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HW # 6

Math 121 B

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UCB extension

Ch 12

$\boxed{16.3}$ Find solution of following DE in terms of Bessel functions by using 16.1 and 16.2.

$xy'' + 2y' + 4y = 0$

(16.1) $y'' + \frac{1-2a}{x} y' + \left[(bc x^{c-1})^2 + \frac{a^2-p^2c^2}{x^2} \right] y = 0$

(16.2) $y = x^a Z_p(bx^c)$

start by dividing DE by x to set it to 16.1 form:

$$y'' + \frac{2}{x} y' + 4\frac{y}{x} = 0$$

compare to 16.1 $\Rightarrow$ $1-2a = 2 \Rightarrow 2a = -1 \Rightarrow \boxed{a = -\frac{1}{2}}$ — ①

$a^2 - p^2c^2 = 0 \rightarrow \frac{1}{4} - p^2c^2 = 0$ — ②

and $(bc)^2 (x^{c-1})^2 = \frac{4}{x}$

so $bc = 2$ — ③

and $x^{2c-2} = x^{-1} \Rightarrow 2c-2 = -1 \Rightarrow \boxed{c = \frac{1}{2}}$ — ④

From ④ and ③ $\Rightarrow$ $b = \frac{2}{c} = \frac{2}{\frac{1}{2}} = 4$ i.e $\boxed{b=4}$

From ④ and ② $\rightarrow$ $\frac{1}{4} - p^2\left(\frac{1}{4}\right) = 0 \Rightarrow 1 = p^2$ i.e $\boxed{p=1}$

So solution $y = x^{-\frac{1}{2}} Z_1(4x^{\frac{1}{2}})$

$$\boxed{y = \frac{1}{\sqrt{x}} Z_1(4\sqrt{x})}$$

i.e general solution $y = \frac{1}{\sqrt{x}}\left[A\, J_1(4\sqrt{x}) + B\, N_1(4\sqrt{x}) \right]$

where A, B are arbitrary constants.

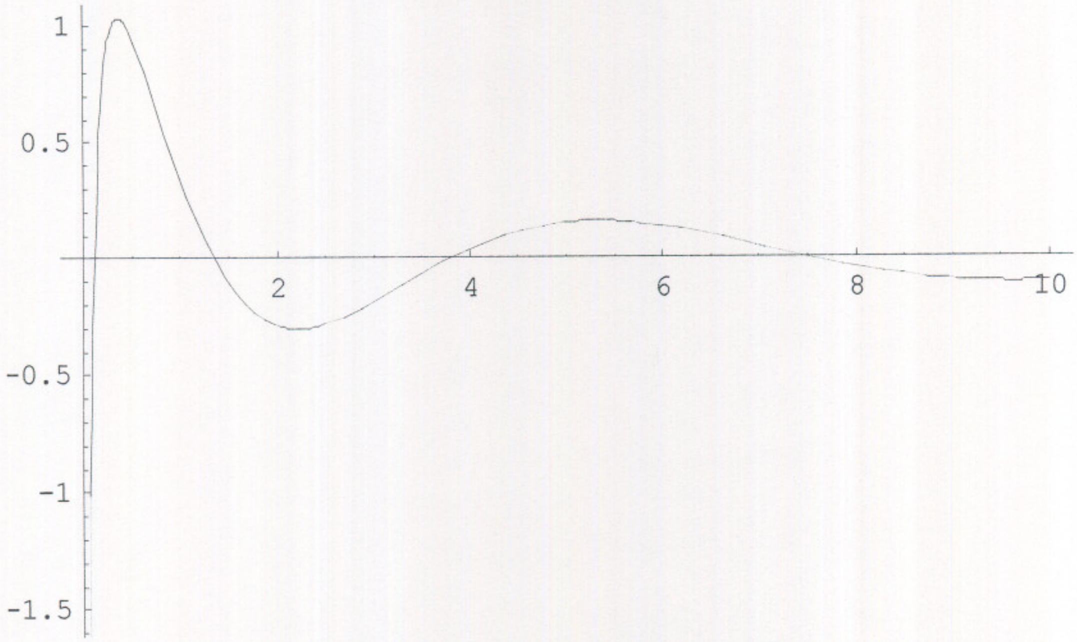
where J_1 is Bessel function order 1.

$N_p(x)$ is $\frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin \pi p}$ (eq. 13.3 in text)

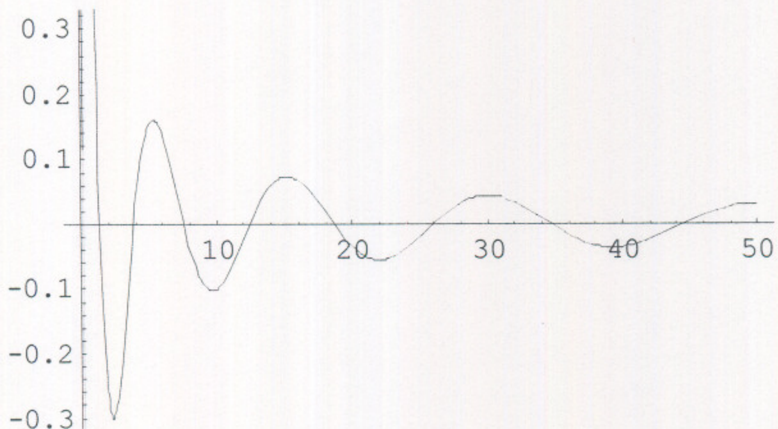
used mathematica to plot the solution for $A=1$, $B=1$.

Please see next.

Plot$\left[\frac{\text{BesselJ}\left[1, 4\sqrt{x}\right]}{\sqrt{x}} + \frac{\text{BesselY}\left[1, 4\sqrt{x}\right]}{\sqrt{x}}, \{x, 0, 10\}\right]$



Plot$\left[\frac{\text{BesselJ}\left[1, 4\sqrt{x}\right]}{\sqrt{x}} + \frac{\text{BesselY}\left[1, 4\sqrt{x}\right]}{\sqrt{x}}, \{x, 0, 50\}\right]$



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ch 12

$\boxed{17.2}$ From problem 12.9, $J_{1/2} = \sqrt{\frac{2}{\pi x}}\ \sin x$. Use (15.2) to obtain $J_{3/2}(x)$ and $J_{5/2}(x)$. substitute your result for the J's into 17.4. to verify the formulas stated for j_0, j_1, j_2 in terms of $\sin x$ and $\cos x$.

15.2 is $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

17.4 for $j_n(x) = \sqrt{\frac{\pi}{2x}}\ J_{\frac{(2n+1)}{2}} = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

since $J_{1/2} = \sqrt{\frac{2}{\pi x}}\ \sin x$, Then using 15.2 $\boxed{J_{p+1} = -\frac{d}{dx}\left[x^{-p} J_p\right]\frac{1}{x^{-p}}}$

let $p = \frac{1}{2}$ $\Rightarrow$ $J_{3/2} = -\frac{d}{dx}\left[x^{-\frac{1}{2}} J_{1/2}\right]\frac{1}{x^{-1/2}}$

i.e $J_{3/2} = -\frac{d}{dx}\left[x^{-\frac{1}{2}}\sqrt{\frac{2}{\pi x}}\sin x\right]\sqrt{x}$

$= -\sqrt{x}\ \frac{d}{dx}\left[\frac{1}{\sqrt{x}}\sqrt{\frac{2}{\pi x}}\ \sin x\right] = -\sqrt{x}\ \frac{d}{dx}\left[\frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x\right] = -\sqrt{\frac{2x}{\pi}}\ \frac{d}{dx}\left[x^{-1}\sin x\right]$

apply product rule of differentiation

$J_{3/2} = -\sqrt{\frac{2x}{\pi}}\left[x^{-1}\cos x + (-1)x^{-2}\sin x\right] = -\sqrt{\frac{2x}{\pi}}\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right]$

i.e $\boxed{J_{3/2} = -\sqrt{\frac{2x}{\pi}}\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right]}$

to find $J_{5/2}$, let $p = 3/2$ and apply 15.2 again.

i.e $J_{p+1} = -x^{p}\ \frac{d}{dx}\left[x^{-p} J_p\right]$

or $J_{5/2} = -x^{3/2}\ \frac{d}{dx}\left[x^{-\frac{3}{2}} J_{3/2}\right] = -x^{\frac{3}{2}}\ \frac{d}{dx}\left[x^{-\frac{3}{2}}\left(-\sqrt{\frac{2x}{\pi}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)\right)\right]$

$J_{5/2} = +x^{\frac{3}{2}}\ \frac{d}{dx}\left[x^{-\frac{3}{2}}\ x^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)\right] = x^{\frac{3}{2}}\ \frac{d}{dx}\left[x^{-1}\sqrt{\frac{2}{\pi}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)\right]$

$= x^{\frac{3}{2}}\sqrt{\frac{2}{\pi}}\ \frac{d}{dx}\left[\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)\right] = x\sqrt{\frac{2x}{\pi}}\ \frac{d}{dx}\left(\frac{\cos x}{x^2} - \frac{\sin x}{x^3}\right)$

$= x\sqrt{\frac{2x}{\pi}}\left(\frac{1}{x^2}(-\sin x) + \cos x(-2x^{-3}) - \left[\frac{1}{x^3}\cos x + \sin x\,(-3x^{-4})\right]\right)$

$= x\sqrt{\frac{2x}{\pi}}\left(-\frac{\sin x}{x^2} - \frac{2\cos x}{x^3} - \frac{\cos x}{x^3} + \frac{3\sin x}{x^4}\right)$ $\longrightarrow$

$\underline{\underline{n=0}}$

$$j_0 = \sqrt{\frac{\pi}{2x}}\, J_{\frac{(2n+1)}{2}} = \sqrt{\frac{\pi}{2x}}\, J_{1/2} = \sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}}\sin x = \sqrt{\frac{2\pi}{2\pi x^2}}\sin x = \boxed{\frac{1}{x}\sin x}$$

using $j_n = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$ we set for $n=0$

$$j_0 = x^0\left(-\frac{1}{x}\frac{d}{dx}\right)^0\left(\frac{\sin x}{x}\right) = \boxed{\frac{\sin x}{x}}$$ which matches

so $\boxed{\text{ok for } n=0}$

$\underline{\underline{n=1}}$

$$j_1 = \sqrt{\frac{\pi}{2x}}\, J_{\left(\frac{2\times1+1}{2}\right)} = \sqrt{\frac{\pi}{2x}}\, J_{3/2} = \sqrt{\frac{\pi}{2x}}\left(-\sqrt{\frac{2x}{\pi}}\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right]\right)$$

$$= -\sqrt{\frac{2\pi x}{2x\pi}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \boxed{-\frac{\cos x}{x} + \frac{\sin x}{x^2}}$$

from derivative formula for j_n, we set

$$j_1 = x^1\left(-\frac{1}{x}\frac{d}{dx}\right)^1\left(\frac{\sin x}{x}\right) = x^1\left(-\frac{1}{x}\left(\frac{1}{x}\cos x + \sin x\left(-1\, x^{-2}\right)\right)\right)$$

$$= x\left(-\frac{1}{x}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)\right) = \boxed{-\frac{\cos x}{x} + \frac{\sin x}{x^2}}$$ which matches

so $\boxed{\text{ok for } n=1}$

$\underline{\underline{n=2}}$

$$j_2 = \sqrt{\frac{\pi}{2x}}\, J_{\left(\frac{2\times2+1}{2}\right)} = \sqrt{\frac{\pi}{2x}}\, J_{5/2} = \sqrt{\frac{\pi}{2x}}\, x\sqrt{\frac{2x}{\pi}}\left(3\frac{\sin x}{x^4} - \frac{\sin x}{x^2} - \frac{3\cos x}{x^3}\right)$$

$$= x\sqrt{\frac{\pi\, 2x}{2x\pi}}\left(3\frac{\sin x}{x^4} - \frac{\sin x}{x^2} - \frac{3\cos x}{x^3}\right) = \boxed{\frac{3\sin x}{x^3} - \frac{\sin x}{x} - \frac{3\cos x}{x^2}}$$

from derivative formula for j_n, we set

$$j_2 = x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^2\left(\frac{\sin x}{x}\right) = x^2\left(-\frac{1}{x}\frac{d}{dx}\left(-\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right)\right)$$

$$= x^2\left(-\frac{1}{x}\frac{d}{dx}\right)\left(-\frac{1}{x}\left[\frac{1}{x}\cos x + \sin x\,(-1)(x^{-2})\right]\right)$$

$$= x^2\left(-\frac{1}{x}\frac{d}{dx}\right)\left(-\frac{\cos x}{x^2} + \frac{\sin x}{x^3}\right) = x^2\left(-\frac{1}{x}\left[-\frac{1}{x^2}(-\sin x) - \cos x\,(-2)x^{-3} + \frac{1}{x^3}\cos x + \sin x\frac{(-3)}{x^4}\right]\right)$$

$$= -x\left[\frac{\sin x}{x^2} + \frac{3\cos x}{x^3} - \frac{3\sin x}{x^4}\right] = \boxed{-\frac{\sin x}{x} - \frac{3\cos x}{x^2} + \frac{3\sin x}{x^3}}$$

which matches above answer also. QED

Ch 12

$\boxed{17.5}$ show from 17.4 that $h_n^{(1)}(x) = -i x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right)$.

17.4: $h_n^{(1)} = j_n(x) + i\, y_n(x)$

where $y_n(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$.

from 17.4: $h_n^{(1)} = j_n(x) + i\, y_n(x)$

replace $j_n(x)$ by $x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$

and $y_n(x)$ by $-x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$

so 17.4 becomes: $h_n^{(1)} = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) - i\, x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$

$$h_n^{(1)} = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{\sin x}{x} - i\frac{\cos x}{x}\right] = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\left(\frac{i}{i}\right)\left(\frac{\sin x}{x} - i\frac{\cos x}{x}\right)\right]$$

$$= x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{1}{i}\left(i\frac{\sin x}{x} - i^2\frac{\cos x}{x}\right)\right] = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{1}{i}\left(\frac{\cos x}{x} + i\frac{\sin x}{x}\right)\right]$$

but $e^{ix} = \cos x + i \sin x$

so $h_n^{(1)} = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{1}{i}\frac{e^{ix}}{x}\right]$ ✓, but $\frac{1}{i} = -i$ so

$$\boxed{h_n^{(1)} = -i x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{e^{ix}}{x}\right]}$$

Ch 12

$\boxed{17.7}$ Solve $xy''=y$ using 16.1 and then express answer in terms of a function I_p by 17.3.

16.1: says that $y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2-p^2c^2}{x^2}\right]y = 0$

has solution $y = x^a Z_p(bx^c)$.

17.3: $I_p(x) = i^{-p} J_p(ix)$

first solve $xy''=y$.

$\Rightarrow y'' - \frac{y}{x} = 0$

compare to 16.1

$1-2a=0$ i.e $2a=1$ or $\boxed{a=\frac{1}{2}}$

$(a^2-p^2c^2)=0$ i.e $\boxed{p^2c^2=\frac{1}{4}}$ — ①

and $b^2c^2x^{2c-2} = -x^{-1} \Rightarrow b^2c^2=-1$

and $2c-2=-1$ i.e $2c=1$ ie $\boxed{c=\frac{1}{2}}$ — ②

Then $b^2\left(\frac{1}{4}\right)=-1$ i.e $b^2=-4$ i.e $\boxed{b=\pm\sqrt{-4} \text{ or } \pm 2i}$

from ① and ② $\Rightarrow p^2\left(\frac{1}{4}\right)=\frac{1}{4}$ i.e $\boxed{p=1}$

So solution is $y = x^a Z_p\, bx^c$

$= \sqrt{x}\, Z_p\, (\pm 2i)\sqrt{x}$

i.e $\boxed{y=\sqrt{x}\, Z_1(2i\sqrt{x}) \quad \text{or} \quad y=\sqrt{x}\, Z_1(-2i\sqrt{x})}$

if I just take the positive root of b, i.e $b=2i$, then general solution is

$$\boxed{y=\sqrt{x}\left[A\, J_1(2i\sqrt{x}) + B\, N_1(2i\sqrt{x})\right]}$$

or $\cancel{y=\sqrt{x}\left[A J_1(2i\sqrt{x}) + B\left(\frac{\cos(\pi)\, J_1(2i\sqrt{x}) - J_{-1}(2i\sqrt{x})}{\sin\pi}\right)\right]}$ — ③

note: because of 0/0 expression, here I'll leave y in terms of J and N and continue from there:

so $y = \sqrt{x}\left[A J_1(2i\sqrt{x}) + B N_1(2i\sqrt{x})\right]$

but $J_p(ix) = \dfrac{I_p(x)}{i^{-p}}$

so $J_1(i\, 2\sqrt{x}) = \dfrac{I_1(2\sqrt{x})}{i^{-1}} = \boxed{i\, I_1(2\sqrt{x})}$

and. $J_{-1}(2i\sqrt{x}) = \dfrac{I_{-1}(2\sqrt{x})}{i} = -i\, I_{-1}(2\sqrt{x})$

so $\boxed{y = \sqrt{x}\left[A \quad i I_1(2\sqrt{x}) + B N_1(2i\sqrt{x})\right]}$

to expand $N(x)$ in terms of J_p using eq. 13·3, I get $\sin\pi$ in denominator. for $x \neq 0$ this has a limit according to book. So I'll show this as well:

$$y = \sqrt{x}\left[A \; i I_1(2\sqrt{x}) + B\left(\frac{\cos\pi \; J_1(2i\sqrt{x}) - J_{-1}(2i\sqrt{x})}{\sin\pi}\right)\right]$$

$$y = \sqrt{x}\left[A \; i I_1(2\sqrt{x}) + B\left(\frac{\cos x \quad i I_1(2\sqrt{x}) + i\, I_{-1}(2\sqrt{x})}{\sin\pi}\right)\right]$$

$$\boxed{y = i\sqrt{x}\left[A \; I_1(2\sqrt{x}) + B\left(\frac{\cos x \; I_1(2\sqrt{x}) + I_{-1}(2\sqrt{x})}{\sin\pi}\right)\right]}$$

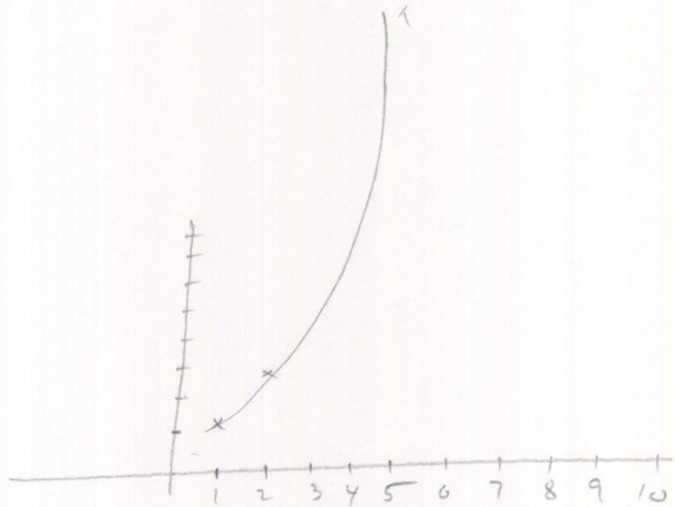
this is general solution in terms of I_p but with indeterminate form for second term. this is valid for $x \neq 0$.

ch 12

$\boxed{17.10}$ Using Tables, sketch graph of the hyperbolic Bessel function $I_0(x)$.

from Table 9.11, page 428, Handbook of math functions, Abramowitz, these are values for $I_0(x)$.

x	$I_0(x)$
1	1.26606
2	2.79
5	27.2
10	2815.7
50	too large.



I now used mma to plot $I_0(x)$: $\longrightarrow$

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Plot[BesselI[0,x],{x,0,5} ]
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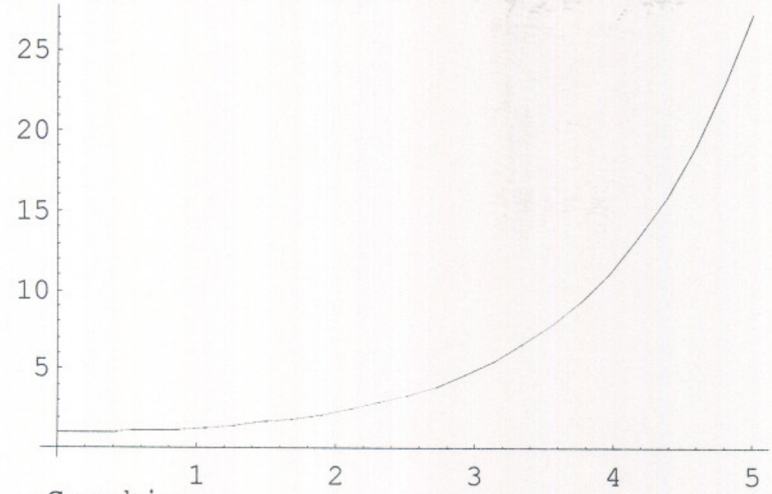

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Plot[BesselI[0,x],{x,0,10} ]
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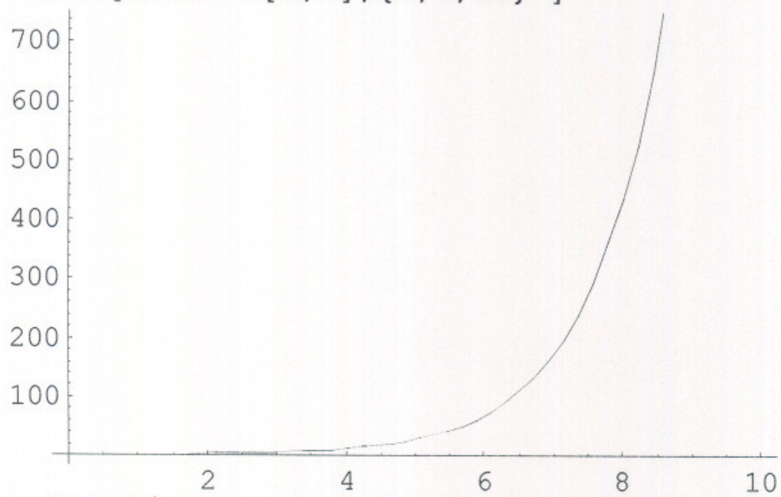

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ch 12

$\boxed{17.12}$ use section 15 recursion relations and 17.4 to obtain the following recursion relations for spherical Bessel functions:

$$j_{n-1}(x) + j_{n+1}(x) = (2n+1)\, j_n(x)/x$$

looking at 15.3 which says

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x).$$

and using (17.4): $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{\frac{2n+1}{2}}(x)$

so from 17.4, find J_{p-1} and J_{p+1} and J_p and sub into 17.4.

so from 17.4, $J_{\frac{2n+1}{2}} = \sqrt{\frac{2x}{\pi}}\, j_n(x)$

or $J_{n+\frac{1}{2}} = \sqrt{\frac{2x}{\pi}}\, j_n$

let $n+\frac{1}{2} = m$.

so $\boxed{J_m = \sqrt{\frac{2x}{\pi}}\, j_{m-\frac{1}{2}}}$

and $J_{m-1} = \sqrt{\frac{2x}{\pi}}\, j_{m-\frac{3}{2}}$

and $J_{m+1} = \sqrt{\frac{2x}{\pi}}\, j_{m+\frac{1}{2}}$

plug into 15.3 $\Rightarrow$

$$\boxed{\sqrt{\frac{2x}{\pi}}\, j_{m-\frac{3}{2}} + \sqrt{\frac{2x}{\pi}}\, j_{m+\frac{1}{2}} = \frac{2\left(m-\frac{1}{2}\right)}{x}\sqrt{\frac{2x}{\pi}}\, j_{m-\frac{1}{2}}}$$

now we want to go back to 'n'. so let $\boxed{m-\frac{3}{2} = n-1} \Rightarrow m+\frac{1}{2} = n+1$

$\Rightarrow m-\frac{1}{2} = n$

$$\Rightarrow \sqrt{\frac{2x}{\pi}}\, j_{n-1} + \sqrt{\frac{2x}{\pi}}\, j_{n+1} = \frac{2}{x}(n-1+1)\sqrt{\frac{2x}{\pi}}\, j_n$$

or $\boxed{j_{n-1} + j_{n+1} = \frac{2n}{x} j_n}$ QED.

Ch 12

$\boxed{19.1}$ Prove equation 19.10 in following way. First note that 19.2 and 19.3 and therfor 19.7 hold whether a and b are zeros of $J_p(x)$ or not. Let 'a' be zero but let 'b' be just any number. from 19.7 shows that

$$\int_0^1 xuv\, dx = \frac{J_p(b)\, a\, J_p'(a)}{b^2-a^2}$$

now let $b \to a$ and evaluate using LHopital rule. here a find

$$\int_0^1 xuv\, dx = \frac{1}{2} J_p'^2(a) \quad \text{for } a=b.$$

Solution

starting From equation 19.7 in book (page 523)

$$(vxu' - uxv')\Big|_0^1 + (a^2-b^2)\int_0^1 xuv\, dx = 0$$

here
$$\boxed{\begin{aligned} u &= J_p(ax) \\ v &= J_p(bx) \end{aligned}}$$

we are given that 'a' is a zero of $J_p(x)$ but 'b' is not.

then the above becomes

$$J_p(bx)\, x\, J_p'(ax) - J_p(ax)\, x\, J_p'(bx)\Big|_0^1 + (a^2-b^2)\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = 0$$

$$\left[J_p(b)(1)(a)J_p'(a) - \underbrace{\cancel{J_p(a)}}_{=0 \text{ since } a \text{ is zero of } J_p}\times 1\times b\times J_p'(b)\right] - [0] + (a^2-b^2)\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = 0$$

so
$$\boxed{\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = \frac{J_p(b)\, a\, J_p'(a)}{b^2-a^2}}$$
follows from above by rearransing

for $b=a$, $J_p(bx) = J_p(ax)$, so LHS in above becomes

$$\int_0^1 x\, J_p^2(ax)\, dx,$$ For RHS, use L Hopital rule $\longrightarrow$

$$\lim_{b \to a} \frac{J_p(b)\, a\, J_p'(a)}{b^2 - a^2} = \lim_{b \to a} \frac{\frac{d}{db}\left(J_p(b)\, a\, J_p'(a) \right)}{\frac{d}{db}\left(b^2 - a^2 \right)}$$

$$= \lim_{b \to a} \frac{J_p'(b)\, a\, J_p'(a)}{2b} = \frac{J_p'(a)\, a\, J_p'(a)}{2a} = \frac{1}{2}\left(J_p'(a)\right)^2$$

so $$\boxed{\int_0^1 x\, J_p^2(ax)\, dx = \frac{1}{2}\left(J_p'(a) \right)^2}$$

now I need to show that LHS also results in $\frac{1}{2} J_{p-1}^2(a)$.

From 15.5

$$J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x)$$

so $J_p'(a) = -\frac{p}{a} J_p(a) + J_{p-1}(a)$, but 'a' is a zero of $J_p(x)$, so $J_p(a) = 0$

so $\left(J_p'(a)\right)^2 = \left(J_{p-1}(a)\right)^2$

i.e $\boxed{\frac{1}{2}\left(J_p'(a)\right)^2 = \frac{1}{2} J_{p-1}^2(a)}$ which is what is required to show.

now need to prove last form, the $\frac{1}{2} J_{p+1}^2(a)$.

From 15.5, $J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$.

so $$J_p'(a) = \frac{p}{a} \underbrace{\cancel{J_p(a)}}_{=0} - J_{p+1}(a)$$

so $$J_p'(a) = -J_{p+1}(a)$$

$$\left(J_p'(a)\right)^2 = J_{p+1}^2(a)$$

so $$\boxed{\frac{1}{2}\left(J_p'(a)\right)^2 = \frac{1}{2} J_{p+1}^2(a)}$$

QED

Ch 12

$\boxed{19.2}$ given $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$

use 19.10 to evaluate $\int_0^1 \left(\frac{\sin ax}{ax} - \cos ax\right)^2 dx$

assume β is a root of J_p . i.e $J_p(\beta) = 0$.

hence now I can use 19.10 (which can only be used if β is root).

i.e I can write $\boxed{\int_0^1 x\, J_p(\beta x)\, J_p(\beta x)\, dx = \frac{1}{2} J_p'^2(\beta)}$

now use the fact that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$, Then above becomes (for $p = 3/2$)

$$\int_0^1 x\left[\sqrt{\frac{2}{\pi \beta x}}\left(\frac{\sin \beta x}{\beta x} - \cos \beta x\right)\right]^2 dx = \frac{1}{2} J_{3/2}'^2(\beta)$$

$$\frac{2}{\beta\pi}\int_0^1 \left(\frac{\sin \beta x}{\beta x} - \cos \beta x\right)^2 dx = \frac{1}{2} J_{3/2}'^2(\beta)$$

$$\boxed{\int_0^1 \left(\frac{\sin \beta x}{\beta x} - \cos \beta x\right)^2 dx = \frac{\beta\pi}{4} J_{3/2}'^2(\beta)}$$

where β is zero of $J_{3/2}(x)$.

befor I use the fact from 'a' being root of $\tan x = x$, let me Calculate RHS of above eqration

$\rightarrow$

$$J'_{3/2}(\beta) = \frac{d}{dx}\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\Bigg|_{x=\beta}$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{1}{\sqrt{x}}\left[\frac{1}{x}\cos x + \sin x\left(-\frac{1}{x^2}\right) - (-\sin x)\right]\right) + \left[\left(\frac{\sin x}{x} - \cos x\right)\left(-\frac{1}{2}\frac{1}{x^{3/2}}\right)\right]\right]$$

$$J'_{3/2}(\beta) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{x}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2} + \sin x\right) + \left(\frac{\sin x}{x} - \cos x\right)\left(\frac{1}{-2x^{3/2}}\right)\right]\Bigg|_{x=\beta}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\cos(\beta)}{\beta^{3/2}} - \frac{\sin\beta}{\beta^{5/2}} + \frac{\sin\beta}{\beta^{1/2}} - \frac{\sin\beta}{2\beta^{5/2}} + \frac{\cos\beta}{2\beta^{3/2}}\right]$$

$$J'_{3/2}\beta = \sqrt{\frac{2}{\pi}}\left[\frac{3}{2}\frac{\cos\beta}{\beta^{3/2}} - \frac{3}{2}\frac{\sin\beta}{\beta^{5/2}} + \frac{\sin\beta}{\beta^{1/2}}\right]$$

$$J'^{2}_{3/2}(\beta) = \frac{2}{\pi}\left[\frac{3}{2}\frac{\cos\beta}{\beta^{3/2}} - \frac{3}{2}\frac{\sin\beta}{\beta^{5/2}} + \frac{\sin\beta}{\beta^{1/2}}\right]^2$$

$$= \frac{2}{\pi}\left[\frac{9}{4}\frac{\cos^2\beta}{\beta^3} + \left(\frac{\sin\beta}{\beta^{1/2}} - \frac{3}{2}\frac{\sin\beta}{\beta^{5/2}}\right)^2 + 2\,\frac{3}{2}\frac{\cos\beta}{\beta^{3/2}}\left(\frac{\sin\beta}{\beta^{1/2}} - \frac{3}{2}\frac{\sin\beta}{\beta^{5/2}}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{9}{4}\frac{\cos^2\beta}{\beta^3} + \left(\frac{\sin^2\beta}{\beta} + \frac{9}{4}\frac{\sin^2\beta}{\beta^5} - 3\frac{\sin\beta}{\beta^{1/2}}\frac{\sin\beta}{\beta^{5/2}}\right) + 3\frac{\cos\beta\sin\beta}{\beta^{3/2}\beta^{1/2}} - \frac{9}{2}\frac{\cos\beta\sin\beta}{\beta^{3/2}\beta^{5/2}}\right]$$

so

$$J'^{2}_{3/2}\beta = \frac{2}{\pi}\left[\frac{9}{4}\frac{\cos^2\beta}{\beta^3} + \frac{\sin^2\beta}{\beta} + \frac{9}{4}\frac{\sin^2\beta}{\beta^5} - 3\frac{\sin^2\beta}{\beta^3} + \frac{3\cos\beta\sin\beta}{\beta^2} - \frac{9}{2}\frac{\cos\beta\sin\beta}{\beta^4}\right]$$

So $$\int_0^1\left(\frac{\sin\beta x}{\beta x} - \cos\beta x\right)^2 dx = \frac{\beta\pi}{4}\left[\quad\right] \qquad (1)$$

$$\boxed{\int_0^1\left(\frac{\sin\beta x}{\beta x} - \cos\beta x\right)^2 dx = \frac{1}{2}\left[\frac{9}{4}\frac{\cos^2\beta}{\beta^2} + \sin^2\beta + \frac{9}{4}\frac{\sin^2\beta}{\beta^4} - 3\frac{\sin^2\beta}{\beta^2} + \frac{3\cos\beta\sin\beta}{\beta} - \frac{9}{2}\frac{\cos\beta\sin\beta}{\beta^3}\right]}$$

the above is valid for β root of J_p. now I need to find how to replace β by 'a' $\longrightarrow$

since 'a' is root of $\tan x = x$, then

$$\frac{\sin x}{\cos x} = x \quad , \text{ or } \frac{\sin x}{x} = \cos x .$$

when $x=a$

From $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

replace x by a $\Rightarrow$ $J_{3/2}(a) = \sqrt{\frac{2}{\pi a}} \left(\frac{\sin a}{a} - \cos a \right)$

$$= \sqrt{\frac{2}{\pi a}} \left(\cos a - \cos a \right)$$

$$= 0$$

good. This means 'a' is a zero of $J_{3/2}(x)$. hence I am allowed now to use equation 19.10. so replace β by 'a' in eqrotion 1 will be valid now $\Rightarrow$

$$\int_0^1 \left(\frac{\sin ax}{ax} - \cos ax \right)^2 dx = \frac{1}{2} \left[\frac{9}{4} \frac{\cos^2 a}{a^2} + \sin^2 a + \frac{9}{4} \frac{\sin^2 a}{a^4} - 3 \frac{\sin^2 a}{a^2} + 3 \frac{\cos a \sin a}{a} - \frac{9}{2} \frac{\cos a \sin a}{a^3} \right]$$

now I can simplify RHS more using $\frac{\sin a}{\cos a} = a$ or $\boxed{\sin a = a \cos a}$

so $$I = \frac{9}{8} \frac{\cos^2 a}{a^2} + \frac{1}{2} a^2 \cos^2 a + \frac{9}{8} \frac{\cos^2 a \, a^2}{a^4} - \frac{3}{2} \frac{a^2 \cos^2 a}{a^2} + \frac{3}{2} \frac{\cos a \; a \cos a}{a} - \frac{9}{4} \frac{\cos a \quad a \cos a}{a^3}$$

$$= \frac{9}{8} \frac{\cos^2 a}{a^2} + \frac{1}{2} a^2 \cos^2 a + \frac{9}{8} \frac{\cos^2 a}{a^2} - \frac{3}{2} \cos^2 a + \frac{3}{2} \cos^2 a - \frac{9}{4} \frac{\cos^2 a}{a^2}$$

$$= \cos^2 a \left[\frac{9}{8a^2} + \frac{a^2}{2} + \frac{9}{8a^2} - \frac{3}{2} + \frac{3}{2} - \frac{9}{4a^2} \right] = \cos^2 a \left[\frac{9}{8a^2} + \frac{a^2}{2} + \frac{9}{8a^2} - \frac{9}{4a^2} \right]$$

$$= \boxed{\frac{a^2}{2} \cos^2 a}$$

Ch 12

$\boxed{19.3}$ Use 17.4 and 19.10 to write the orthogonality condition and the normalization integral for the spherical Bessel function $j_n(x)$.

from 19.10

$$\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = \begin{cases} 0 & a \neq b \\ \frac{1}{2} J_{p+1}^2(a) = \frac{1}{2} J_{p-1}^2(a) = \frac{1}{2} J_p'^2(a) & a = b \end{cases}$$

when a, b are roots of $J_p(x)$.

from (17.4) $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{\frac{(2n+1)}{2}}(x)$

so $J_{\frac{2n+1}{2}}(x) = j_n(x)\sqrt{\frac{2x}{\pi}}$

let $p = \frac{2n+1}{2} \implies \frac{2p-1}{2} = n$.

so $J_p(x) = j_{\frac{2p-1}{2}}(x)\sqrt{\frac{2x}{\pi}} \implies \boxed{J_p(ax) = j_{\frac{2p-1}{2}}(ax)\sqrt{\frac{2ax}{\pi}}}$

so 19.10 becomes

$$\int_0^1 x\, j_{\frac{2p-1}{2}}(ax)\, j_{\frac{2p-1}{2}}(bx) \sqrt{\frac{2ax}{\pi}}\sqrt{\frac{2bx}{\pi}}\, dx = \Big\{ \quad \text{see below.}$$

$$\int_0^1 \sqrt{\frac{4abx^2}{\pi^2}}\, x\, j_{\frac{2p-1}{2}}(ax)\, j_{\frac{2p-1}{2}}(bx)\, dx = \Big\{ \quad \text{see later.}$$

$$\int_0^1 \frac{2x}{\pi}\sqrt{ab}\, x\, j_{\frac{2p-1}{2}}(ax)\, j_{\frac{2p-1}{2}}(bx)\, dx = \Big\{ \quad \text{see later}$$

$$\frac{2}{\pi}\sqrt{ab} \int_0^1 x^2\, j_{\frac{2p-1}{2}}(ax)\, j_{\frac{2p-1}{2}}(bx)\, dx = \begin{cases} 0 & a \neq b \\ \frac{1}{2} j_{p}(a) = \frac{1}{2} j_{p-1}(a) = \frac{1}{2} j'^2_{\frac{2p-1}{2}}(a) & a = b \end{cases}$$

QED.

Ch 12

$\boxed{20.1}$ use table to evaluate the following limit

$$\lim_{x\to 0} \frac{J_4(x)}{J_2^{\,2}(x)}$$

this is for small x (since $x\to 0$). so use approximation for small x from table.

for small x $\quad J_p(x) = \frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p + O(x^{p+2})$

hence above limit can be written as

$$\lim_{x\to 0} \frac{\frac{1}{\Gamma(4+1)}\left(\frac{x}{2}\right)^4 + O(x^{4+2})}{\left[\frac{1}{\Gamma(2+1)}\left(\frac{x}{2}\right)^2 + O(x^{2+2})\right]^2} = \lim_{x\to 0} \frac{\frac{1}{\Gamma(5)}\left(\frac{x}{2}\right)^4 + O(x^6)}{\left[\frac{1}{\Gamma(3)}\left(\frac{x}{2}\right)^2 + O(x^4)\right]^2}$$

$$\lim_{x\to 0} \frac{\frac{1}{\Gamma(5)}\left(\frac{x}{2}\right)^4 + O(x^6)}{\frac{1}{\Gamma^2(3)}\left(\frac{x}{2}\right)^4 + O^2(x^4) + 2\,\frac{1}{\Gamma(3)}\left(\frac{x}{2}\right)^2 O(x^4)}$$

now, for $x\to 0$, $[O(x^4)]^2$ is very small, since x^8 term. the same can be said as $O(x^6)$ and $O(x^4)$. These terms can be ignored in the limit.

$$\text{so} \lim_{x\to 0} = \frac{\frac{1}{\Gamma(5)}\left(\frac{x}{2}\right)^4}{\frac{1}{\Gamma^2(3)}\left(\frac{x}{2}\right)^4} = \frac{\frac{1}{\Gamma(5)}}{\frac{1}{\Gamma^2(3)}} = \frac{\Gamma^2(3)}{\Gamma(5)}$$

$$= \frac{(2!)^2}{(4!)} = \frac{2^2}{2\times3\times4} = \frac{1}{2\times3} = \boxed{\frac{1}{6}}$$

Ch 12

$\boxed{21.13}$ Solve using Frobenius method. show that conditions of Fuchs's theorem as satisfied.

$$x^2 y'' - x y' + y = 0$$

$\rightarrow y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots + a_n x^{n+s} + \cdots$

$y' = a_0 s x^{s-1} + a_1 (s+1) x^s + a_2 (s+2) x^{s+1} + \cdots + a_n (s+n) x^{(s+n-1)}$

$\rightarrow xy' = a_0 s x^s + a_1 (s+1) x^{s+1} + a_2 (s+2) x^{s+2} + \cdots + a_n (s+n) x^{(s+n)}$

$y'' = a_0 s(s-1) x^{s-2} + a_1 (s+1) s x^{s-1} + a_2 (s+2)(s+1) x^s + \cdots$

$\rightarrow x^2 y'' = a_0 s(s-1) x^s + a_1 (s+1) s x^{s+1} + a_2 (s+2)(s+1) x^{s+2} + \cdots$

set up table:

	x^s	x^{s+1}	x^{s+2}	x^{s+n}
y	a_0	a_1	a_2	a_n
$-xy'$	$-a_0 s$	$-a_1(s+1)$	$-a_2(s+2)$	$-a_n(s+n)$
x^2y''	$a_0 s(s-1)$	$a_1(s+1)s$	$a_2(s+2)(s+1)$	$a_n(s+n)(s+n-1)$

First solve indicial equation.

from first column we get

$$a_0 - a_0 s + a_0 s(s-1) = 0$$

$1 - s + s^2 - s = 0$ for $a_0 \neq 0$ by hypothesis

so $s^2 - 2s + 1 = 0 \Rightarrow s = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2 \pm \sqrt{4-4}}{2}$

$s = 1 \pm 0 = 1$

observe $\underline{\text{one}}$ solution only since s has one value.

$\longrightarrow$

for $s=1$, Find recursive formula

$$a_n - a_n(s+n) + a_n(s+n)(s+n-1) = 0$$

i.e $\quad a_n - a_n(1+n) + a_n(1+n)(n) = 0$

so $\quad a_n(1-(1+n)+n+n^2) = 0$

$a_n(1-1-n+n+n^2) = 0$

$a_n(n^2) = 0 \implies a_n = 0$ for $n > 0$.

so solution is only

$y = a_0 x^1$

$\boxed{y_1 = a_0 x}$ this is the first solution.

now need to show that condition of fuchs's are met.

our equation are written as $y'' + f(x)y' + g(x)y = 0$

so $\left.\begin{array}{l} f(x) = -\frac{1}{x} \\ g(x) = \frac{1}{x^2} \end{array}\right\}$ for this problem. $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$

$\left.\begin{array}{lcl} x f(x) & = & -1 \\ x^2 g(x) & = & 1 \end{array}\right\}$ these are expandable as power series

as $-1 = a_0 x^0$, $a_0 = -1$

as $1 = a_0 x^0$, $a_0 = 1$

hence fuchs's condition are met. this means second

solution $\boxed{y_2(x) = \quad y_1(x)\ln(x) + \text{ another Frobenius Series}}$

so $\boxed{y_2(x) = a_0 x \ln(x) + \text{another Frobenius series}}$

now I find the second solution $\longrightarrow$

The second solution series can be found by

writing $y = a_0 x \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+s}$ and sub

this into the DE given to solve for b_n terms. However, a

faster method is to use the substitution

$$y_2(x) = u(x)\, v(x)$$

where $u(x)$ is the first solution.

so let $y_2(x) = \underbrace{a_0 x}_{\substack{\text{first} \\ \text{solution.}}} v(x)$

from $x^2 y'' - xy' + y = 0$ ——— ①

$$y' = a_0 x\, v'(x) + a_0 v(x)$$

$$y'' = a_0 x\, v''(x) + v'(x)\, a_0 + a_0 v'(x)$$

so ① becomes

$$x^2 \left[a_0 x\, v''(x) + v'(x)\, a_0 + a_0 v'(x) \right] - x \left[a_0 x\, v'(x) + a_0 v(x) \right] + a_0 x\, v(x) = 0$$

$$a_0 x^3 v'' + a_0 x^2 v' + a_0 x^2 v' - a_0 x^2 v' - a_0 x v + a_0 x v = 0$$

$$a_0 x^3 v'' + a_0 x^2 v' = 0$$

$$x v'' + v' = 0$$

i.e. $x \dfrac{d^2 v(x)}{dx^2} + v(x) = 0$. This is separable equation whose solution as given in book page 529 $\boxed{v(x) = A + B \ln(x)}$

So second solution is $y_2 = u(x)\, v(x)$

$$y_2 = a_0 x \left[A + B \ln(x) \right] = A_1 x + A_2 x \ln(x)$$

So general solution $y = y_1 + y_2 = \overbrace{a_0 x}^{y_1} + \overbrace{A_1 x + A_2 x \ln(x)}^{y_2}$

$$\boxed{= K_1 x + A_2 x \ln(x)}$$

ch 12

$\boxed{21.15}$ solve $xy'' + xy' - 2y = 0$

3/5

let $y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots$

$y' = a_0 s x^{s-1} + a_1 (s+1) x^s + a_2 (s+2) x^{s+1} + \cdots$

$\rightarrow xy' = a_0 s x^s + a_1 (s+1) x^{s+1} + a_2 (s+2) x^{s+2} + \cdots$

$y'' = a_0 s(s-1) x^{s-2} + a_1 (s+1) s x^{s-1} + a_2 (s+2)(s+1) x^s + \cdots$

$\rightarrow xy'' = a_0 s(s-1) x^{s-1} + a_1 (s+1) s x^s + a_2 (s+2)(s+1) x^{s+1}$

$\rightarrow 2y = 2a_0 x^s + 2a_1 x^{s+1} + 2a_2 x^{s+2} + \cdots$

Set up Table

	x^{s-1}	x^s	x^{s+1}	x^{s+n}
xy''	$a_0 s(s-1)$	$a_1 (s+1) s$	$a_2 (s+2)(s+1)$	$a_{n+1} (s+n+1)(s+n)$
xy'		$a_0 s$	$a_1 (s+1)$	$a_n (s+n)$
$-2y$		$-2a_0$	$-2a_1$	$-2a_n$

so from first Column, $a_0 s(s-1) = 0$

i.e $s^2 - s = 0$ for $a_0 \neq 0$

so $s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-0}}{2} = \frac{1 \pm 1}{2}$

$= \frac{1+1}{2}$ or 0 i.e $\boxed{s = 1 \text{ or } 0}$

since the two values of s differ by an iteger, then the larger value of s gives the one solution.

so use $\boxed{s = 1}$ $\longrightarrow$

The recursive formula is

$$a_{n+1}(s+n+1)(s+n) + a_n(s+n) - 2a_n = 0$$

$$a_{n+1} = \frac{2a_n - a_n(s+n)}{(s+n+1)(s+n)} = a_n\frac{(2-(s+n))}{(s+n+1)(s+n)}.$$

but $s=1$. so

$$a_{n+1} = \frac{a_n(1-n)}{(2+n)(1+n)}.$$

For $n=0$, $\quad a_1 = \frac{a_0}{2}$

for $n=1$, $\quad a_2 = \frac{a_1(0)}{3\times 2} = 0$

for $n=2$, $\quad a_3 = \frac{a_2(1-2)}{4\times 3} = 0 \quad$ since $a_2=0$.

so all a_n, $n>1$ are zero.

so first solution is $\quad y_1 = a_0 x + \frac{a_0}{2}x^2$

$$\boxed{y_1 = a_0\left(x + \frac{x^2}{2}\right)}$$

now need to show that condition of Fuchs's are met.

write DE as $y'' + f(x)y' + g(x)y = 0$

so $y'' + y' - \frac{2}{x}y = 0$.

i.e $\boxed{\begin{array}{l} f(x) = 1 \\ g(x) = -\frac{2}{x} \end{array}}$

i.e $\underline{xf(x)}$ and $\underline{x^2g(x)}$ both are expandable in power series

this means second solution is

$y_2(x) = y_1(x)\ln(x) +$ another Frobenius series

$\longrightarrow$

now to find the second solution.

let $\quad y_2(x) = u(x)\ v(x)$ $\quad$ where $u(x) = y_1(x)$.

so $\quad y_2(x) = a_0\left(x + \frac{x^2}{2}\right) v(x)$.

Why are you abandoning the form $Y_2 = Y_1 \cdot \ln x + \Sigma b_n x^n$?

sub into the DE $\Rightarrow$

$$x y_2''(x) + x y_2'(x) - 2 y_2(x) = 0$$

$$y_2'(x) = (a_0 + a_0 x) v(x) + a_0\left(x + \frac{x^2}{2}\right) v'(x)$$

$$y_2''(x) = (a_0 + a_0 x) v' + v(a_0) + a_0\left(x + \frac{x^2}{2}\right) v'' + v'(a_0 + a_0 x)$$

$$x y_2'(x) = (a_0 x + a_0 x^2) v + a_0\left(x^2 + \frac{x^3}{2}\right) v'$$

$$x y_2''(x) = (a_0 x + a_0 x^2) v' + a_0 x v + a_0\left(x^2 + \frac{x^3}{2}\right) v'' + v'(a_0 x + a_0 x^2)$$

DE is

$$a_0\left(x^2 + \frac{x^3}{2}\right) v'' + 2(a_0 x + a_0 x^2) v' + a_0 x v + a_0\left(x^2 + \frac{x^3}{2}\right) v' + (a_0 x + a_0 x^2) v$$
$$- 2 a_0\left(x + \frac{x^2}{2}\right) v = 0$$

$$v''\left(a_0 x^2 + a_0 \frac{x^3}{2}\right) + v'\left(2a_0 x + 2a_0 x^2 + a_0 x^2 + \frac{a_0 x^3}{2}\right) + v\left(a_0 x + a_0 x + a_0 x^2 - 2a_0 x - a_0 x^2\right) = 0$$

$$v''\left(a_0 x^2 + a_0 \frac{x^3}{2}\right) + v'\left(2a_0 x + 3a_0 x^2 + \frac{1}{2} a_0 x^3\right) + v(0) = 0.$$

$$\boxed{v''\left(a_0 x + a_0 \frac{x^2}{2}\right) + v'\left(2a_0 + 3a_0 x + \frac{1}{2} a_0 x^2\right) = 0}$$

or $\boxed{v''\left(x + \frac{x^2}{2}\right) + v'\left(2 + 3x + \frac{1}{2} x^2\right) = 0}$ $\quad$ since $a_0 \neq 0$.

Solve for $v(x)$

$\longrightarrow$

$$v'' + v'\left(\frac{2+3x+\frac{1}{2}x^2}{x+\frac{x^2}{2}}\right) = 0$$

$$v'' = -v'\left(\frac{2+3x+\frac{1}{2}x^2}{x+\frac{x^2}{2}}\right) = -v'\left(\frac{4+6x+x^2}{2x+x^2}\right)$$

ie $$\frac{d^2v}{dx^2} = -\left(\frac{4+6x+x^2}{2x+x^2}\right)\frac{dv}{dx}$$

let $\frac{dv}{dx} = Y(x)$

so $$\frac{dY}{dx} = -\left(\frac{4+6x+x^2}{2x+x^2}\right)Y$$

so $$\frac{dY}{Y} = -\left(\frac{4+6x+x^2}{2x+x^2}\right)dx$$

so $$\ln Y = -\int \frac{4+6x+x^2}{2x+x^2}\,dx = -\int \frac{4+6x+x^2}{x(x+2)}\,dx$$

do partial fraction

$$\frac{A}{x} + \frac{Bx+C}{(x+2)} = \frac{A(x+2)+Bx^2+Cx}{x(x+2)} = \frac{4+6x+x^2}{x(x+2)}$$

$$Ax+2A+Bx^2+Cx = 4+6x+x^2$$

so
$$\left.\begin{aligned} A+C&=6\\ 2A&=4\\ B&=1\end{aligned}\right\}\quad A=2 \Rightarrow C=6-A=4.$$

so $$-\int \frac{2}{x} + \frac{x+4}{x+2}\,dx = \ln Y$$

integration constant

$$A\ln Y = -\left[2\ln(x) + x + 2\ln(x+2)\right]$$

so $$Y = B\exp(2\ln(x)+x+2\ln(x+2)]$$

i.e. $$\frac{dv}{dx} = B\exp(2\ln(x)+x+2\ln(x+2)]$$

$\longrightarrow$

so $v = B\int \exp(2\ln(x) + x + 2\ln(x+2))\,dx$

$$v(x) = B\int e^{(2\ln(x))}\, e^{x}\, e^{2\ln(x+2)}\,dx$$

$$= B\int e^{x}\,(e^{\ln x})^{2}\,(e^{\ln(x+2)})^{2}\,dx$$

$$= B\int e^{x}\,x^{2}\,(x+2)^{2}\,dx$$

$$= B\int e^{x}\,x^{2}(x^{2}+4x+4)\,dx$$

$$= B\int e^{x}(x^{4}+4x^{3}+4x^{2})\,dx$$

$$= B\left[\int x^{4}e^{x} + 4\int e^{x}x^{3} + 4\int e^{x}x^{2}\right]$$

$$\boxed{v(x) = B e^{x}(8-8x+4x^{2}+x^{4}) + C}$$

C: integration constant.

so second solution is

$$y_2 = u(x)\,v(x)$$

$$y_2 = a_0\left(x+\frac{x^2}{2}\right)\left(B e^{x}(8-8x+4x^{2}+x^{4})+C\right)$$

so general solution is

$$y = y_1(x) + y_2(x)$$

$$= a_0\left(x+\frac{x^2}{2}\right) + a_0\left(x+\frac{x^2}{2}\right)\left(B e^{x}(8-8x+4x^{2}+x^{4})+C\right)$$

$$\boxed{y = K_1\left(x+\frac{x^2}{2}\right) + K_2\left(x+\frac{x^2}{2}\right)e^{x}(8-8x+4x^{2}+x^{4})}$$

K_1, K_2 arbitrary constants.

→

ch 12

$\boxed{21.18}$ solve $x^2y'' - 3xy' + 4y = 0$, $u = x^2$

here first solution is given.

so first look at Fuchs's Conditions.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0.$$

$$\left.\begin{array}{l} f(x) = -\frac{3}{x} \Rightarrow xf(x) = -3 \\ g(x) = \frac{4}{x^2} \Rightarrow x^2g(x) = 4 \end{array}\right\}$$ regular or has nonessential singularity at origion.

let $y_2 = u(x)\,v(x) = x^2 v(x)$

so $y_2' = x^2 v' + 2xv$

$y_2'' = x^2v'' + 2xv' + 2xv' + 2v$

$x^2y_2'' = x^4v'' + 4x^3v' + 2x^2v$

$-3xy_2' = -3x^3v' - 6x^2v$

so DE becomes

$$(x^4v'' + 4x^3v' + 2x^2v) - 3x^3v' - 6x^2v + 4x^2v = 0$$

$$v''(x^4) + v'(4x^3 - 3x^3) + v(2x^2 - 6x^2 + 4x^2) = 0$$

$$\boxed{x^4v'' + x^3v' = 0}$$

$$v'' = -\frac{x^3}{x^4}v'$$

$$v'' = -\frac{1}{x}v'$$

$$\frac{d^2v}{dx^2} = -\frac{1}{x}\frac{dv}{dx}$$. let $\frac{dv}{dx} = Y(x)$

$$\frac{dY}{dx} = -\frac{1}{x}Y \Rightarrow \frac{dY}{Y} = -\frac{1}{x}dx \longrightarrow$$

$\ln Y(x) = -\ln(x) + C$

so $e^{-\ln(x)+C} = Y(x)$.

i.e. $\frac{dv}{dx} = A e^{-\ln(x)} = A \frac{1}{x}$

so $dv = A \frac{dx}{x}$

so $\boxed{v(x) = A \ln(x) + B}$

So second solution $y_2 = x^2 v(x)$

$y_2 = x^2 (A \ln(x) + B)$

So general solution

$y = y_1 + y_2$

$= x^2 + x^2 (A \ln(x) + B)$

$= x^2 + A x^2 \ln(x) + B x^2$

$\boxed{y(x) = C x^2 + A x^2 \ln(x)}$