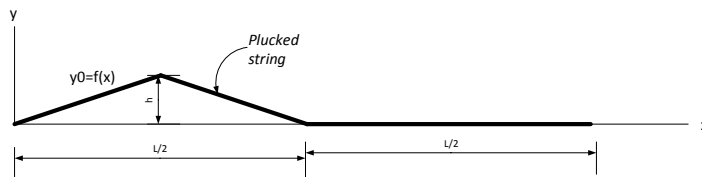
$$y = \frac{8h}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi vt}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi vt}{L}\right) + \dots\right)$$

The above is the result we are asked to show.

3   chapter 13, problem 4.2. Mary Boas, second edition

A string of length L has zero initial velocity and a displacement $y_0(x)$ as shown. Find the displacement as a function of x and t.

Solution

The PDE that governs this problem is the wave equation $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

The candidate solutions are

$$y = \begin{cases} \sin(kx)\,\sin(\omega t) \\ \sin(kx)\,\cos(\omega t) \\ \cos(kx)\,\sin(\omega t) \\ \cos(kx)\,\cos(\omega t) \end{cases}$$

where $\omega = kv$ and $k = \frac{2\pi}{\lambda}$ where λ is the wave length

Now we discard solutions that contains $\cos kx$ since the string is fixed at $x = 0$.

So we are left with

$$y = \begin{cases} \sin(kx)\,\sin(\omega t) \\ \sin(kx)\,\cos(\omega t) \end{cases}$$

Now, $y = 0$ at $x = L$ then from $\sin kx = 0$ or $\sin kL = 0$ we need $k = \frac{n\pi}{L}$

Hence solutions become

$$y = \begin{cases} \sin(\frac{n\pi}{L}x)\,\sin(\frac{n\pi}{L}vt) \\ \sin(\frac{n\pi}{L}x)\,\cos(\frac{n\pi}{L}vt) \end{cases}$$

Applying initial conditions, which says that at time $t = 0$, velocity is zero.

Hence from above, after taking $\frac{\partial y}{\partial t}$, we get

$$\frac{\partial y}{\partial t} = \begin{cases} \frac{n\pi v}{L}\sin(\frac{n\pi}{L}x)\,\cos(\frac{n\pi v}{L}t) \\ -\frac{n\pi v}{L}\sin(\frac{n\pi}{L}x)\,\sin(\frac{n\pi v}{L}t) \end{cases}$$

For the above to be zero at $t = 0$ then we discard first solution above with $\cos t$ in it. Hence final general solution is now

$$y = \{\ \sin(\frac{n\pi}{L}x)\,\cos(\frac{n\pi}{L}vt)$$

A general solution is a linear combination of the above solutions, hence

$$y = \sum_{n=1}^{\infty} b_n \, \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi}{L}vt\right) \qquad (1)$$

To find b_n, we apply the second initial condition, which is $y = y_0 = f(x)$

(Notice that we use two initial conditions, i.e. at time t=0 we are looking at speed and position, this is because we started with a PDE with $\frac{\partial^2 y}{\partial t^2}$ in it, which is a second order in t.)

At t=0, (1) becomes

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \qquad (2)$$

To find $f(x)$ from diagram, we see that for $0 \le x \le \frac{L}{4}$, $y = x\frac{h}{L/4} = \frac{4h}{L}x$

For $\frac{L}{4} < x \le \frac{L}{2}$, $y = -\left(x - \frac{L}{4}\right)\frac{h}{L/4} + h = -\left(x - \frac{L}{4}\right)\frac{4h}{L} + h = -x\frac{4h}{L} + \frac{L}{4}\frac{4h}{L} + h = -x\frac{4h}{L} + 2h$

For $\frac{L}{2} < x \le L$, $y = 0$

Hence

$$y = \begin{cases} \frac{4h}{L}x & 0 \le x \le \frac{L}{4} \\[2mm] 2h - x\frac{4h}{L} & \frac{L}{4} < x \le \frac{L}{2} \\[2mm] 0 & \frac{L}{2} < x \le L \end{cases}$$

Do the inner product on both sides of equation (2) w.r.t. $\sin\frac{n\pi}{L}x$

$$b_n \int_0^L \sin^2\frac{n\pi}{L}x \, dx = \int_0^L f(x) \sin\frac{n\pi}{L}x \, dx$$

$$b_n\frac{L}{2} = \int_0^{\frac{L}{4}} f(x) \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} f(x) \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{2}}^{L} f(x) \sin\frac{n\pi}{L}x \, dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} \left(2h - x\frac{4h}{L}\right) \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{2}}^{L} 0 \sin\frac{n\pi}{L}x \, dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} 2h \sin\left(\frac{n\pi}{L}x\right) - x\frac{4h}{L} \sin\left(\frac{n\pi}{L}x\right) \, dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \sin\frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} 2h \sin\left(\frac{n\pi}{L}x\right) dx - \int_{\frac{L}{4}}^{\frac{L}{2}} x\frac{4h}{L} \sin\left(\frac{n\pi}{L}x\right) \, dx$$

$$= \frac{4h}{L}\int_0^{\frac{L}{4}} x \sin\frac{n\pi}{L}x \, dx + 2h \int_{\frac{L}{4}}^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right)dx - \frac{4h}{L}\int_{\frac{L}{4}}^{\frac{L}{2}} x \sin\left(\frac{n\pi}{L}x\right) \, dx$$

$$b_n = \frac{8h}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right)$$

Looking at few values of b_n

$$b_n = \frac{8h}{1^2\pi^2}\left(2\sin\left(\frac{\pi}{4}\right) - \sin\frac{\pi}{2}\right), \frac{8h}{2^2\pi^2}\left(2\sin\left(\frac{2\pi}{4}\right) - \sin\frac{2\pi}{2}\right), \frac{8h}{3^2\pi^2}\left(2\sin\left(\frac{3\pi}{4}\right) - \sin\frac{3\pi}{2}\right), \dots$$

$$= \frac{8h}{\pi^2}\left[\left(2\sin\left(\frac{\pi}{4}\right) - \sin\frac{\pi}{2}\right), \frac{1}{2^2}\left(2\sin\frac{2\pi}{4} - \sin\frac{2\pi}{2}\right), \frac{1}{3^2}\left(2\sin\frac{3\pi}{4} - \sin\frac{3\pi}{2}\right), \dots\right]$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left\{\left(2\sin\left(\frac{\pi}{4}\right) - \sin\frac{\pi}{2}\right), \left(2\sin\frac{2\pi}{4} - \sin\frac{2\pi}{2}\right), \left(2\sin\frac{3\pi}{4} - \sin\frac{3\pi}{2}\right), \dots\right\}\right]$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right)\right]$$

Hence from equation (1) above, we get

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\cos\frac{n\pi}{L}vt$$

$$= \sum_{n=1}^{\infty} \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right)\right]\sin\frac{n\pi}{L}x\cos\frac{n\pi}{L}vt$$

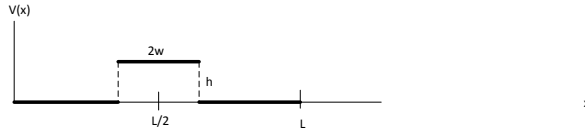$$= \frac{8h}{\pi^2}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\cos\frac{n\pi}{L}vt$$

Where

$$B_n = \frac{1}{n^2}\left(2\,\sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right)$$

The above is the result required to show.

4 chapter 13, problem 4.6. Mary Boas, second edition

A string of length L is initially stretched straight, its ends are fixed for all time t. At time t=0 its points are given the velocity $V(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0}$ as shown in diagram below. Determine the shape of the string at time t.

Solution

The PDE that governs this problem is the wave equation $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

The candidate solutions are

$$
y = \begin{cases}
\sin(kx)\,\sin(\omega t) \\
\sin(kx)\,\cos(\omega t) \\
\cos(kx)\,\sin(\omega t) \\
\cos(kx)\,\cos(\omega t)
\end{cases}
$$

Where $\omega = kv$ and $k = \frac{2\pi}{\lambda}$ where λ is the wave length

Now we discard solutions that contains $\cos kx$ since the string is fixed at $x = 0$.

So we are left with

$$
y = \begin{cases}
\sin(kx)\,\sin(\omega t) \\
\sin(kx)\,\cos(\omega t)
\end{cases}
$$

Now, $y = 0$ at $x = L$ then from $\sin kx = 0$ or $\sin kL = 0$ we need $k = \frac{n\pi}{L}$

Hence solutions become

$$
y = \begin{cases}
\sin\frac{n\pi}{L}x \sin\frac{n\pi}{L}vt \\
\sin\frac{n\pi}{L}x \cos\frac{n\pi}{L}vt
\end{cases}
$$

Applying initial conditions, which says that at time $t = 0$, velocity is given by $V(x)$

Hence from above, after taking $\frac{\partial y}{\partial t}$, we get

$$
\frac{\partial y}{\partial t} = \begin{cases}
\frac{n\pi v}{L}\sin\frac{n\pi}{L}x \cos\frac{n\pi v}{L}t \\
-\frac{n\pi v}{L}\sin\frac{n\pi}{L}x \sin\frac{n\pi v}{L}t
\end{cases}
$$

For the above we discard velocity solution above with $\sin t$ in it since that will give zero velocity at time t=0, which is not the case here. Hence we discard y solution with $\cos t$ in it, then the final general solution for y is now

$$
y = \sin\frac{n\pi}{L}x \sin\frac{n\pi}{L}vt
$$

A general solution is a linear combination of the above solutions, hence

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sin\frac{n\pi vt}{L} \qquad (1)$$

To find b_n, we apply the velocity initial condition. Hence differentiate equation (1) and set t=0, we have

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi vt}{L}$$

Setting t=0

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L} \sin\frac{n\pi x}{L} = V_{t=0} \qquad (2)$$

Now to find $V_{t=0}$. From diagram, we see that for $0 \le x \le \frac{L}{2} - w$, $V_{t=0} = 0$

For $\frac{L}{2} - w < x \le \frac{L}{2} + w$, $V_{t=0} = h$

For $\frac{L}{2} + w < x \le L$, $V_{t=0} = 0$

Hence

$$V_{t=0} = \begin{cases} 0 & 0 \le x \le \frac{L}{2} - w \\[2mm] h & \frac{L}{2} - w < x \le \frac{L}{2} + w \\[2mm] 0 & \frac{L}{2} + w < x \le L \end{cases}$$

Do the inner product on both sides of equation (2) w.r.t. $\sin\frac{n\pi}{L}x$

$$b_n \frac{n\pi v}{L} \int_0^L \sin^2\frac{n\pi}{L}x \ dx = \int_0^L V(x) \sin\frac{n\pi x}{L} \ dx$$

$$b_n \frac{n\pi v}{2} = \int_0^{\frac{L}{2}-w} 0 \sin\frac{n\pi}{L}x \ dx + \int_{\frac{L}{2}-w}^{\frac{L}{2}+w} h \sin\frac{n\pi x}{L} \ dx + \int_{\frac{L}{2}+w}^L 0 \sin\frac{n\pi}{L}x \ dx$$

$$b_n \frac{n\pi v}{2} = \int_{\frac{L}{2}-w}^{\frac{L}{2}+w} h \sin\frac{n\pi x}{L} \ dx$$

$$b_n \frac{n\pi v}{2} = -h \frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}-w}^{\frac{L}{2}+w}$$

$$b_n \frac{n\pi v}{2} = -h \frac{L}{n\pi}\left[\cos\frac{n\pi\left(\frac{L}{2}+w\right)}{L} - \cos\frac{n\pi\left(\frac{L}{2}-w\right)}{L}\right]$$

$$b_n \frac{n\pi v}{2} = -h \frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{2} + \frac{n\pi w}{L}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi w}{L}\right)\right]$$

$$b_n = -\frac{2hL}{n^2\pi^2 v}\left[\cos\left(\frac{n\pi}{2} + \frac{n\pi w}{L}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi w}{L}\right)\right]$$

But $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

and $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

Let $a = \frac{n\pi}{2}, b = \frac{n\pi w}{L}$

Hence b_n becomes

$$b_n = -\frac{2hL}{n^2\pi^2 v}[\cos(a+b) - \cos(a-b)]$$

$$= -\frac{2hL}{n^2\pi^2 v}[\cos(a)\cos(b) - \sin(a)\sin(b) - \{\cos(a)\cos(b) + \sin(a)\sin(b)\}]$$

$$= -\frac{2hL}{n^2\pi^2 v}[\cos(a)\cos(b) - \sin(a)\sin(b) - \cos(a)\cos(b) - \sin(a)\sin(b)]$$

$$= -\frac{2hL}{n^2\pi^2 v}[-\sin(a)\sin(b) - \sin(a)\sin(b)]$$

$$= \frac{4hL}{n^2\pi^2 v}\sin(a)\sin(b)$$

$$= \frac{4hL}{n^2\pi^2 v}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi w}{L}\right)$$

For even n, the term $\sin\left(\frac{n\pi}{2}\right)$ is zero. For n odd $\sin\left(\frac{n\pi}{2}\right) = 1$ when $n = 1, 5, 9, \dots$ and $\sin\left(\frac{n\pi}{2}\right) = -1$ when $n = 3, 7, 11, \dots$ Hence

$$b_n = A(n)\frac{4hL}{n^2\pi^2 v}\sin\left(\frac{n\pi w}{L}\right) \qquad n = 1, 3, 5, 7, \dots$$

And $A(n)$ is a function which returns 1 when $n = 1, 5, 9, \dots$ and returns -1 when $n = 3, 7, 11, \dots$

Hence now we have b_n we can substitute in (1)

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sin\frac{n\pi v t}{L}$$

$$y = \sum_{n\ odd}^{\infty} A(n)\frac{4hL}{n^2\pi^2 v}\sin\left(\frac{n\pi w}{L}\right)\left[\sin\frac{n\pi x}{L} \sin\frac{n\pi v t}{L}\right]$$

$$y = \frac{4hL}{\pi^2 v}\sum_{n\ odd}^{\infty} A(n)\frac{1}{n^2}\sin\left(\frac{n\pi w}{L}\right)\left[\sin\frac{n\pi x}{L} \sin\frac{n\pi v t}{L}\right]$$

Which is the general solution. Looking at few expanded terms in the series we get

$$y = \frac{4hL}{\pi^2 v}\left\{\sin\left(\frac{\pi w}{L}\right)\sin\frac{\pi x}{L}\sin\frac{\pi v t}{L} - \frac{1}{9}\sin\left(\frac{3\pi w}{L}\right)\sin\frac{3\pi x}{L}\sin\frac{3\pi v t}{L} + \frac{1}{25}\sin\left(\frac{5\pi w}{L}\right)\sin\frac{5\pi x}{L}\sin\frac{n\pi v t}{L}\right\}$$

Which is the result required.

5   chapter 13, problem 5.1. Mary Boas, second edition

Compute numerically the coefficients $c_m = \dfrac{200}{k_m J_1(k_m)}$ for the first 3 terms of the series

$u = \displaystyle\sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-k_m z}$ for the steady state temp. in a solid semi-infinite cylinder when $u = 0$ at $r = 1$ and $u = 100$ at $z = 0$. find u at $r = 1/2, z = 1$

Solution

Here, we are looking at the solution for temp. inside a semi-infinite cylinder. This solution is for the case of a uniform temp. distribution on the boundary $z = 0$ is given by u equation shown above. Note that in the expression $c_m = \dfrac{200}{k_m J_1(k_m)}$, the k_m are the zeros of J_0 not J_1.

Need to find c_1, c_2, c_3 where $c_1 = \dfrac{200}{k_1 J_1(k_1)}$

To find k_1 and $J_1(k_1)$ I used mathematica.

I plotted $J_0(x)$ to see where the zeros are located first

In[16]:= Plot[BesselJ[0, x], {x, 0, 10}]

Out[16]= - Graphics -
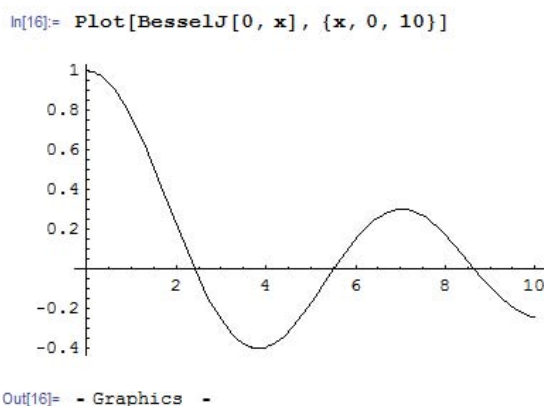
So I see there is a zero near 2,5, and 9. I use mathematica to find these:

In[20]:= k1 = FindRoot[BesselJ[0, x] = 0, {x, 2}]

Out[20]= {x → 2.40483 }

In[21]:= k2 = FindRoot[BesselJ[0, x] = 0, {x, 5}]

Out[21]= {x → 5.52008 }

In[22]:= k3 = FindRoot[BesselJ[0, x] = 0, {x, 9}]

Out[22]= {x → 8.65373 }

Now I need to find $J_1(k_m)$. This is the result for 3 terms:

In[37]:= BesselJ[1, k1[[1, 2]]]

Out[37]= 0.519147

In[38]:= BesselJ[1, k2[[1, 2]]]

Out[38]= -0.340265

In[39]:= BesselJ[1, k3[[1, 2]]]

Out[39]= 0.271452

Hence, now the c_m terms can be found:

$$c_1 = \frac{200}{k_1 J_1(k_1)} = \frac{200}{(2.404)(0.519)} = 160.30$$

$$c_2 = \frac{200}{k_2 J_1(k_2)} = \frac{200}{(5.52)(-0.34)} = -106.56$$

$$c_3 = \frac{200}{k_3 J_1(k_3)} = \frac{200}{(8.65)(0.27)} = 85.635$$

Evaluating $u = \sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-k_m z}$ for the first 3 terms when $r = 1/2, z = 1$

$$u = c_1 J_0(k_1 r)e^{-k_1 z} + c_2 J_0(k_2 r)e^{-k_2 z} + c_3 J_0(k_3 r)e^{-k_3 z}$$

$$= c_1 J_0\left(k_1 \frac{1}{2}\right)e^{-k_1} + c_2 J_0\left(k_2 \frac{1}{2}\right)e^{-k_2} + c_3 J_0\left(k_3 \frac{1}{2}\right)e^{-k_3}$$

$$= (160.30)J_0\left(2.404\frac{1}{2}\right)e^{-2.404} - (106.56)J_0\left(5.52\frac{1}{2}\right)e^{-5.52} + (85.635)J_0\left(8.65\frac{1}{2}\right)e^{-8.65}$$

$$= (160.30)J_0(1.202)e^{-2.404} - (106.56)J_0(2.76)e^{-5.52} + (85.635)J_0(4.325)e^{-8.65}$$

Mathematica was used to evaluate J_0 values above.

```
In[40]:= BesselJ[0, 1.202]

Out[40]= 0.670136

In[41]:= BesselJ[0, 2.76]

Out[41]= -0.168385

In[42]:= BesselJ[0, 4.325]

Out[42]= -0.356614
```

Hence

$$u = (160.30)(0.67)e^{-2.404} - (106.56)(-0.168)e^{-5.52} + (85.635)(-0.356)e^{-8.65}$$

$$u = 9.7043 + 7.1713 \times 10^{-2} - 5.3389 \times 10^{-3}$$

$$u = 9.7707 \text{ degrees}$$

6 chapter 13, problem 5.2. Mary Boas, second edition

Find the solution for the steady state temp. distribution in a solid semi-infinite cylinder if the boundary temp. are $u = 0$ at $r = 1$ and $u = y = r \sin\theta$ at $z = 0$.

Solution

The candidate solutions are given by the solution to the Laplace equation in cylindrical coordinates which are

$$
u = \begin{cases} J_n(k\,r)\sin(n\theta)e^{-k\,z} & (1) \\[2em] J_n(k\,r)\cos(n\theta)e^{-k\,z} & (2) \end{cases}
$$

Where k is a zero of J_n (This is because we have used the B.C. of $u = 0$ at $r = 1$ to determine that the $k's$ have to be the zeros of J_n) when deriving the above solutions. See book page 560.

From boundary conditions we want $u = r\sin\theta$ when $z = 0$, hence we need to keep the solution (1) above, with $n = 1$. Hence a solution is

$$
u = J_1(k\,r)\sin(\theta)e^{-k\,z} \qquad (3)
$$

A general solution is a linear series combinations (eigenfunctions) of (3), each eigenfunction for each of the zeros of J_1. Call these zeros k_m

$$
u = \sum_{m=1}^{\infty} c_m J_1(k_m\,r)\sin(\theta)e^{-k_m\,z} \qquad (4)
$$

We now apply B.C. at $z = 0$ to find c_m. From (4) when $z = 0$

$$
r\sin\theta = \sum_{m=1}^{\infty} c_m J_1(k_m\,r)\sin(\theta) \qquad (5)
$$

We use (5) to find c_m and then substitute into (4) to obtain the final solution.

To find c_m from (5), take the inner product of each side with respect to $rJ_1(k_u\,r)$ from $r = 0$ to $r = 1$

$$
\int_0^1 r\sin\theta[rJ_1(k_u\,r)]\,dr = \sum_{m=1}^{\infty} c_m \left(\int_0^1 J_1(k_m\,r)\sin(\theta)[rJ_1(k_u\,r)]\,dr \right)
$$

$$
\sin\theta \int_0^1 r^2 J_1(k_u\,r)\,dr = \sum_{m=1}^{\infty} c_m \sin(\theta)\left(\int_0^1 J_1(k_m\,r)[rJ_1(k_u\,r)]\,dr \right)
$$

Dividing each side by $\sin\theta$

$$
\int_0^1 r^2 J_1(k_u\,r)\,dr = \sum_{m=1}^{\infty} c_m \left(\int_0^1 J_1(k_m\,r)[rJ_1(k_u\,r)]\,dr \right)
$$

From orthogonality of Bessel function, we know that

$$
\int_0^1 J_p(k_m\,r)rJ_p(k_u\,r)dr = 0
$$

If $m \neq u$. Hence in above equation all terms on the right drop except for one when $u = m$. We get

$$\int_0^1 r^2 J_1(k_m\, r)\, dr = c_m \int_0^1 r\ J_1(k_m\, r) J_1(k_m\, r) dr$$

Or

$$c_m = \frac{\int_0^1 r^2 J_1(k_m\, r)\, dr}{\int_0^1 r\ J_1(k_m\, r)\ J_1(k_m\, r)\, dr} \qquad (6)$$

The integral in the denominator above is found from equation 19.10 in text on page 523 which gives

$$\int_0^1 r\ J_1(k_m\, r)\ J_1(k_m\, r)\, dr = \frac{1}{2}[J_2(k_m)]^2 \qquad (7)$$

Now, we need to find the integral of the numerator in equation (6).

Using equation 15.1 in text, page 514, which says

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

Putting $p = 2$ above, and letting $x = k_m r$ gives

$$\frac{1}{k_m}\frac{d}{dr}\left[(k_m r)^2 J_2(k_m r)\right] = (k_m r)^2 J_1(k_m r)$$

$$\frac{1}{k_m}\frac{d}{dr}\left[k_m^2 r^2 J_2(k_m r)\right] = k_m^2 r^2 J_1(k_m r)$$

$$\frac{1}{k_m}\frac{d}{dr}\left[r^2 J_2(k_m r)\right] = r^2 J_1(k_m r)$$

Integrating each side w.r.t r from $0\ldots 1$

$$\frac{1}{k_m}\int_0^1 \frac{d}{dr}\left[r^2 J_2(k_m r)\right] dr = \int_0^1 r^2 J_1(k_m r)\, dr$$

$$\frac{1}{k_m}\left[r^2 J_2(k_m r)\right]_0^1 = \int_0^1 r^2 J_1(k_m r)\, dr$$

$$\frac{1}{k_m}\left[J_2(k_m) - 0\right] = \int_0^1 r^2 J_1(k_m r)\, dr$$

$$\frac{1}{k_m} J_2(k_m) = \int_0^1 r^2 J_1(k_m r)\, dr \qquad (8)$$

Substituting (7) and (8) into (6)

$$c_m = \frac{\int_0^1 r^2 J_1(k_m\, r)\, dr}{\int_0^1 r\ J_1(k_m\, r)\ J_1(k_m\, r)\, dr}$$

$$= \frac{\frac{1}{k_m} J_2(k_m)}{\frac{1}{2}[J_2(k_m)]^2}$$

$$= \frac{2}{k_m J_2(k_m)}$$

Substituting this into (4) above, we get

$$u = \sum_{m=1}^{\infty} c_m \, J_1(k_m \, r) \sin(\theta) e^{-k_m \, z}$$

$$u = \sum_{m=1}^{\infty} \frac{2}{k_m \, J_2(k_m)} \, J_1(k_m \, r) \, \sin(\theta) \, e^{-k_m \, z}$$

where k_m are zeros of J_1

The above is the result we are asked to show.

7   chapter 13, problem 5.4. Mary Boass, second edition

A flat circular plate of radius 1 is initially at temp. 100^0. From $t = 0$ on, the circumference of the plate is held at 0^0. Find the time-dependent temp distribution $u(r, \theta, t)$

<u>Solution</u>

First convert heat equation from Cartesian coordinates to polar.

heat equation in 2D Cartesian is

$$\nabla^2 u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial x^2} u = \frac{1}{\alpha^2} \frac{\partial}{\partial t} u$$

First need to express Laplacian operator ∇^2 in polar coordinates:

$$x = r \cos \theta$$
$$y = r \sin \theta$$

Hence

$$\frac{\partial x}{\partial r} = \cos \theta \tag{A}$$
$$\frac{\partial y}{\partial r} = \sin \theta$$

And

$$\frac{\partial x}{\partial \theta} = -r \, \sin \theta \tag{B}$$
$$\frac{\partial y}{\partial \theta} = r \, \cos \theta$$
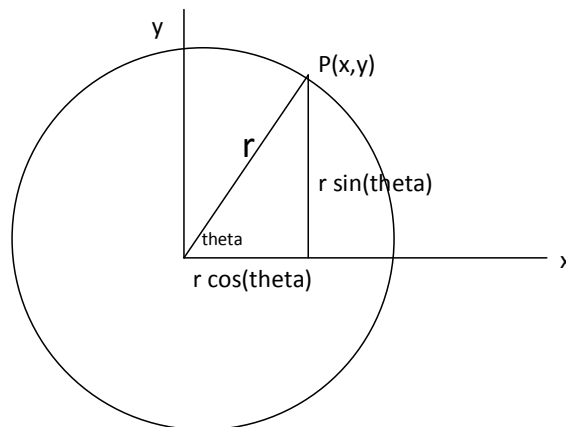
From geometry, we also know that

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan \frac{y}{x}$$

The above 2 relations imply

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y}$$

Hence we can express the above, using equations (A) and (B) as follows:

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

Multiply each side by r

$$r\frac{\partial}{\partial r} = r\cos\theta \frac{\partial}{\partial x} + r\sin\theta \frac{\partial}{\partial y}$$

$$= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{1}$$

Squaring each sides of (1) gives

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2$$

$$r\left(r\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right) = x\frac{\partial}{\partial x}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}y\frac{\partial}{\partial y}$$

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) + 2x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right)$$

$$= x\left(x\frac{\partial^2}{\partial x^2} + \overset{=1}{\overbrace{\frac{\partial}{\partial x}}}x\frac{\partial}{\partial x}\right) + y\left(y\frac{\partial^2}{\partial y^2} + \overset{=1}{\overbrace{\frac{\partial}{\partial y}}}y\frac{\partial}{\partial y}\right) + 2x\left(y\frac{\partial^2}{\partial x\partial y} + \overset{=0}{\overbrace{\frac{\partial}{\partial x}}}y\frac{\partial}{\partial y}\right)$$

$$= x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y} \tag{2}$$

Notice that when manipulating of differential operators, $x\frac{\partial}{\partial x} \neq \frac{\partial}{\partial x}x$. Similarly

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}$$

$$= -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{3}$$

Squaring each side of (3) gives

$$\left(\frac{\partial}{\partial\theta}\right)^2 = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)^2$$

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} = -y\frac{\partial}{\partial x}\left(-y\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) + x\frac{\partial}{\partial y}\left(-y\frac{\partial}{\partial x}\right)$$

$$\frac{\partial^2}{\partial\theta^2} = -y\left(-y\frac{\partial^2}{\partial x^2} + \overbrace{\frac{\partial}{\partial x}(-y)}^{=0}\frac{\partial}{\partial x}\right) + x\left(x\frac{\partial^2}{\partial y^2} + \overbrace{\frac{\partial}{\partial y}(x)}^{=0}\frac{\partial}{\partial y}\right)$$

$$- y\left(x\frac{\partial^2}{\partial y\partial x} + \overbrace{\frac{\partial}{\partial x}x}^{=1}\frac{\partial}{\partial y}\right) + x\left(-y\frac{\partial^2}{\partial x\partial y} + \overbrace{\frac{\partial}{\partial y}y}^{=1}\frac{\partial}{\partial x}\right)$$

$$= y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial x\partial y} - x\frac{\partial}{\partial x}$$

$$= y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - 2yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} \tag{4}$$

Adding equation (2) and (4) and carry cancellations

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2} = \left(x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y}\right) + \left(y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - 2yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right)$$

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2} = \left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2}\right) + \left(y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}\right)$$

Hence we get

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2} = x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}$$

$$= \left(x^2 + y^2\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Dividing by r^2

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{\left(x^2 + y^2\right)}{r^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

But $r^2 = x^2 + y^2$ hence

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \nabla^2$$

Now that we have the Laplacian in polar coordinates, we can solve the problem by applying separation of variables on the heat PDE expressed in polar coordinates.

$$\frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u = \frac{1}{\alpha^2}\frac{\partial}{\partial t}u \tag{5}$$

Let solution $u(r, \theta, t)$ be a linear combination of functions each depends on only r, θ, or t

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t) \tag{6}$$

Substitute (6) in (5). First evaluate the various derivatives:

$$\frac{\partial}{\partial r}u = \Theta(\theta)T(t)\frac{\partial}{\partial r}R(r)$$

$$\frac{\partial^2}{\partial r^2}u = \Theta(\theta)T(t)\frac{\partial^2}{\partial r^2}R(r)$$

$$\frac{\partial}{\partial \theta}u = R(r)T(t)\frac{\partial}{\partial \theta}\Theta(\theta)$$

$$\frac{\partial^2}{\partial \theta^2}u = R(r)T(t)\frac{\partial^2}{\partial \theta^2}\Theta(\theta)$$

$$\frac{\partial}{\partial t}u = R(r)\Theta(\theta)\frac{\partial}{\partial t}T(t)$$

Hence equation (5) becomes

$$\frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u = \frac{1}{\alpha^2}\frac{\partial}{\partial t}u$$

$$\Theta(\theta)T(t)\frac{d^2}{dr^2}R(r) + \frac{1}{r}\Theta(\theta)T(t)\frac{d}{dr}R(r) + \frac{1}{r^2}R(r)T(t)\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}R(r)\Theta(\theta)\frac{d}{dt}T(t)$$

Divide by $R(r)\Theta(\theta)T(t)$

$$\frac{1}{R(r)}\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{1}{R(r)}\frac{d}{dr}R(r) + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t)$$

$$\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t)$$

We notice that the RHS depends only on t and the LHS depends only on r, θ and they equal to each others, hence they both must be constant. Let this constant be $-k^2$

Hence

$$\frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t) = -k^2 \tag{7}$$

$$\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = -k^2 \tag{8}$$

equation (7) is a linear first order ODE with constant coeff. $\frac{d}{dt}T(t) = -\alpha^2 T(t)k^2$ or $\frac{dT(t)}{T(t)} = -\alpha^2 k^2 dt$

Integrating to solve gives

$$\int \frac{dT(t)}{T(t)} = \int -\alpha^2 k^2 dt$$
$$\ln T(t) = -\alpha^2 k^2 t$$

or

$$T(t) = e^{-\alpha^2 k^2 t} \tag{9}$$

Looking at equation (8). Multiply each sides by r^2 we get

$$\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = -r^2k^2$$

$$\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + r^2k + \frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = 0$$

$$r^2\left(\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + k^2\right) + \frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = 0 \tag{10}$$

The second term depends only on θ and the first term depends only on r and they are equal, hence they must be both constant. Let this constant be $-n^2$ hence

$$\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = -n^2$$

$$\frac{d^2}{d\theta^2}\Theta(\theta) = -n^2\Theta(\theta)$$

This is a second order linear ODE with constant coeff. Solution is

$$\Theta(\theta) = \begin{cases} \sin n\theta \\ \\ \cos n\theta \end{cases} \tag{11}$$

From (10) we now have

$$r^2\left(\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + k^2\right) - n^2 = 0$$

$$\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + r^2k^2 - n^2 = 0$$

$$r^2\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \left(r^2k^2 - n^2\right)R(r) = 0$$

$$r^2\frac{d^2}{dr^2}R(r) + r\frac{d}{dr}R(r) + \left(r^2k^2 - n^2\right)R(r) = 0 \tag{12}$$

Equation (12) is the Bessel D.E., its solutions are $J_n(kr)$ and $N_n(kr)$. As described on book on page 560, we can not use the $N_n(kr)$ solution since plate contains the origin and $N_n(0)$ is not defined. So we use solution $R(r) = J_n(kr)$. From boundary conditions, we want solution to be zero at $r = 1$, hence we want $J_n(k) = 0$, hence the k's are the zeros of J_n

Putting these solutions together, we get from (6)

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t)$$

$$= \begin{cases} J_n(kr)\sin n\theta e^{-\alpha^2k^2t} \\ \\ J_n(kr)\cos n\theta e^{-\alpha^2k^2t} \end{cases}$$

From symmetry of plate, the solution can not depend on the angle θ, hence let $n = 0$ and so as not to get $u = 0$, we must pick the solution with $\cos n\theta$ term. Hence our solution now is

$$u(r,t) = J_0(kr)\, e^{-\alpha^2 k^2 t}$$

Where k is a zero of J_0

The general solution is a linear combination of this eigenfunction for all zeros of J_0, hence

$$u(r,t) = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \tag{13}$$

We find c_m by using initial condition. When $t = 0$, temp. was 100^0 hence

$$100 = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)$$

Applying inner product w.r.t. $rJ_0(k_u r)$ from $0 \ldots 1$

$$\int_0^1 100\, rJ_0(k_u r)\, dr = \int_0^1 \left(\sum_{m=1}^{\infty} c_m\, J_0(k_m r) \right) rJ_0(k_u r)\, dr$$

$$100 \int_0^1 rJ_0(k_u r)\, dr = \sum_{m=1}^{\infty} c_m \int_0^1 J_0(k_m r)\, rJ_0(k_u r)\, dr$$

From orthogonality of $J_0(k_m r)$ and $J_0(k_u r)$, all terms drop expect when $m = u$

$$100 \int_0^1 rJ_0(k_u r)\, dr = c_u \int_0^1 r[J_0(k_u r)]^2\, dr$$

From here we can follow the book on page 561 to get

$$c_m = \frac{200}{k_m J_1(k_m)}$$

Substitute this in equation 13

$$
\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} c_m\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \\
&= \sum_{m=1}^{\infty} \frac{200}{k_m J_1(k_m)}\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \\
&= 200 \sum_{m=1}^{\infty} \frac{1}{k_m J_1(k_m)}\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t}
\end{aligned}
$$

Where k_m are zeros of J_0

Notice that final solution does not depend on θ

8   chapter 13, problem 5.11. Mary Boas, second edition

Solve

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2 R$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

Solution

First equation, use power series method.

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2 R$$

$$r\left(r\frac{d^2 R}{dr^2} + \frac{dR}{dr}\right) - n^2 R = 0$$

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0$$

Let $R = a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots$ then

$$R = a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots$$

$$-n^2 R = -n^2 a_0 r^s - n^2 a_1 r^{s+1} - n^2 a_2 r^{s+2} - n^2 a_3 r^{s+3} - n^2 a_4 r^{s+4} - \cdots$$

$$\frac{dR}{dr} = s\, a_0 r^{s-1} + (s+1)\, a_1 r^s + (s+2)\, a_2 r^{s+1} + (s+3)\, a_3 r^{s+2} + \cdots$$

$$r\frac{dR}{dr} = s\, a_0 r^s + (s+1)\, a_1 r^{s+1} + (s+2)\, a_2 r^{s+2} + (s+3)\, a_3 r^{s+3} + \cdots$$

$$\frac{d^2 R}{dr^2} = (s-1)s\, a_0 r^{s-2} + s(s+1)\, a_1 r^{s-1} + (s+1)(s+2)\, a_2 r^s + (s+2)(s+3)\, a_3 r^{s+1} + \cdots$$

$$r^2\frac{d^2 R}{dr^2} = (s-1)s\, a_0 r^s + s(s+1)\, a_1 r^{s+1} + (s+1)(s+2)\, a_2 r^{s+2} + (s+2)(s+3)\, a_3 r^{s+3} + \cdots$$

Table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2 R$	$-n^2 a_0$	$-n^2 a_1$	$-n^2 a_2$	$-n^2\, a_m$
$r\dfrac{dR}{dr}$	$s\, a_0$	$(s+1)\, a_1$	$(s+2)\, a_2$	$(s+m)a_m$
$r^2\dfrac{d^2 R}{dr^2}$	$(s-1)s\, a_0$	$s(s+1)\, a_1$	$(s+1)(s+2)\, a_2$	$(s+m-1)(s+m)\, a_m$

Hence, from first column we see , and since $a_0 \neq 0$ we solve for s

$$-n^2 a_0 + s\, a_0 + (s-1)s\, a_0 = 0$$

$$a_0\left(-n^2 + s + (s-1)s\right) = 0$$

$$-n^2 + s + (s-1)s = 0$$

$$-n^2 + s^2 = 0$$

$$s = \pm n$$

We see from second column, $a_1\left(-n^2 + (s+1) + s^2 + s\right) = 0$ or $a_1\left(-s^2 + 2s + 1 + s^2\right) = 0$, hence $a_1(2s + 1) = 0$

For $a_1 \neq 0$ then $s = -\frac{1}{2}$, this means n is not an integer since $s = \pm n$. hence a_1 must be zero.

The same applies to all a_m , $m > 0$ Hence solution contains only a_0

$$R = a_0 r^{\pm n}$$

$$R = \begin{cases} a_0 r^{-n} \\ \\ a_0 r^{+n} \end{cases}$$

For some constant a_0. This solution is when $n \neq 0$

If $n = 0$, table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2 R$	0	0	0	0
$r\frac{dR}{dr}$	$s\, a_0$	$(s+1)\, a_1$	$(s+2)\, a_2$	$(s+m)a_m$
$r^2\frac{d^2R}{dr^2}$	$(s-1)s\, a_0$	$s(s+1)\, a_1$	$(s+1)(s+2)\, a_2$	$(s+m-1)(s+m)\, a_m$

From first column:

$$sa_0 + s^2 a_0 - sa_0 = 0$$
$$a_0\left(s + s^2 - s\right) = 0$$
$$s^2 = 0$$
$$s = 0$$

And all other $a's$ are zero. Hence $R = a_0$ or R is constant.

Now for the second ODE

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$

Let $R = a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots$ then

$$R = a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots$$
$$-l(l+1)R = -l(l+1)a_0 r^s - l(l+1)a_1 r^{s+1} - l(l+1)a_2 r^{s+2} - l(l+1)a_3 r^{s+3} - l(l+1)a_4 r^{s+4} - \cdots$$
$$\frac{dR}{dr} = s\, a_0 r^{s-1} + (s+1)\, a_1 r^s + (s+2)\, a_2 r^{s+1} + (s+3)\, a_3 r^{s+2} + \cdots$$
$$2r\frac{dR}{dr} = 2s\, a_0 r^s + 2(s+1)\, a_1 r^{s+1} + 2(s+2)\, a_2 r^{s+2} + 2(s+3)\, a_3 r^{s+3} + \cdots$$
$$\frac{d^2R}{dr^2} = (s-1)s\, a_0 r^{s-2} + s(s+1)\, a_1 r^{s-1} + (s+1)(s+2)\, a_2 r^s + (s+2)(s+3)\, a_3 r^{s+1} + \cdots$$
$$r^2\frac{d^2R}{dr^2} = (s-1)s\, a_0 r^s + s(s+1)\, a_1 r^{s+1} + (s+1)(s+2)\, a_2 r^{s+2} + (s+2)(s+3)\, a_3 r^{s+3} + \cdots$$

Table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2 R$	$-l(l+1)a_0$	$-l(l+1)a_1$	$-l(l+1)a_2$	$-l(l+1)\,a_m$
$2r\dfrac{dR}{dr}$	$2s\,a_0$	$2(s+1)\,a_1$	$2(s+2)\,a_2$	$2(s+m)a_m$
$r^2\dfrac{d^2R}{dr^2}$	$(s-1)s\,a_0$	$s(s+1)\,a_1$	$(s+1)(s+2)\,a_2$	$(s+m-1)(s+m)\,a_m$

From first column:

$$-l(l+1)a_0 + 2s\,a_0 + (s-1)s\,a_0 = 0$$
$$a_0(-l(l+1) + 2s + (s-1)s) = 0$$
$$-l(l+1) + 2s + (s-1)s = 0$$
$$-l(l+1) + s + s^2 = 0$$
$$(s-l)(s-(-l-1)) = 0$$

Hence $s = l$ or $s = -l - 1$.

We also see that all other $a's$ will be zero, since recursive formula has only a_m in it and no other a. Hence

$$R = a_0 r^s$$

$$R = \begin{cases} a_0 r^l \\ \\ a_0 r^{-l-1} \end{cases}$$

For some constant a_0