HW 9

MATH 121B

Spring 2004
UC BERKELEY

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chapter 13

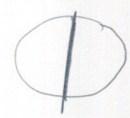
6.1

will use $\boxed{z = J_n(k_{mn} r)\cos n\theta \cos k_{mn} v t}$ to

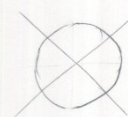
Complete Figure 6.1.

k_{mn} is the m^{th} zero of J_n.

Circle is divided into as many sectors as $2n$. for example, when $n=1$, we will set



when $n=2$, we will set



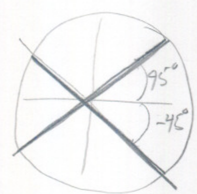
this is because we want a solution

$$\cos n\theta = \pm\frac{\pi}{2}$$

which means $\theta = \pm\frac{\pi}{2n}$

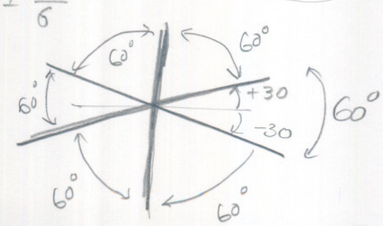
so for $n=1$, $\theta = +\frac{\pi}{2}, -\frac{\pi}{2}$

$n=2$ $\quad 2\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{4}$



for $n=3$ $\quad 3\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{6}$

so



etc...

now, for changing of the m.

from $z = J_n(k_{mn} r) \cos n\theta \cos k_{mn} v t$

we want $J_n(k_{mn} r)$ to zero.

as m increases, $k_{m_2 n} > k_{m_1 n}$ when $m_2 > m_1$

so r becomes smaller for each m increasing.

if original radius of drum is 1, then

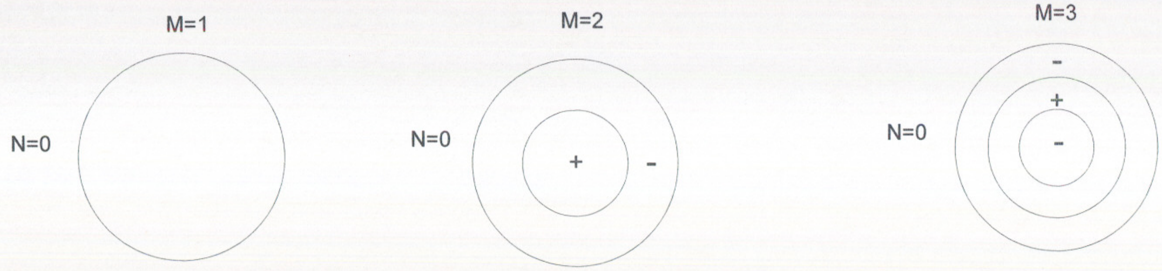
r for $m=1 \Rightarrow 1$

r for $m=2 \Rightarrow \frac{k_{10}}{k_{20}} < 1$

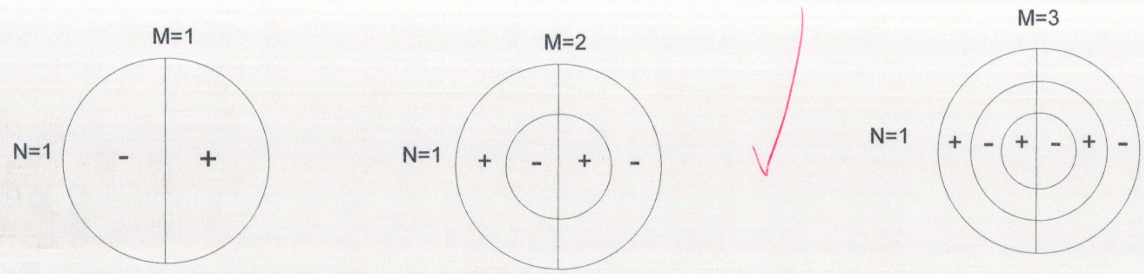
r for $m=3 \rightarrow \frac{k_{10}}{k_{30}} < \frac{k_{10}}{k_{20}}$

So each time m increases by 1 we add a smaller radius inside. so now I show the complete figure for $n=0,1,2$, $m=1,2,3$.

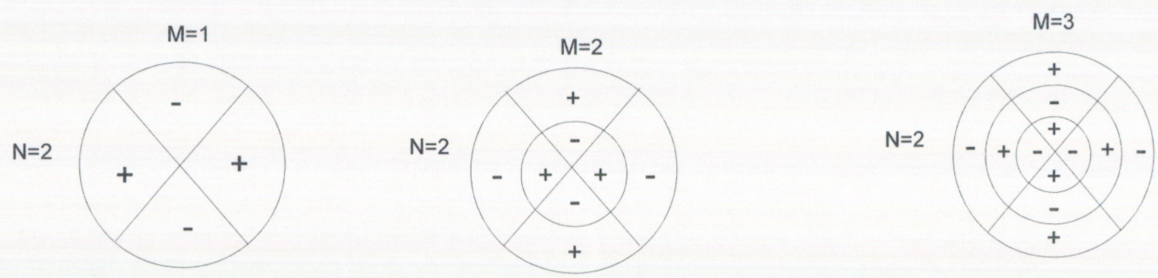
$\longrightarrow$



$z = J_0(k_{10}r)\cos(k_{10}vt)$ $\quad$ $z = J_0(k_{20}r)\cos(k_{20}vt)$ $\quad$ $z = J_0(k_{30}r)\cos(k_{30}vt)$



$z = J_1(k_{11}r)\cos\theta\cos(k_{11}vt)$ $\quad$ $z = J_1(k_{21}r)\cos\theta\cos(k_{21}vt)$ $\quad$ $z = J_1(k_{31}r)\cos\theta\cos(k_{31}vt)$



$z = J_2(k_{12}r)\cos 2\theta\cos(k_{12}vt)$ $\quad$ $z = J_2(k_{22}r)\cos 2\theta\cos(k_{22}vt)$ $\quad$ $z = J_2(k_{32}r)\cos 2\theta\cos(k_{32}vt)$

chapter 13, 6.2

look up first 3 zeros of k_{mn} of each Bessel function J_0, J_1, J_2, J_3. Find the first 6 frequencies of a vibrating circular membrane as multiples of fundamental frequency.

Solution

frequency is given by $\omega_{mn} = k_{mn} v$

Fundamental frequency is $\omega_{10}\, v$

First zero of J_0

so ratio of frequency ω_{mn} to Fundamental is

$$\boxed{\frac{\omega_{mn}}{\omega_{10}} = \frac{k_{mn}}{k_{10}}}$$

First zero of J_0

so $$\boxed{\omega_{mn} = \omega_{10} \frac{k_{mn}}{k_{10}}}$$

using this I can find frequencies as multiple of Fundamental freq.

$\longrightarrow$

First 3 Zeros of Bessel function. using Handbook.

J_0

		Zero
m=1	$\longrightarrow$	2.4
m=2	$\longrightarrow$	5.65
m=3	$\longrightarrow$	8.55

J_1

m=1	$\longrightarrow$	3.75
m=2	$\longrightarrow$	7.25
m=3	$\longrightarrow$	10.05

J_2

m=1	$\longrightarrow$	5.05
m=2	$\longrightarrow$	8.45
m=3	$\longrightarrow$	11.55

J_3

m=1	$\longrightarrow$	6.3
m=2	$\longrightarrow$	13.1
m=3	$\longrightarrow$	16.3

So now I can find the first 6 frequencies.

First $= \omega_{10}$ This is Fundemental Frequencies.

Second $= \omega_{11} = \omega_{10} \frac{K_{11}}{K_{10}} = \omega_0 \frac{3.75}{2.4} = \boxed{1.56\, \omega_0}$

Third $= \omega_{12} = \omega_{10} \frac{K_{12}}{K_{10}} = \omega_{10} \frac{5.05}{2.4} = \boxed{2.1\, \omega_0}$

Fourth $= \omega_{20} = \omega_{10} \frac{K_{20}}{K_{10}} = \omega_{10} \frac{5.65}{2.4} = \boxed{2.35\, \omega_0}$

Fifth $= \omega_{21} = \omega_{10} \frac{K_{21}}{K_{10}} = \omega_{10} \frac{7.25}{2.4} = \boxed{3.01\, \omega_0}$

Sixth $= \omega_{22} = \omega_{10} \frac{K_{22}}{K_{10}} = \omega_{10} \frac{8.45}{2.4} = \boxed{3.5\, \omega_0}$

2 chapter 13, problem 4.1. Mary Boas, second edition

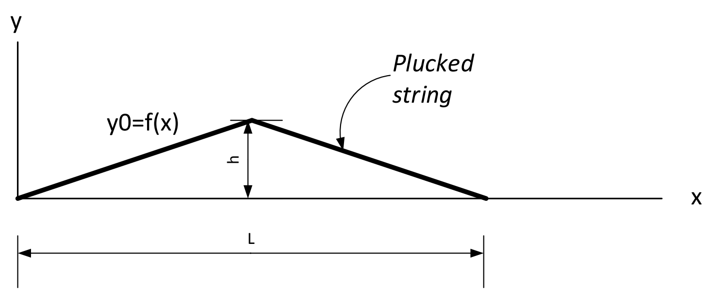
Complete the plucked string problem to get equation 4.0

Solution

Here we start with the solution given in 4.8

$$y_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad (1)$$

Where $f(x)$ represents the initial position (shape) of the string.



Now need to find b_n

First need to define $f(x)$, from diagram we see that from $x = 0$ to $x = L/2$ the slope is $\frac{h}{L/2} = \frac{2h}{L}$ hence from equation of line we get $y = \frac{2h}{L}x$

From $x = L/2$ to $x = L$, slope is $-\frac{2h}{L}$, so $y = h - \frac{2h}{L}\left(x - \frac{L}{2}\right) = h - \frac{2h}{L}x + h = 2h - \frac{2h}{L}x = 2\left(h - \frac{hx}{L}\right)$

so we have

$$f(x) = \begin{cases} \frac{2h}{L}x & 0 \leq x \leq \frac{L}{2} \\ 2\left(h - \frac{hx}{L}\right) & \frac{L}{2} < x \leq L \end{cases}$$

so from (1) we get, after applying inner product w.r.t. $\sin\left(\frac{n\pi x}{L}\right)$

$$
\begin{aligned}
b_n \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) &= \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\\
b_n\,\frac{L}{2} &= \int_0^{\frac{L}{2}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\\
b_n\,\frac{L}{2} &= \int_0^{\frac{L}{2}} \frac{2h}{L}x\ \sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L} 2\left(h-\frac{hx}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\\
b_n\,\frac{L}{2} &= \frac{2h}{L}\int_0^{\frac{L}{2}} x\ \sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L} 2h\ \sin\left(\frac{n\pi x}{L}\right)dx - \int_{\frac{L}{2}}^{L} \frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)dx\\
b_n\,\frac{L}{2} &= \frac{2h}{L}\int_0^{\frac{L}{2}} x\ \sin\left(\frac{n\pi x}{L}\right)dx + 2h\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)dx - \frac{2h}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx\\
b_n\,\frac{L}{2} &= \frac{16\,h\,L\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}\\
b_n &= \frac{32\,h\ \cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}
\end{aligned}
$$

so

$$
b_n = \frac{32\,h\,L\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{4}\right)^3}{n^2\pi^2}
$$

Looking at few values of n to see the pattern

$$
\begin{aligned}
b_n &= \frac{32\,h\ \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)^3}{1^2\pi^2}, \frac{32\,h\ \cos\left(\frac{2\pi}{4}\right)\sin\left(\frac{2\pi}{4}\right)^3}{2^2\pi^2}, \frac{32\,h\ \cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)^3}{3^2\pi^2}, \dots\\
&= \frac{8h}{\pi^2}, 0, -\frac{8h}{9\,\pi^2}, 0, \frac{8h}{25\,\pi^2}, \dots\\
&= \frac{8h}{\pi^2}\left(1, 0, -\frac{1}{9}, 0, \frac{1}{25}, \dots\right)
\end{aligned}
$$

Notice that we have terms for only odd n.

Now, substituting the above in the general solution given in equation 4.7 in book, which is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)$$

Gives

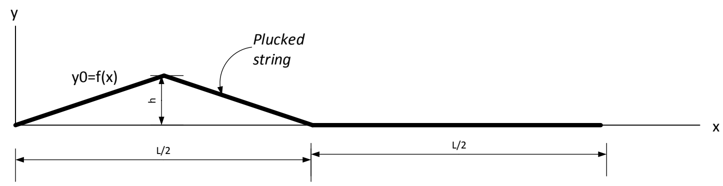
$$y = \frac{8h}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi v t}{L}\right) + 0 + -\frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi v t}{L}\right) + 0 + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi v t}{L}\right) + \ldots\right)$$

$$y = \frac{8h}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi v t}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi v t}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi v t}{L}\right) + \ldots\right)$$

The above is the result we are asked to show.

3 chapter 13, problem 4.2. Mary Boas, second edition

A string of length L has zero initial velocity and a displacement $y_0(x)$ as shown. Find the displacement as a function of x and t.



Solution

The PDF that governs this problem is the wave equation $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

The candidate solutions are

$$y = \begin{cases} \sin(kx)\ \sin(\omega t) \\ \sin(kx)\ \cos(\omega t) \\ \cos(kx)\ \sin(\omega t) \\ \cos(kx)\ \cos(\omega t) \end{cases}$$

where $\omega = kv$ and $k = \frac{2\pi}{\lambda}$ where λ is the wave length

Now we discard solutions that contains $\cos kx$ since the string is fixed at $x = 0$.

So we are left with

$$y = \begin{cases} \sin(kx)\ \sin(\omega t) \\ \sin(kx)\ \cos(\omega t) \end{cases}$$

Now, $y = 0$ at $x = L$ then from $\sin kx = 0$ or $\sin kL = 0$ we need $k = \frac{n\pi}{L}$

Hence solutions become

$$y = \begin{cases} \sin(\frac{n\pi}{L}x)\ \sin(\frac{n\pi}{L}vt) \\ \sin(\frac{n\pi}{L}x)\ \cos(\frac{n\pi}{L}vt) \end{cases}$$

Applying initial conditions, which says that at time $t = 0$, velocity is zero.

Hence from above, after taking $\frac{\partial y}{\partial t}$, we get

$$\frac{\partial y}{\partial t} = \begin{cases} \frac{n\pi v}{L}\ \sin(\frac{n\pi}{L}x)\ \cos(\frac{n\pi v}{L}t) \\ -\frac{n\pi v}{L}\ \sin(\frac{n\pi}{L}x)\ \sin(\frac{n\pi v}{L}t) \end{cases}$$

For the above to be zero at $t = 0$ then we discard first solution above with $\cos t$ in it. Hence final general solution is now

$$y = \{\ \sin(\frac{n\pi}{L}x)\ \cos(\frac{n\pi}{L}vt)$$

A general solution is a linear combination of the above solutions, hence

$$y = \sum_{n=1}^{\infty} b_n\ \sin\left(\frac{n\pi}{L}x\right)\ \cos\left(\frac{n\pi}{L}vt\right) \qquad (1)$$

To find b_n, we apply the second initial condition, which is $y = y_0 = f(x)$

(Notice that we use two initial conditions, i.e. at time t=0 we are looking at speed and position, this is because we started with a PDE with $\frac{\partial^2 y}{\partial t^2}$ in it, which is a second order in t.)

At t=0, (1) becomes

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \qquad (2)$$

To find $f(x)$ from diagram, we see that for $0 \le x \le \frac{L}{4}, y = x\frac{h}{L/4} = \frac{4h}{L}x$

For $\frac{L}{4} < x \le \frac{L}{2}, y = -\left(x - \frac{L}{4}\right)\frac{h}{L/4} + h = -\left(x - \frac{L}{4}\right)\frac{4h}{L} + h = -x\frac{4h}{L} + \frac{L}{4}\frac{4h}{L} + h = -x\frac{4h}{L} + 2h$

For $\frac{L}{2} < x \le L, y = 0$

Hence

$$y = \begin{cases} \frac{4h}{L}x & 0 \le x \le \frac{L}{4} \\ 2h - x\frac{4h}{L} & \frac{L}{4} < x \le \frac{L}{2} \\ 0 & \frac{L}{2} < x \le L \end{cases}$$

Do the inner product on both sides of equation (2) w.r.t. $\sin \frac{n\pi}{L}x$

$$b_n \int_0^L \sin^2 \frac{n\pi}{L}x \ \ dx = \int_0^L f(x) \ \sin \frac{n\pi}{L}x \ \ dx$$

$$b_n \frac{L}{2} = \int_0^{\frac{L}{4}} f(x) \ \sin \frac{n\pi}{L}x \, dx \ + \ \int_{\frac{L}{4}}^{\frac{L}{2}} f(x) \ \sin \frac{n\pi}{L}x \, dx + \int_{\frac{L}{2}}^{L} f(x) \ \sin \frac{n\pi}{L}x \, dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \ \sin \frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} \left(2h - x\frac{4h}{L}\right) \sin \frac{n\pi}{L}x \, dx + \int_{\frac{L}{2}}^{L} 0 \ \sin \frac{n\pi}{L}x \, dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \ \sin \frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} 2h \sin\left(\frac{n\pi}{L}x\right) - x\frac{4h}{L} \sin\left(\frac{n\pi}{L}x\right) \ dx$$

$$= \int_0^{\frac{L}{4}} \frac{4h}{L}x \ \sin \frac{n\pi}{L}x \, dx + \int_{\frac{L}{4}}^{\frac{L}{2}} 2h \sin\left(\frac{n\pi}{L}x\right) dx - \int_{\frac{L}{4}}^{\frac{L}{2}} x\frac{4h}{L} \sin\left(\frac{n\pi}{L}x\right) \ dx$$

$$= \frac{4h}{L} \int_0^{\frac{L}{4}} x \ \sin \frac{n\pi}{L}x \, dx + 2h \int_{\frac{L}{4}}^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) dx - \frac{4h}{L} \int_{\frac{L}{4}}^{\frac{L}{2}} x \sin\left(\frac{n\pi}{L}x\right) \ dx$$

$$b_n = \frac{8h}{n^2\pi^2}\left(2 \sin\left(\frac{n\pi}{4}\right) - \sin \frac{n\pi}{2}\right)$$

Looking at few values of b_n

$$b_n = \frac{8h}{1^2\pi^2}\left(2 \sin\left(\frac{\pi}{4}\right) - \sin \frac{\pi}{2}\right), \frac{8h}{2^2\pi^2}\left(2 \sin\left(\frac{2\pi}{4}\right) - \sin \frac{2\pi}{2}\right), \frac{8h}{3^2\pi^2}\left(2 \sin\left(\frac{3\pi}{4}\right) - \sin \frac{3\pi}{2}\right), \ldots$$

$$= \frac{8h}{\pi^2}\left[\left(2 \sin\left(\frac{\pi}{4}\right) - \sin \frac{\pi}{2}\right), \frac{1}{2^2}\left(2 \sin \frac{2\pi}{4} - \sin \frac{2\pi}{2}\right), \frac{1}{3^2}\left(2 \sin \frac{3\pi}{4} - \sin \frac{3\pi}{2}\right), \ldots \right]$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left\{\left(2 \sin\left(\frac{\pi}{4}\right) - \sin \frac{\pi}{2}\right), \left(2 \sin \frac{2\pi}{4} - \sin \frac{2\pi}{2}\right), \left(2 \sin \frac{3\pi}{4} - \sin \frac{3\pi}{2}\right), \ldots \right\}\right]$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left(2 \sin\left(\frac{n\pi}{4}\right) - \sin \frac{n\pi}{2}\right)\right]$$

Hence from equation (1) above, we get

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x \cos \frac{n\pi}{L}vt$$

$$= \sum_{n=1}^{\infty} \frac{8h}{\pi^2}\left[\frac{1}{n^2}\left(2 \sin\left(\frac{n\pi}{4}\right) - \sin \frac{n\pi}{2}\right)\right] \sin \frac{n\pi}{L}x \cos \frac{n\pi}{L}vt$$

$$= \frac{8h}{\pi^2}\sum_{n=1}^{\infty} B_n \ \sin \frac{n\pi}{L}x \cos \frac{n\pi}{L}vt$$

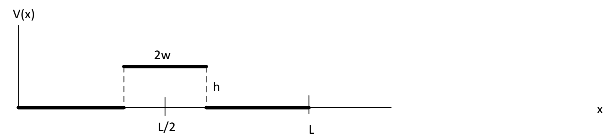
Where

$$B_n = \frac{1}{n^2}\left(2\ \sin\left(\frac{n\pi}{4}\right) - \sin\frac{n\pi}{2}\right)$$

The above is the result required to show.

4 chapter 13, problem 4.6. Mary Boas, second edition

A string of length L is initially stretched straight, its ends are fixed for all time t. At time t=0 its points are given the velocity $V(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0}$ as shown in diagram below. Determine the shape of the string at time t.



<u>Solution</u>

The PDE that governs this problem is the wave equation $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

The candidate solutions are

$$y = \begin{cases} \sin(kx)\ \sin(\omega t) \\ \sin(kx)\ \cos(\omega t) \\ \cos(kx)\ \sin(\omega t) \\ \cos(kx)\ \cos(\omega t) \end{cases}$$

Where $\omega = kv$ and $k = \frac{2\pi}{\lambda}$ where λ is the wave length

Now we discard solutions that contains $\cos kx$ since the string is fixed at $x = 0$.

So we are left with

$$y = \begin{cases} \sin(kx)\ \sin(\omega t) \\ \sin(kx)\ \cos(\omega t) \end{cases}$$

Now, $y = 0$ at $x = L$ then from $\sin kx = 0$ or $\sin kL = 0$ we need $k = \frac{n\pi}{L}$

Hence solutions become

$$y = \begin{cases} \sin \frac{n\pi}{L} x \sin \frac{n\pi}{L} vt \\ \sin \frac{n\pi}{L} x \cos \frac{n\pi}{L} vt \end{cases}$$

Applying initial conditions, which says that at time $t = 0$, velocity is given by $V(x)$

Hence from above, after taking $\frac{\partial y}{\partial t}$, we get

$$\frac{\partial y}{\partial t} = \begin{cases} \frac{n\pi v}{L} \sin \frac{n\pi}{L} x \cos \frac{n\pi v}{L} t \\ -\frac{n\pi v}{L} \sin \frac{n\pi}{L} x \sin \frac{n\pi v}{L} t \end{cases}$$

For the above we discard velocity solution above with $\sin t$ in it since that will give zero velocity at time t=0, which is not the case here. Hence we discard y solution with $\cos t$ in it, then the final general solution for y is now

$$y = \sin \frac{n\pi}{L} x \sin \frac{n\pi}{L} vt$$

A general solution is a linear combination of the above solutions, hence

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sin \frac{n\pi vt}{L} \qquad (1)$$

To find b_n, we apply the velocity initial condition. Hence differentiate equation (1) and set t=0, we have

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}$$

Setting t=0

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L} \sin \frac{n\pi x}{L} = V_{t=0} \qquad (2)$$

Now to find $V_{t=0}$. From diagram, we see that for $0 \le x \le \frac{L}{2} - w$, $V_{t=0} = 0$

For $\frac{L}{2} - w < x \le \frac{L}{2} + w$,$V_{t=0} = h$

For $\frac{L}{2} + w < x \le L$, $V_{t=0} = 0$

Hence

$$V_{t=0} = \begin{cases} 0 & 0 \le x \le \frac{L}{2} - w \\ h & \frac{L}{2} - w < x \le \frac{L}{2} + w \\ 0 & \frac{L}{2} + w < x \le L \end{cases}$$

Do the inner product on both sides of equation (2) w.r.t. $\sin\frac{n\pi}{L}x$

$$b_n\frac{n\pi v}{L}\int_0^L \sin^2\frac{n\pi}{L}x\ \ dx = \int_0^L V(x)\ \sin\frac{n\pi x}{L}\ \ dx$$

$$b_n\frac{n\pi v}{2} = \int_0^{\frac{L}{2}-w} 0\ \sin\frac{n\pi}{L}x\,dx\ +\ \int_{\frac{L}{2}-w}^{\frac{L}{2}+w} h\ \sin\frac{n\pi x}{L}\,dx + \int_{\frac{L}{2}+w}^{L} 0\ \sin\frac{n\pi}{L}x\,dx$$

$$b_n\frac{n\pi v}{2} = \int_{\frac{L}{2}-w}^{\frac{L}{2}+w} h\ \sin\frac{n\pi x}{L}\,dx$$

$$b_n\frac{n\pi v}{2} = -h\ \frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}-w}^{\frac{L}{2}+w}$$

$$b_n\frac{n\pi v}{2} = -h\ \frac{L}{n\pi}\left[\cos\frac{n\pi\left(\frac{L}{2}+w\right)}{L} - \cos\frac{n\pi\left(\frac{L}{2}-w\right)}{L}\right]$$

$$b_n\frac{n\pi v}{2} = -h\ \frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{2}+\frac{n\pi w}{L}\right) - \cos\left(\frac{n\pi}{2}-\frac{n\pi w}{L}\right)\right]$$

$$b_n = -\frac{2hL}{n^2\pi^2 v}\left[\cos\left(\frac{n\pi}{2}+\frac{n\pi w}{L}\right) - \cos\left(\frac{n\pi}{2}-\frac{n\pi w}{L}\right)\right]$$

But $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

and $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

Let $a = \frac{n\pi}{2}, b = \frac{n\pi w}{L}$

Hence b_n becomes

$$\begin{aligned}
b_n &= -\frac{2hL}{n^2\pi^2 v}[\cos(a+b) - \cos(a-b)] \\
&= -\frac{2hL}{n^2\pi^2 v}[\cos(a)\cos(b) - \sin(a)\sin(b) - \{\cos(a)\cos(b) + \sin(a)\sin(b)\}] \\
&= -\frac{2hL}{n^2\pi^2 v}[\cos(a)\cos(b) - \sin(a)\sin(b) - \cos(a)\cos(b) - \sin(a)\sin(b)] \\
&= -\frac{2hL}{n^2\pi^2 v}[-\sin(a)\sin(b)\ \ - \sin(a)\sin(b)] \\
&= \frac{4hL}{n^2\pi^2 v}\sin(a)\sin(b) \\
&= \frac{4hL}{n^2\pi^2 v}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi w}{L}\right)
\end{aligned}$$

For even n, the term $\sin\left(\frac{n\pi}{2}\right)$ is zero. For n odd $\sin\left(\frac{n\pi}{2}\right) = 1$ when $n = 1,5,9,\dots$ and $\sin\left(\frac{n\pi}{2}\right) = -1$ when $n = 3,7,11,\dots$ Hence

$$b_n = A(n)\frac{4hL}{n^2\pi^2 v}\sin\left(\frac{n\pi w}{L}\right) \qquad n = 1,3,5,7,\dots$$

And $A(n)$ is a function which returns 1 when $n = 1,5,9,..$ and returns -1 when $n = 3,7,11,\dots$

Hence now we have b_n we can substitute in (1)

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sin\frac{n\pi vt}{L}$$
$$y = \sum_{n\ odd}^{\infty} A(n)\frac{4hL}{n^2\pi^2 v} \sin\left(\frac{n\pi w}{L}\right)\left[\sin\frac{n\pi x}{L} \sin\frac{n\pi vt}{L}\right]$$
$$y = \frac{4hL}{\pi^2 v}\sum_{n\ odd}^{\infty} A(n)\frac{1}{n^2} \sin\left(\frac{n\pi w}{L}\right)\left[\sin\frac{n\pi x}{L} \sin\frac{n\pi vt}{L}\right]$$

Which is the general solution. Looking at few expanded terms in the series we get

$$y = \frac{4hL}{\pi^2 v}\left\{\sin\left(\frac{\pi w}{L}\right) \sin\frac{\pi x}{L} \sin\frac{\pi vt}{L} - \frac{1}{9}\sin\left(\frac{3\pi w}{L}\right) \sin\frac{3\pi x}{L} \sin\frac{3\pi vt}{L} + \frac{1}{25}\sin\left(\frac{5\pi w}{L}\right) \sin\frac{5\pi x}{L} \sin\frac{n\pi vt}{L}\right\}$$

Which is the result required.

5 chapter 13, problem 5.1. Mary Boas, second edition

Compute numerically the coefficients $c_m = \frac{200}{k_m J_1(k_m)}$ for the first 3 terms of the series $u = \sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-k_m z}$ for the steady state temp. in a solid semi-infinite cylinder when $u = 0$ at $r = 1$ and $u = 100$ at $z = 0$. find u at $r = 1/2, z = 1$

Solution

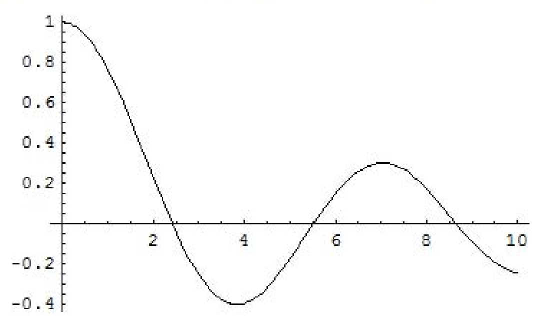
Here, we are looking at the solution for temp. inside a semi-infinite cylinder. This solution is for the case of a uniform temp. distribution on the boundary $z = 0$ is given by u equation shown above. Note that in the expression $c_m = \frac{200}{k_m J_1(k_m)}$, the k_m are the zeros of J_0 not J_1.

Need to find c_1, c_2, c_3 where $c_1 = \frac{200}{k_1 J_1(k_1)}$

To find k_1 and $J_1(k_1)$ I used mathematica.

I plotted $J_0(x)$ to see where the zeros are located first

```
In[16]:= Plot[BesselJ[0, x], {x, 0, 10}]
```




```
Out[16]= - Graphics -
```

So I see there is a zero near 2,5, and 9. I use mathematica to find these:

```
In[20]:= k1 = FindRoot[BesselJ[0, x] == 0, {x, 2}]

Out[20]= {x → 2.40483 }

In[21]:= k2 = FindRoot[BesselJ[0, x] == 0, {x, 5}]

Out[21]= {x → 5.52008 }

In[22]:= k3 = FindRoot[BesselJ[0, x] == 0, {x, 9}]

Out[22]= {x → 8.65373 }
```

Now I need to find $J_1(k_m)$. This is the result for 3 terms:

```
In[37]:= BesselJ[1, k1[[1, 2]]]
Out[37]= 0.519147
In[38]:= BesselJ[1, k2[[1, 2]]]
Out[38]= -0.340265
In[39]:= BesselJ[1, k3[[1, 2]]]
Out[39]= 0.271452
```

Hence, now the c_m terms can be found:

$$\begin{aligned}
c_1 &= \frac{200}{k_1 J_1(k_1)} = \frac{200}{(2.404)(0.519)} = 160.30 \\
c_2 &= \frac{200}{k_2 J_1(k_2)} = \frac{200}{(5.52)(-0.34)} = -106.56 \\
c_3 &= \frac{200}{k_3 J_1(k_3)} = \frac{200}{(8.65)(0.27)} = 85.635
\end{aligned}$$

Evaluating $u = \sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-k_m z}$ for the first 3 terms when $r = 1/2, z = 1$

$$\begin{aligned}
u &= c_1 J_0(k_1 r) e^{-k_1 z} + c_2 J_0(k_2 r) e^{-k_2 z} + c_3 J_0(k_3 r) e^{-k_3 z} \\
&= c_1 J_0\left(k_1 \frac{1}{2}\right) e^{-k_1} + c_2 J_0\left(k_2 \frac{1}{2}\right) e^{-k_2} + c_3 J_0\left(k_3 \frac{1}{2}\right) e^{-k_3} \\
&= (160.30) J_0\left(2.404 \frac{1}{2}\right) e^{-2.404} - (106.56) J_0\left(5.52 \frac{1}{2}\right) e^{-5.52} + (85.635) J_0\left(8.65 \frac{1}{2}\right) e^{-8.65} \\
&= (160.30) J_0(1.202) e^{-2.404} - (106.56) J_0(2.76) e^{-5.52} + (85.635) J_0(4.325) e^{-8.65}
\end{aligned}$$

Mathematica was used to evaluate J_0 values above.

```
In[40]:= BesselJ[0, 1.202]
Out[40]= 0.670136
In[41]:= BesselJ[0, 2.76]
Out[41]= -0.168385
In[42]:= BesselJ[0, 4.325]
Out[42]= -0.356614
```

Hence

$$
\begin{aligned}
u &= (160.30)(0.67)e^{-2.404} - (106.56)(-0.168)e^{-5.52} + (85.635)(-0.356)e^{-8.65} \\
u &= 9.7043 + 7.1713 \times 10^{-2} - 5.3389 \times 10^{-3} \\
u &= 9.7707 \text{ degrees}
\end{aligned}
$$

6 chapter 13, problem 5.2. Mary Boas, second edition

Find the solution for the steady state temp. distribution in a solid semi-infinite cylinder if the boundary temp. are $u = 0$ at $r = 1$ and $u = y = r\sin\theta$ at $z = 0$.

Solution

The candidate solutions are given by the solution to the Laplace equation in cylindrical coordinates which are

$$u = \begin{cases} J_n(k\,r)\sin(n\theta)e^{-k\,z} & (1) \\ \\ J_n(k\,r)\cos(n\theta)e^{-k\,z} & (2) \end{cases}$$

Where k is a zero of J_n (This is because we have used the B.C. of $u = 0$ at $r = 1$ to determine that the $k's$ have to be the zeros of J_n) when deriving the above solutions. See book page 560.

From boundary conditions we want $u = r\sin\theta$ when $z = 0$, hence we need to keep the solution (1) above, with $n = 1$. Hence a solution is

$$u = J_1(k\,r)\sin(\theta)e^{-k\,z} \qquad (3)$$

A general solution is a linear series combinations (eigenfunctions) of (3), each eigenfunction for each of the zeros of J_1. Call these zeros k_m

$$u = \sum_{m=1}^{\infty} c_m\, J_1(k_m\,r)\sin(\theta)e^{-k_m\,z} \qquad (4)$$

We now apply B.C. at $z = 0$ to find c_m. From (4) when $z = 0$

$$r\sin\theta = \sum_{m=1}^{\infty} c_m\, J_1(k_m\,r)\sin(\theta) \qquad (5)$$

We use (5) to find c_m and then substitute into (4) to obtain the final solution.

To find c_m from (5), take the inner product of each side with respect to $rJ_1(k_u\,r)$ from $r = 0$ to $r = 1$

$$\int_0^1 r\sin\theta[rJ_1(k_u\,r)]\,dr = \sum_{m=1}^{\infty} c_m\left(\int_0^1 J_1(k_m\,r)\sin(\theta)[rJ_1(k_u\,r)]\,dr\right)$$
$$\sin\theta\int_0^1 r^2 J_1(k_u\,r)\,dr = \sum_{m=1}^{\infty} c_m\ \sin(\theta)\left(\int_0^1 J_1(k_m\,r)[rJ_1(k_u\,r)]\,dr\right)$$

Dividing each side by $\sin\theta$

$$\int_0^1 r^2 J_1(k_u\, r)\, dr = \sum_{m=1}^{\infty} c_m \left(\int_0^1 J_1(k_m\, r)[r J_1(k_u\, r)]\, dr \right)$$

From orthogonality of Bessel function, we know that

$$\int_0^1 J_p(k_m\, r) r J_p(k_u\, r) dr = 0$$

If $m \neq u$. Hence in above equation all terms on the right drop except for one when $u = m$. We get

$$\int_0^1 r^2 J_1(k_m\, r)\, dr = \ c_m \ \int_0^1 r \ \ J_1(k_m\, r) J_1(k_m\, r) dr$$

Or

$$c_m = \frac{\int_0^1 r^2 J_1(k_m\, r)\, dr}{\int_0^1 r \ \ J_1(k_m\, r) \ \ J_1(k_m\, r)\, dr} \qquad (6)$$

The integral in the denominator above is found from equation 19.10 in text on page 523 which gives

$$\int_0^1 r \ \ J_1(k_m\, r) \ \ J_1(k_m\, r)\, dr = \frac{1}{2}[J_2(k_m)]^2 \qquad (7)$$

Now, we need to find the integral of the numerator in equation (6).

Using equation 15.1 in text, page 514, which says

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

Putting $p = 2$ above, and letting $x = k_m r$ gives

$$\begin{aligned}
\frac{1}{k_m}\frac{d}{dr}\left[(k_m r)^2 J_2(k_m r)\right] &= (k_m r)^2 J_1(k_m r) \\
\frac{1}{k_m}\frac{d}{dr}\left[k_m^2 r^2 J_2(k_m r)\right] &= k_m^2 r^2 J_1(k_m r) \\
\frac{1}{k_m}\frac{d}{dr}\left[r^2 J_2(k_m r)\right] &= r^2 J_1(k_m r)
\end{aligned}$$

Integrating each side w.r.t r from 0 ... 1

$$\begin{aligned}
\frac{1}{k_m}\int_0^1 \frac{d}{dr}\left[r^2 J_2(k_m r)\right] dr &= \int_0^1 r^2 J_1(k_m r)\, dr \\
\frac{1}{k_m}\left[r^2 J_2(k_m r)\right]_0^1 &= \int_0^1 r^2 J_1(k_m r)\, dr \\
\frac{1}{k_m}\left[J_2(k_m) - 0\right] &= \int_0^1 r^2 J_1(k_m r)\, dr \\
\frac{1}{k_m} J_2(k_m) &= \int_0^1 r^2 J_1(k_m r)\, dr \qquad (8)
\end{aligned}$$

Substituting (7) and (8) into (6)

$$\begin{aligned}
c_m &= \frac{\int_0^1 r^2 J_1(k_m\, r)\, dr}{\int_0^1 r\; J_1(k_m\, r)\; J_1(k_m\, r)\, dr} \\
&= \frac{\frac{1}{k_m} J_2(k_m)}{\frac{1}{2}[J_2(k_m)]^2} \\
&= \frac{2}{k_m\, J_2(k_m)}
\end{aligned}$$

Substituting this into (4) above, we get

$$\begin{aligned}
u &= \sum_{m=1}^{\infty} c_m\, J_1(k_m\, r)\sin(\theta) e^{-k_m\, z} \\
u &= \sum_{m=1}^{\infty} \frac{2}{k_m\, J_2(k_m)}\, J_1(k_m\, r)\;\sin(\theta)\, e^{-k_m\, z}
\end{aligned}$$

where k_m are zeros of J_1

The above is the result we are asked to show.

7 chapter 13, problem 5.4. Mary Boass, second edition

A flat circular plate of radius 1 is initially at temp. 100^0. From $t = 0$ on, the circumference of the plate is held at 0^0. Find the time-dependent temp distribution $u(r, \theta, t)$

Solution

First convert heat equation from Cartesian coordinates to polar.

heat equation in 2D Cartesian is

$$\nabla^2 u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial x^2} u = \frac{1}{\alpha^2} \frac{\partial}{\partial t} u$$

First need to express Laplacian operator ∇^2 in polar coordinates:

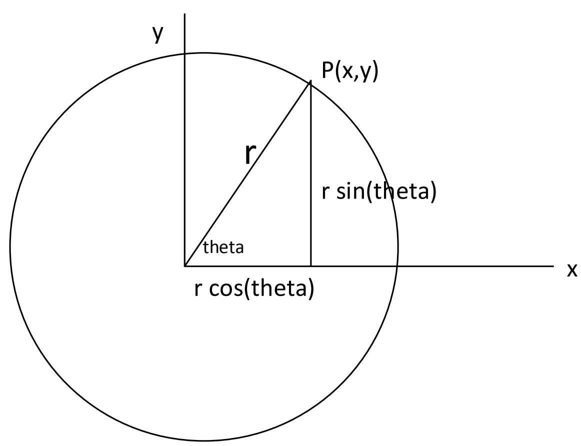
$$x = r \cos\theta$$
$$y = r \sin\theta$$

Hence

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \text{(A)}$$
$$\frac{\partial y}{\partial r} = \sin\theta$$

And

$$\frac{\partial x}{\partial \theta} = -r\ \sin\theta \qquad \text{(B)}$$
$$\frac{\partial y}{\partial \theta} = r\ \cos\theta$$



From geometry, we also know that

$$
\begin{aligned}
r &= \sqrt{x^2+y^2}\\
\theta &= \arctan\frac{y}{x}
\end{aligned}
$$

The above 2 relations imply

$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$

Hence we can express the above, using equations (A) and (B) as follows:

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\\
&= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}
\end{aligned}
$$

Multiply each side by r

$$
\begin{aligned}
r\frac{\partial}{\partial r} &= r\cos\theta\frac{\partial}{\partial x} + r\sin\theta\frac{\partial}{\partial y}\\
&= x\,\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad (1)
\end{aligned}
$$

Squaring each sides of (1) gives

$$
\begin{aligned}
r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) &= \left(x\,\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2\\
r\left(r\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\right) &= x\frac{\partial}{\partial x}\,x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}\,y\frac{\partial}{\partial y}\\
r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} &= x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) + 2x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right)\\
&= x\left(x\frac{\partial^2}{\partial x^2} + \overbrace{\frac{\partial}{\partial x}x}^{=1}\frac{\partial}{\partial x}\right) + y\left(y\frac{\partial^2}{\partial y^2} + \overbrace{\frac{\partial}{\partial y}y}^{=1}\frac{\partial}{\partial y}\right) + 2x\left(y\frac{\partial^2}{\partial x\partial y} + \overbrace{\frac{\partial}{\partial x}y}^{=0}\frac{\partial}{\partial y}\right)\\
&= x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y} \qquad (2)
\end{aligned}
$$

Notice that when manipulating of differential operators, $x\frac{\partial}{\partial x} \neq \frac{\partial}{\partial x}x$. Similarly

$$\begin{aligned}
\frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} \\
&= -r\ \sin\theta\frac{\partial}{\partial x} + r\ \cos\theta\frac{\partial}{\partial y} \\
&= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad (3)
\end{aligned}$$

Squaring each side of (3) gives

$$\begin{aligned}
\left(\frac{\partial}{\partial\theta}\right)^2 &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)^2 \\
\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} &= -y\frac{\partial}{\partial x}\left(-y\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) + x\frac{\partial}{\partial y}\left(-y\frac{\partial}{\partial x}\right) \\
\frac{\partial^2}{\partial\theta^2} &= -y\left(-y\frac{\partial^2}{\partial x^2} + \overbrace{\frac{\partial}{\partial x}(-y)}^{=0}\frac{\partial}{\partial x}\right) + x\left(x\frac{\partial^2}{\partial y^2} + \overbrace{\frac{\partial}{\partial y}(x)}^{=0}\frac{\partial}{\partial y}\right) \\
&\quad - y\left(x\frac{\partial^2}{\partial y\partial x} + \overbrace{\frac{\partial}{\partial x}x}^{=1}\frac{\partial}{\partial y}\right) + x\left(-y\frac{\partial^2}{\partial x\partial y} + \overbrace{\frac{\partial}{\partial y}y}^{=1}\frac{\partial}{\partial x}\right) \\
&= y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial x\partial y} - x\frac{\partial}{\partial x} \\
&= y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - 2yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} \qquad (4)
\end{aligned}$$

Adding equation (2) and (4) and carry cancellations

$$r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2} = \left(x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y}\right)+\left(y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2} - 2yx\frac{\partial^2}{\partial y\partial x} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right)$$

$$r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2} = \left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2}\right) + \left(y^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}\right)$$

Hence we get

$$
\begin{aligned}
r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2} &= x^2\frac{\partial^2}{\partial x^2}+y^2\frac{\partial^2}{\partial y^2}+y^2\frac{\partial^2}{\partial x^2}+x^2\frac{\partial^2}{\partial y^2}\\
&= \left(x^2+y^2\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)
\end{aligned}
$$

Dividing by r^2

$$
\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}=\frac{\left(x^2+y^2\right)}{r^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)
$$

But $r^2=x^2+y^2$ hence

$$
\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\nabla^2
$$

Now that we have the Laplacian in polar coordinates, we can solve the problem by applying separation of variables on the heat PDE expressed in polar coordinates.

$$
\frac{\partial^2}{\partial r^2}u+\frac{1}{r}\frac{\partial}{\partial r}u+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u=\frac{1}{\alpha^2}\frac{\partial}{\partial t}u \tag{5}
$$

Let solution $u(r,\theta,t)$ be a linear combination of functions each depends on only r,θ, or t

$$
u(r,\theta,t)=R(r)\Theta(\theta)T(t) \tag{6}
$$

Substitute (6) in (5). First evaluate the various derivatives:

$$
\frac{\partial}{\partial r}u=\Theta(\theta)T(t)\frac{\partial}{\partial r}R(r)
$$

$$
\frac{\partial^2}{\partial r^2}u=\Theta(\theta)T(t)\frac{\partial^2}{\partial r^2}R(r)
$$

$$
\frac{\partial}{\partial\theta}u=R(r)T(t)\frac{\partial}{\partial\theta}\Theta(\theta)
$$

$$
\frac{\partial^2}{\partial\theta^2}u=R(r)T(t)\frac{\partial^2}{\partial\theta^2}\Theta(\theta)
$$

$$\frac{\partial}{\partial t}u = R(r)\Theta(\theta)\frac{\partial}{\partial t}T(t)$$

Hence equation (5) becomes

$$\frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u = \frac{1}{\alpha^2}\frac{\partial}{\partial t}u$$

$$\Theta(\theta)T(t)\frac{d^2}{dr^2}R(r) + \frac{1}{r}\Theta(\theta)T(t)\frac{d}{dr}R(r) + \frac{1}{r^2}R(r)T(t)\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}R(r)\Theta(\theta)\frac{d}{dt}T(t)$$

Divide by $R(r)\Theta(\theta)T(t)$

$$\frac{1}{R(r)}\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{1}{R(r)}\frac{d}{dr}R(r) + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t)$$

$$\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = \frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t)$$

We notice that the RHS depends only on t and the LHS depends only on r, θ and they equal to each others, hence they both must be constant. Let this constant be $-k^2$

Hence

$$\frac{1}{\alpha^2}\frac{1}{T(t)}\frac{d}{dt}T(t) = -k^2 \tag{7}$$

$$\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r) + \frac{1}{r}\frac{d}{dr}R(r)\right] + \frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) = -k^2 \tag{8}$$

equation (7) is a linear first order ODE with constant coeff. $\frac{d}{dt}T(t) = -\alpha^2 T(t)k^2$ or $\frac{dT(t)}{T(t)} = -\alpha^2 k^2 dt$

Integrating to solve gives

$$\int \frac{dT(t)}{T(t)} = \int -\alpha^2 k^2 dt$$

$$\ln T(t) = -\alpha^2 k^2 t$$

or

$$T(t) = e^{-\alpha^2 k^2 t} \tag{9}$$

Looking at equation (8). Multiply each sides by r^2 we get

$$
\begin{aligned}
\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) &= -r^2k^2\\
\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+r^2k+\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) &= 0\\
r^2\left(\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+k^2\right)+\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) &= 0 \qquad (10)
\end{aligned}
$$

The second term depends only on θ and the first term depends only on r and they are equal, hence they must be both constant. Let this constant be $-n^2$ hence

$$
\begin{aligned}
\frac{1}{\Theta(\theta)}\frac{d^2}{d\theta^2}\Theta(\theta) &= -n^2\\
\frac{d^2}{d\theta^2}\Theta(\theta) &= -n^2\Theta(\theta)
\end{aligned}
$$

This is a second order linear ODE with constant coeff. Solution is

$$
\Theta(\theta)=\begin{cases}\sin n\theta\\ \\ \cos n\theta\end{cases} \qquad (11)
$$

From (10) we now have

$$
\begin{aligned}
r^2\left(\frac{1}{R(r)}\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+k^2\right)-n^2 &= 0\\
\frac{r^2}{R(r)}\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+r^2k^2-n^2 &= 0\\
r^2\left[\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)\right]+\left(r^2k^2-n^2\right)R(r) &= 0\\
r^2\frac{d^2}{dr^2}R(r)+r\frac{d}{dr}R(r)+\left(r^2k^2-n^2\right)R(r) &= 0 \qquad (12)
\end{aligned}
$$

Equation (12) is the Bessel D.E., its solutions are $J_n(kr)$ and $N_n(kr)$. As described on book on page 560, we can not use the $N_n(kr)$ solution since plate contains the origin and $N_n(0)$ is not defined. So we use solution $R(r) = J_n(kr)$. From boundary conditions, we want solution to be zero at $r = 1$, hence we want $J_n(k) = 0$, hence the k's are the zeros of J_n

Putting these solutions together, we get from (6)

$$
\begin{aligned}
u(r,\theta,t) &= R(r)\Theta(\theta)T(t)\\
&= \begin{cases} J_n(kr)\sin n\theta e^{-\alpha^2 k^2 t} \\ \\ J_n(kr)\cos n\theta e^{-\alpha^2 k^2 t} \end{cases}
\end{aligned}
$$

From symmetry of plate, the solution can not depend on the angle θ, hence let $n = 0$ and so as not to get $u = 0$, we must pick the solution with $\cos n\theta$ term. Hence our solution now is

$$u(r,t) = J_0(kr)\, e^{-\alpha^2 k^2 t}$$

Where k is a zero of J_0

The general solution is a linear combination of this eigenfunction for all zeros of J_0, hence

$$u(r,t) = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \tag{13}$$

We find c_m by using initial condition. When $t = 0$, temp. was 100^0 hence

$$100 = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)$$

Applying inner product w.r.t. $rJ_0(k_u r)$ from $0 \dots 1$

$$
\begin{aligned}
\int_0^1 100\, rJ_0(k_u r)\, dr &= \int_0^1 \left(\sum_{m=1}^{\infty} c_m\, J_0(k_m r)\right) rJ_0(k_u r)\, dr\\
100\int_0^1 rJ_0(k_u r)\, dr &= \sum_{m=1}^{\infty} c_m \int_0^1 J_0(k_m r)\, rJ_0(k_u r)\, dr
\end{aligned}
$$

From orthogonality of $J_0(k_m r)$ and $J_0(k_u r)$, all terms drop expect when $m = u$

$$100\int_0^1 rJ_0(k_u r)\, dr = c_u \int_0^1 r[J_0(k_u r)]^2\, dr$$

From here we can follow the book on page 561 to get

$$c_m = \frac{200}{k_m J_1(k_m)}$$

Substitute this in equation 13

$$\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} c_m\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \\
&= \sum_{m=1}^{\infty} \frac{200}{k_m J_1(k_m)}\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t} \\
&= 200 \sum_{m=1}^{\infty} \frac{1}{k_m J_1(k_m)}\, J_0(k_m r)\, e^{-\alpha^2 k_m^2\, t}
\end{aligned}$$

Where k_m are zeros of J_0

Notice that final solution does not depend on θ

8 chapter 13, problem 5.11. Mary Boas, second edition

Solve

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2R$$
$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)R$$

Solution

First equation, use power series method.

$$
\begin{aligned}
r\frac{d}{dr}\left(r\frac{dR}{dr}\right)&=n^2R\\
r\left(r\frac{d^2R}{dr^2}+\frac{dR}{dr}\right)-n^2R&=0\\
r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-n^2R&=0
\end{aligned}
$$

Let $R=a_0r^s+a_1r^{s+1}+a_2r^{s+2}+a_3r^{s+3}+a_4r^{s+4}+\cdots$ then

$$
\begin{aligned}
R&=a_0r^s+a_1r^{s+1}+a_2r^{s+2}+a_3r^{s+3}+a_4r^{s+4}+\cdots\\
-n^2R&=-n^2a_0r^s-n^2a_1r^{s+1}-n^2a_2r^{s+2}-n^2a_3r^{s+3}-n^2a_4r^{s+4}-\cdots\\
\frac{dR}{dr}&=s\ a_0r^{s-1}+(s+1)\ a_1r^s+(s+2)\ a_2r^{s+1}+(s+3)\ a_3r^{s+2}+\ \cdots\\
r\frac{dR}{dr}&=s\ a_0r^s+(s+1)\ a_1r^{s+1}+(s+2)\ a_2r^{s+2}+(s+3)\ a_3r^{s+3}+\ \cdots\\
\frac{d^2R}{dr^2}&=\ (s-1)s\ a_0r^{s-2}+s(s+1)\ a_1r^{s-1}+(s+1)(s+2)\ a_2r^s+(s+2)(s+3)\ a_3r^{s+1}+\ \cdots\\
r^2\frac{d^2R}{dr^2}&=\ (s-1)s\ a_0r^s+s(s+1)\ a_1r^{s+1}+(s+1)(s+2)\ a_2r^{s+2}+(s+2)(s+3)\ a_3r^{s+3}+\ \cdots
\end{aligned}
$$

Table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2R$	$-n^2a_0$	$-n^2a_1$	$-n^2a_2$	$-n^2\ a_m$
$r\frac{dR}{dr}$	$s\ a_0$	$(s+1)\ a_1$	$(s+2)\ a_2$	$(s+m)a_m$
$r^2\frac{d^2R}{dr^2}$	$(s-1)s\ a_0$	$s(s+1)\ a_1$	$(s+1)(s+2)\ a_2$	$(s+m-1)(s+m)\ a_m$

Hence, from first column we see , and since $a_0 \neq 0$ we solve for s

$$
\begin{aligned}
-n^2 a_0 + s\, a_0 + (s-1)s\, a_0 &= 0 \\
a_0\left(-n^2 + s + (s-1)s\right) &= 0 \\
-n^2 + s + (s-1)s &= 0 \\
-n^2 + s^2 &= 0 \\
s &= \pm n
\end{aligned}
$$

We see from second column, $a_1\left(-n^2 + (s+1) + s^2 + s\right) = 0$ or $a_1\left(-s^2 + 2s + 1 + s^2\right) = 0$, hence $a_1(2s+1) = 0$

For $a_1 \neq 0$ then $s = -\frac{1}{2}$, this means n is not an integer since $s = \pm n$. hence a_1 must be zero.

The same applies to all a_m , $m > 0$ Hence solution contains only a_0

$$
R = a_0 r^{\pm n}
$$

$$
R = \begin{cases} a_0 r^{-n} \\ \\ a_0 r^{+n} \end{cases}
$$

For some constant a_0. This solution is when $n \neq 0$

If $n = 0$, table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2 R$	0	0	0	0
$r\frac{dR}{dr}$	$s\, a_0$	$(s+1)\, a_1$	$(s+2)\, a_2$	$(s+m)a_m$
$r^2\frac{d^2R}{dr^2}$	$(s-1)s\, a_0$	$s(s+1)\, a_1$	$(s+1)(s+2)\, a_2$	$(s+m-1)(s+m)\, a_m$

From first column:

$$
\begin{aligned}
s a_0 + s^2 a_0 - s a_0 &= 0 \\
a_0\left(s + s^2 - s\right) &= 0 \\
s^2 &= 0 \\
s &= 0
\end{aligned}
$$

And all other $a's$ are zero. Hence $R = a_0$ or R is constant.

Now for the second ODE

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$
$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$

Let $R = a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots$ then

$$\begin{aligned}
R &= a_0 r^s + a_1 r^{s+1} + a_2 r^{s+2} + a_3 r^{s+3} + a_4 r^{s+4} + \cdots \\
-l(l+1)R &= -l(l+1)a_0 r^s - l(l+1)a_1 r^{s+1} - l(l+1)a_2 r^{s+2} - l(l+1)a_3 r^{s+3} - l(l+1)a_4 r^{s+4} - \cdots \\
\frac{dR}{dr} &= s\ a_0 r^{s-1} + (s+1)\ a_1 r^s + (s+2)\ a_2 r^{s+1} + (s+3)\ a_3 r^{s+2} + \ \cdots \\
2r\frac{dR}{dr} &= 2s\ a_0 r^s + 2(s+1)\ a_1 r^{s+1} + 2(s+2)\ a_2 r^{s+2} + 2(s+3)\ a_3 r^{s+3} + \ \cdots \\
\frac{d^2R}{dr^2} &= \ (s-1)s\ a_0 r^{s-2} + s(s+1)\ a_1 r^{s-1} + (s+1)(s+2)\ a_2 r^s + (s+2)(s+3)\ a_3 r^{s+1} + \ \cdots \\
r^2\frac{d^2R}{dr^2} &= \ (s-1)s\ a_0 r^s + s(s+1)\ a_1 r^{s+1} + (s+1)(s+2)\ a_2 r^{s+2} + (s+2)(s+3)\ a_3 r^{s+3} + \ \cdots
\end{aligned}$$

Table is

	r^s	r^{s+1}	r^{s+2}	r^{s+m}
$-n^2R$	$-l(l+1)a_0$	$-l(l+1)a_1$	$-l(l+1)a_2$	$-l(l+1)\ a_m$
$2r\frac{dR}{dr}$	$2s\ a_0$	$2(s+1)\ a_1$	$2(s+2)\ a_2$	$2(s+m)a_m$
$r^2\frac{d^2R}{dr^2}$	$(s-1)s\ a_0$	$s(s+1)\ a_1$	$(s+1)(s+2)\ a_2$	$(s+m-1)(s+m)\ a_m$

From first column:

$$\begin{aligned}
-l(l+1)a_0 + 2s\ a_0 + (s-1)s\ a_0 &= 0 \\
a_0(-l(l+1) + 2s\ + (s-1)s\ ) &= 0 \\
-l(l+1) + 2s\ + (s-1)s &= 0 \\
-l(l+1) + s\ + s^2 &= 0 \\
(s-l)(s-(-l-1)) &= 0
\end{aligned}$$

Hence $s = l$ or $s = -l-1$.

We also see that all other $a's$ will be zero, since recursive formula has only a_m in it and no other a. Hence

$$R = a_0 r^s$$

$$R = \begin{cases} a_0 r^l \\ \\ a_0 r^{-l-1} \end{cases}$$

For some constant a_0