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HW # 4

Math 121 B

NASSER ABBASI

UCB extension

ch 12.

$\boxed{6.4}$ show that functions $f(x)$ and $g(x)$ are orthogonal on $(-a, a)$ if $f(x)$ is even and $g(x)$ is odd.

$$\text{inner product} = \int_{-a}^{a} f(x)\, g(x)\, dx$$

$$= \int_{-a}^{a} \text{even function} \times \text{odd function} \; dx$$

$$= \int_{-a}^{a} \text{odd function} \; dx.$$

$$\text{but} \int_{-a}^{a} \text{odd function}\, dx = 0$$

$$\text{hence} \int_{-a}^{a} f(x)\, g(x)\, dx = 0$$

hence $f(x), g(x)$ are orthogonal to each other over $(-a, a)$.

$\boxed{6.5}$ evaluate $\int_{-1}^{1} P_0(x)\, P_2(x)\, dx$ to show they are orthogonal.

$P_0(x) = 1$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\Rightarrow \int_{-1}^{1} 1 \cdot \tfrac{1}{2}(3x^2-1)\, dx = \frac{1}{2}\left[\left(\frac{3x^3}{3}\right)_{-1}^{1} - (x)_{-1}^{1} \right]$$

$$= \frac{1}{2}\left[(1^3 - (-1)^3) - (1 - (-1)) \right]$$

$$= \frac{1}{2}\left[(1 - (-1)) - (1+1) \right]$$

$$= \frac{1}{2}\left[2 - 2 \right] = 0$$

hence $P_0(x), P_2(x)$ are orthogonal over $(-1, 1)$

$\boxed{6.6}$ show in two ways that $P_\ell(x)$ and $P'_\ell(x)$ are orthogonal on $(-1,1)$.

first way:

$P_\ell(x)$ is either odd or even function. if $\ell=1,3,5,\ldots$ i.e. ℓ is odd, then $P_\ell(x)$ is odd. if $\ell=2,4,6,\ldots$ i.e. even, then $P_\ell(x)$ is even.

now if $P_\ell(x)$ is even, then $P'_\ell(x)$ is odd. this is because $P_\ell(x)$ is a polynomial of terms in x with exponents that are all either even (if $P_\ell(x)$ is even) or odd (if $P_\ell(x)$ is odd). similarly if $P_\ell(x)$ is odd then $P'_\ell(x)$ is even.

$$\text{hence } \int_{-1}^{1} P_\ell(x)\, P'_\ell(x)\, dx \quad \begin{cases} \int_{-1}^{1} \text{even} \times \text{odd} \\ \text{or} \\ \int_{-1}^{1} \text{odd} \times \text{even}. \end{cases}$$

in both cases we get $\int_{-1}^{1} \text{odd function } dx = 0$

hence $P_\ell(x), P'_\ell(x)$ are orthogonal over $(-1,1)$.

Second way: using relationship $\int_{-1}^{1} x^m P_\ell(x)\, dx = 0$ if $m<\ell$.

now degree of $P'_\ell(x)$ is one less than degree of $P_\ell(x)$, since derivative lowers degree (order) of polynomial.

now I integrate $\int_{-1}^{1} P'_\ell(x)\, P_\ell(x)\, dx$ term by term as this:

$$\int_{-1}^{1} \left(\text{first term in } P'_\ell(x) + \underset{\text{in } P'(x)}{\text{second term}} + \cdots + \underset{\text{in } P'_\ell(x)}{\text{last term}}\right) P_\ell(x)\, dx$$

now each term of $P'_\ell(x)$ is of the form ax^m. since $m<\ell$ from as mentioned above, we get

$$\int_{-1}^{1} (P'_\ell \ 1^{st} \text{ term}) P_\ell(x) + \int_{-1}^{1} (P'_\ell \ 2^{nd} \text{ term}) P_\ell(x) + \cdots$$

and all these integrals are of the form

$$\text{some constant} \int_{-1}^{1} x^m \, P_\ell(x) \, dx$$

where $m < \ell$. (actually largest m value is $\ell - 1$)

hence we get

$$0 + 0 + 0 \cdots \quad = 0$$

i.e $\int_{-1}^{1} P_\ell'(x) \, P_\ell(x) \, dx = 0$

hence P_ℓ' and P_ℓ are orthogonal on $(-1, 1)$.

(PS. please see problem 7.3 solution next for the same argument (written more clearly than here) which shows that $\int_{-1}^{1} P_m P_\ell = 0$ when $m < \ell$

and this is the same as saying $\int_{-1}^{1} P_\ell' P_\ell \, dx = 0$ since order of P' is less than order of P_ℓ.)

Ch 12 $\boxed{7.3}$ use problem 4.4 to show that $\int_{-1}^{1} P_m\, P_\ell\, dx = 0$ if $m<\ell$.

in 4.4, we showed that $\int_{-1}^{1} x^m P_\ell\, dx = 0$ if $m < \ell$.

write $P_m(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$.

where each $f(x)$ is ~~of~~ the form $a x^u$, where a is some constant, and u is an exponent which can be zero for f_1 term. (if P_m is even, Then $f_1(x)$ is a Constant, if P_m is odd than $f_1(x) = ax$).

so now we write $\int_{-1}^{1} P_m P_\ell\, dx = \int_{-1}^{1} (f_1+f_2+f_3+\cdots+f_m)\, P_\ell\, dx$

$$= \int_{-1}^{1} f_1(x) P_\ell(x)\, dx + \int_{-1}^{1} f_2(x) P_\ell(x)\, dx + \cdots + \int_{-1}^{1} f_m(x)\, P_\ell(x)\, dx.$$

$$= a_1 \int_{-1}^{1} x^{n_1} P_\ell(x)\, dx + a_2 \int_{-1}^{1} x^{n_2} P_\ell(x)\, dx + \cdots + a_m \int_{-1}^{1} x^{n_m} P_\ell(x)\, dx.$$

where we have each exponent $n_i < \ell$ as given.

hence by applying 4.4, we get 0 for each integral.

if the first term of $P_m(x)$ was a constant (i.e $P_m(x)$ was even), Then we get $\int_{-1}^{1} P_\ell(x)\, dx$ for the first integral, which is also zero. (see proof of this last part in problem 7.5 next)

hence $\int_{-1}^{1} P_m\, P_\ell\, dx = 0$ if $m < \ell$.

Ch 12

$\boxed{7.5}$ show that $\int_{-1}^{1} P_\ell(x)\, dx = 0 \qquad \ell > 0$

This can be show immediatly by applying result of problem 4.4 which said $\int_{-1}^{1} x^m P_\ell \, dx = 0$ when $m < \ell$.

let $m = 0$ here and the result follows.

another proof is to write $\int_{-1}^{1} P_\ell \, dx$ as

$\int_{-1}^{1} P_0 P_\ell \, dx$, since $P_0 = 1$.

now apply orthogonality principle for Legendre polynomials which says that $\int_{-1}^{1} P_n P_m \, dx = 0 \quad n \neq m$,

and since here $n = 0$ and $m = \ell > 0$, then $m \neq n$,

hence $\int_{-1}^{1} P_\ell \, dx = 0$.

ch 12

$\boxed{8.2}$ Find the norm of each following function on given interval and state the normalized function.

$P_2(x)$ on $(-1,1)$

$$N^2 = \int_{-1}^{1} P_2^*(x)\, P_2(x)\, dx = \frac{2}{2(2)+1} = \frac{2}{5}$$

so Norm of $P_2(x) = \boxed{\sqrt{\frac{2}{5}}}$

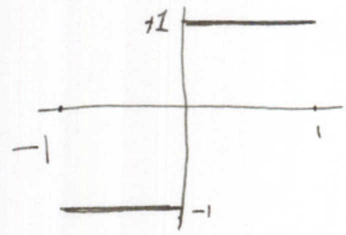
hence the normalized $P_2(x) = \dfrac{P_2(x)}{N} = \dfrac{P_2(x)}{\sqrt{\frac{2}{5}}}$

$$= \sqrt{\frac{5}{2}}\left(\frac{1}{2}(3x^2-1)\right) = \boxed{\frac{\sqrt{10}}{4}(3x^2-1)}$$

Ch 12

$\boxed{9.1}$ Expand following function in Legendre series.

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$



use 3 terms.

$$f(x) = a_0 P_0 + a_1 P_1 + a_2 P_2$$

where
$$a_i = \frac{\int_{-1}^{1} f(x) P_i(x)\, dx}{\int_{-1}^{1} P_i(x) P_i(x)\, dx} = \frac{\int_{-1}^{0_-} (-1) P_i(x)\, dx + \int_{0_+}^{1} (+1) P_i(x)\, dx}{\frac{2}{2i+1}}$$

$$a_0 = \frac{-\int_{-1}^{0_-} P_0\, dx + \int_{0_+}^{1} P_0\, dx}{\frac{2}{1}} = \frac{-\int_{-1}^{0_-} 1\, dx + \int_{0_+}^{1} 1\, dx}{2} = \frac{-[x]_{-1}^{0} + [x]_0^1}{2}$$

$$= \frac{-[+1] + [1]}{2} = \boxed{0}$$

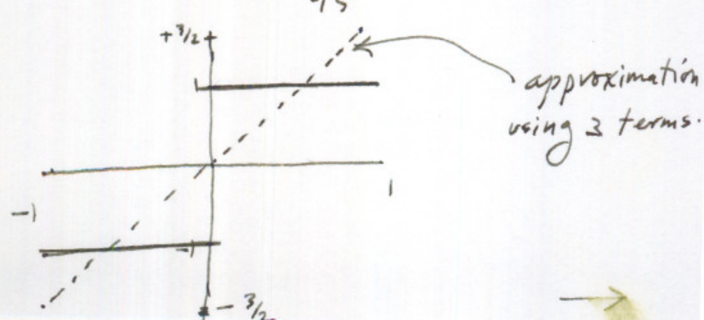
$$a_1 = \frac{-\int_{-1}^{0} P_1\, dx + \int_0^1 P_1\, dx}{\frac{2}{2+1}} = \frac{-\int_{-1}^{0} x\, dx + \int_0^1 x\, dx}{2/3} = \frac{-\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1}{2/3}$$

$$= \frac{-\frac{1}{2}[0-1] + \frac{1}{2}[1-0]}{2/3} = \frac{\frac{1}{2} + \frac{1}{2}}{2/3} = \boxed{\frac{3}{2}}$$

$$a_2 = \frac{-\int_{-1}^{0} \frac{1}{2}(3x^2-1)\, dx + \int_0^1 \frac{1}{2}(3x^2-1)\, dx}{\frac{2}{4+1}} = \frac{-\frac{1}{2}\left[\frac{3x^3}{3} - x\right]_{-1}^{0} + \frac{1}{2}\left[\frac{3x^3}{3} - x\right]_0^1}{2/5}$$

$$= \frac{-\frac{1}{2}\left[(0-0) - (-1^3 - (-1))\right] + \frac{1}{2}\left[(1-1) - (0-0)\right]}{2/5} = \frac{-\frac{1}{2}[0] + \frac{1}{2}[0]}{2/5} = \boxed{0}$$

So $f(x) \simeq \frac{3}{2} P_1 = \frac{3}{2} x$



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let me try and see if we add one more term

$$a_3 = \frac{-\int_{-1}^{0} P_3(x)\,dx + \int_0^1 P_3(x)\,dx}{\frac{2}{6+1}} = \frac{-\int_{-1}^{0}\left(-\frac{3x}{2}+\frac{5x^3}{2}\right)dx + \int_0^1\left(-\frac{3x}{2}+\frac{5x^3}{2}\right)dx}{2/7}$$

I just relized that I could do this faster by noting that

$$\int_{-1}^{0} \text{even } P_\ell = \int_0^1 \text{even } P_\ell .$$

so all even a's $= 0$
~~so I only need to integrate once over~~ $(-1,0)$ or $(0,1)$.

also $$\int_{-1}^{0} \text{odd } P_\ell = -\int_0^1 \text{odd } P_\ell .$$ so all a's that are odd need integrate only from $(-1,0)$ and multiply by 2 to get result. (sorry for not seeing this earlier!)

$$a_3 = \frac{2\int_{-1}^{0} P_3(x)\,dx}{2/7} = \frac{2\int_{-1}^{0}\left(-\frac{3x}{2}+\frac{5x^3}{2}\right)dx}{2/7}$$

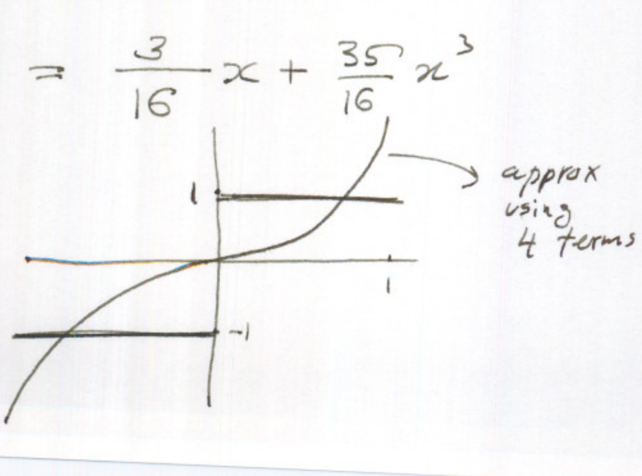
$$= \frac{2\left[\left[-\frac{3}{2}\frac{x^2}{2}+\frac{5}{2}\frac{x^4}{4}\right]_{-1}^{0}\right]}{2/7} = \frac{2\left[\left(-\frac{3}{4}(0)+\frac{5}{8}(0)\right)-\left(-\frac{3}{4}(-1)^2+\frac{5}{8}(-1)^4\right)\right]}{2/7}$$

$$= 7\left[0-\left(\frac{-3}{4}+\frac{5}{8}\right)\right] = 7\left[\frac{3}{4}-\frac{5}{8}\right] = 7\left[\frac{1}{8}\right] = 7/8$$

so $$f(x) \cong \frac{3}{2}x + 7/8\left(-\frac{3}{2}x+\frac{5x^3}{2}\right)$$

$$\cong \frac{3}{2}x - \frac{21}{16}x + \frac{35}{16}x^3 = \frac{3}{16}x + \frac{35}{16}x^3$$

$$= \frac{1}{16}\left(3x+35x^3\right)$$



ch 12 9.6

Expand by legendre series using 2 terms.

$$f(x) = \begin{cases} 0 & (-1,0) \\ \left(\ln \frac{1}{x}\right)^2 & (0,1) \end{cases}$$

$$\boxed{\tilde{f}(x) = a_0 P_0 + a_1 P_1} \qquad \text{where} \qquad \boxed{a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\frac{2}{2n+1}}}$$

since $f(x) = 0$ over $(-1,0)$, integration is only needed from $(0,1)$.

$$a_0 = \frac{\int_0^1 f(x) P_0(x)\, dx}{2} = \frac{1}{2}\int_0^1 \ln^2\left(\tfrac{1}{x}\right) \cdot 1 \; dx$$

let $\ln \frac{1}{x} = u \longrightarrow \frac{du}{dx} = -\frac{1}{x} \longrightarrow dx = -x\,du$

$e^u = \frac{1}{x} \Rightarrow x = e^{-u}$ so $dx = -e^{-u}\,du$.

when $x=0$, $u = \infty$. when $x=1$, $u=0$

$$\text{so } a_0 = \frac{1}{2}\int_\infty^0 u^2(-e^{-u})\, du = \frac{1}{2}\int_0^\infty u^2 e^{-u}\, du = \frac{1}{2}\Gamma(3) = \frac{1}{2}\, 2! = \boxed{1}$$

To find a_1

$$a_1 = \frac{\int_0^1 f(x) P_1(x)\, dx}{2/3} = \frac{3}{2}\int_0^1 \ln^2\left(\tfrac{1}{x}\right) x\, dx$$

let $x = e^{-u} \rightarrow \frac{dx}{du} = -e^{-u}$.

when $x=0 \longrightarrow u = \infty$, when $x = 1 \rightarrow u = 0$

$$\ln \frac{1}{x} = \ln \frac{1}{e^{-u}} = \ln e^u = u$$

$$\text{so } a_1 = \frac{3}{2}\int_\infty^0 u^2 e^{-u}(-e^{-u})\, du = \frac{3}{2}\int_0^\infty u^2 e^{-2u}\, du$$

but $\boxed{\int_0^\infty u^n e^{-au}\, du = \frac{n!}{a^{n+1}}}$ from equation 2.2 page 458 book.

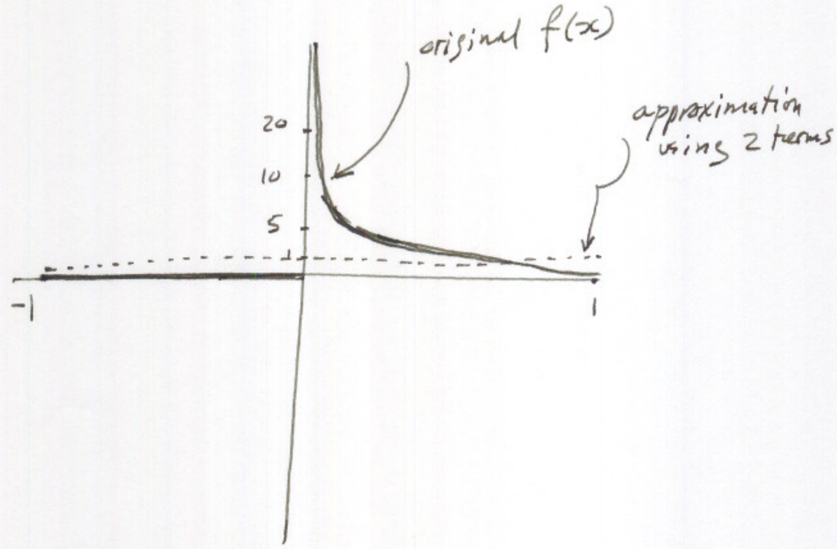
$$\text{so } a_1 = \frac{3}{2}\left(\frac{2!}{2^3}\right) = \frac{3}{2}\left(\frac{1}{4}\right) = \boxed{\frac{3}{8}}.$$

$$\text{hence } \tilde{f}(x) = a_0 P_0 + a_1 P_1 = \boxed{1 + \frac{3}{8}x}$$

$\longrightarrow$

hence $f(x) \simeq a_0 P_0 + a_1 P_1$

$$= 1 \cdot 1 + \frac{3}{8} x = 1 + \frac{3}{8} x$$



when $x = -1$ $\quad \tilde{f}(x) = 1 - \frac{3}{8} = \frac{5}{8}$

when $x = 0$ $\quad \tilde{f}(x) = 1$

when $x = 1$ $\quad \tilde{f}(x) = 1 + \frac{3}{8} = 1\frac{3}{8}$

ch 12

$\boxed{9.11}$ Expand in Legendre polynomials

$$f(x) = 7x^4 - 3x + 1$$

This is a degree 4 polynomial. hence best fit will only require using upto P_4 legendre Basis. (This is because $\int_{-1}^{1} P_\ell \,(\text{any poly degree} < \ell) = 0$ hence no need to try more than P_5 and higher)

i.e $\tilde{f}(x) = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4$

where

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\frac{2}{2n+1}}$$

$$a_0 = \frac{\int_{-1}^{1} (7x^4 - 3x + 1) \cdot 1 \, dx}{2} = \frac{1}{2}\left(\left[\frac{7}{5}x^5\right]_{-1}^{1} - 3\left[\frac{x^2}{2}\right]_{-1}^{1} + \left[x\right]_{-1}^{1} \right)$$

$$= \frac{1}{2}\left(\frac{7}{5}\left(1-(-1)^5\right) - \frac{3}{2}\left(1-(-1)^2\right) + 2\right) = \frac{1}{2}\left(\frac{7}{5}(2) - \frac{3}{2}(0) + 2\right) = \boxed{\frac{12}{5}}$$

$$a_1 = \frac{\int_{-1}^{1} (7x^4 - 3x + 1)\, x}{\frac{2}{3}} = \frac{3}{2}\left[7\left(\frac{x^6}{6}\right)_{-1}^{1} - 3\left(\frac{x^3}{3}\right)_{-1}^{1} + \left(\frac{x^2}{2}\right)_{-1}^{1} \right]$$

$$= \frac{3}{2}\left[\frac{7}{6}\left(1-(-1)^6\right) - \left(1-(-1)^3\right) + \frac{1}{2}\left(1-(-1)^2\right)\right] = \frac{3}{2}\left[\frac{7}{6}(0) - (2) + \frac{1}{2}(0)\right]$$

$$= \boxed{-3}$$

$$a_2 = \frac{\int_{-1}^{1} (7x^4 - 3x + 1) \cdot \frac{1}{2}(3x^2 - 1)}{\frac{2}{5}} = \frac{5}{2}\frac{1}{2}\int_{-1}^{1} 21x^6 - 7x^4 - 9x^3 + 3x + 3x^2 - 1 \, dx$$

$$= \frac{5}{4}\left(21\left(\frac{x^7}{7}\right)_{-1}^{1} - 7\left(\frac{x^5}{5}\right)_{-1}^{1} - 9\left(\frac{x^4}{4}\right)_{-1}^{1} + 3\left(\frac{x^2}{2}\right)_{-1}^{1} + 3\left(\frac{x^3}{3}\right)_{-1}^{1} - \left(x\right)_{-1}^{1} \right)$$

$$= \frac{5}{4}\left(\frac{21}{7}\left(1-(-1)^7\right) - \frac{7}{5}\left(1-(-1)^5\right) - \frac{9}{4}\left(1-(-1)^4\right) + \frac{3}{2}\left(1-(-1)^2\right) + \left(1-(-1)^3\right) - \left(1-(-1)\right)\right)$$

$$= \frac{5}{4}\left(\frac{21}{7}(2) - \frac{7}{5}(2) - \frac{9}{4}(0) + \frac{3}{2}(0) + (2) - (2)\right) = \frac{5}{4}\left(\frac{42}{7} - \frac{14}{5}\right) = \frac{5}{4}\left(\frac{210}{35} - \frac{98}{35}\right)$$

$$= \frac{5}{4}\left(\frac{112}{35}\right) = \boxed{\cancel{4.45}}\ \boxed{4}$$

→

$$a_3 = \frac{\int_{-1}^{1} (7x^4-3x+1)\left(-\frac{3}{2}x + \frac{5}{2}x^3\right)\,dx}{\frac{2}{7}}$$

$$= \frac{7}{2}\frac{1}{2}\int_{-1}^{1} -21x^5 + 35x^7 + 9x^2 - 15x^4 - 3x + 5x^3 \quad dx$$

$$= \frac{7}{4}\left(-2\left(\frac{x^6}{6}\right)\Big|_{-1}^{1} + 35\left(\frac{x^8}{8}\right)\Big|_{-1}^{1} + 9\left(\frac{x^3}{3}\right)\Big|_{-1}^{1} - 15\left(\frac{x^5}{5}\right)\Big|_{-1}^{1} - 3\left(\frac{x^2}{2}\right)\Big|_{-1}^{1} + 5\left(\frac{x^4}{4}\right)\Big|_{-1}^{1}\right)$$

$$= \frac{7}{4}\left(-\frac{1}{3}(1-(-1)^6) + \frac{35}{8}(1-(-1)^8) + 3(\cancel{1}-(-1)^3) - 3(1-(-1)^5) - \frac{3}{2}(1-(-1)^2) + \frac{5}{4}(0)\right)$$

$$= \frac{7}{4}\left(-\frac{1}{3}(0) + \frac{35}{8}(0) + 3(2) - 3(2) - \frac{3}{2}(0) + \frac{5}{4}(0)\right) = \frac{7}{4}(6-6) = \boxed{0}$$

$$a_4 = \frac{\int_{-1}^{1} (7x^4-3x+1)\left(\frac{1}{8}(3-30x^2+35x^4)\right)dx}{\frac{2}{9}}$$

$$= \frac{9}{2}\frac{1}{8}\int_{-1}^{1} 21x^4 - 210x^6 + 245x^8 - 9x + 90x^3 - 95x^5 + 3 - 30x^2 + 35x^4 \quad dx$$

$$= \frac{9}{16}\left[21\left(\frac{2}{5}\right) - 210\left(\frac{2}{7}\right) + 245\left(\frac{2}{9}\right) - 9(0) + 90(0) - 95(0) + 3(2) - 30\left(\frac{2}{3}\right) + 35\left(\frac{2}{5}\right)\right)$$

$$= \frac{9}{16}\left(\frac{42}{5} - \frac{420}{7} + \frac{490}{9} + 6 - 20 + 14\right) = \frac{9}{16}\left(\frac{128}{45}\right) = \boxed{\frac{8}{5}}$$

so $\tilde{f}(x) = a_0P_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 = \boxed{\frac{12}{5}P_0 - 3P_1 + 4P_2 + 0P_3 + \frac{8}{5}P_4}$

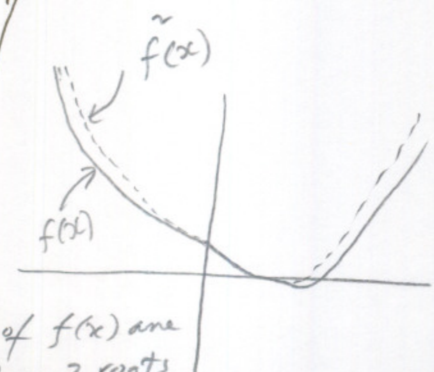
$$= \frac{12}{5}(1) - 3(x) + 4\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + 0 + \frac{8}{5}\left(\frac{1}{8}(3-30x^2+35x^4)\right)$$

$$\cancel{= \frac{7}{5} - 3x + 6x^2 - 2 + \frac{3}{5} - \frac{30}{5}x^2 + 7x^4}$$

$$\cancel{\tilde{f}(x) = 7x^4 - x^2 - 3x}$$

$$= \frac{12}{5} - 3x + 6x^2 - 2 + \frac{3}{5} - \frac{30}{5}x^2 + 7x^4$$

$$\boxed{\tilde{f}(x) = 7x^4 - x^2 - 3x + 1}$$

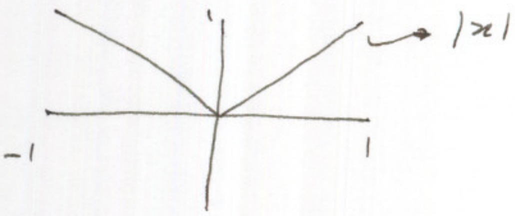


note: 2 roots of $f(x)$ are real. the other 2 roots are complex.

ch 112

$\boxed{9.14}$ Find the best fit (in Least square sense) Second order degree polynomial approximation to each of these functions for $-1 < x < 1$

$|x|$



Least square fit of a polynomial is the same as expanding the polynomial using legendre polynomials. this was proofed in class notes and in problem 9.16, page 504.

So I need to expand $|x|$ in legendre polynomials. since we want a Second degree polynomial, then

$$\tilde{f}(x) = a_0 P_0 + a_1 P_1 + a_2 P_2 \qquad \text{where } a_n = \frac{\int_{-1}^{1} f(x) P_n}{\frac{2}{2n+1}}$$

$$a_0 = \frac{\int_{-1}^{1} |x| P_0 \, dx}{2} = \frac{1}{2}\left(2 \int_0^1 x \, dx\right) = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}[1-0] = \boxed{\frac{1}{2}}$$

$$a_1 = \frac{\int_{-1}^{1} |x| P_1 \, dx}{\frac{2}{3}} = \frac{3}{2} \int_{-1}^{1} \text{even } f \times \overbrace{\text{odd } f}^{\text{odd}} \, dx = 0$$

$$a_2 = \frac{\int_{-1}^{1} |x| P_2}{\frac{2}{5}} = \frac{5}{2}\left(\int_{-1}^{0} -x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \int_0^1 x\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right)$$

$$= \frac{5}{2}\left(-\left[\left(\frac{3}{2}\frac{x^4}{4}\right)_{-1}^{0} + \frac{1}{2}\left(\frac{x^2}{2}\right)_{-1}^{0}\right] + \left[\left(\frac{3}{2}\frac{x^4}{4}\right)_0^1 - \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1\right]\right)$$

$$= \frac{5}{2}\left(-\left[\frac{3}{8}\left(0-(-1)^4\right) + \frac{1}{4}\left(0-(-1)^2\right)\right] + \left[\frac{3}{8}(1-0) - \frac{1}{4}(1-0)\right]\right)$$

$$= \frac{5}{2}\left(-\left[\frac{3}{8}(-1) + \frac{1}{4}(-1)\right] + \left[\frac{3}{8} - \frac{1}{4}\right]\right) = \frac{5}{2}\left[-\left(-\frac{3}{8} - \frac{1}{4}\right) + \left(\frac{1}{8}\right)\right]$$

$$= \frac{5}{2}\left(-\left(\frac{-5}{8}\right) + \frac{1}{8}\right) = \frac{5}{2}\left(\frac{5}{8} + \frac{1}{8}\right) = \frac{5}{2}\left(\frac{6}{8}\right) = \boxed{\frac{15}{8}}$$

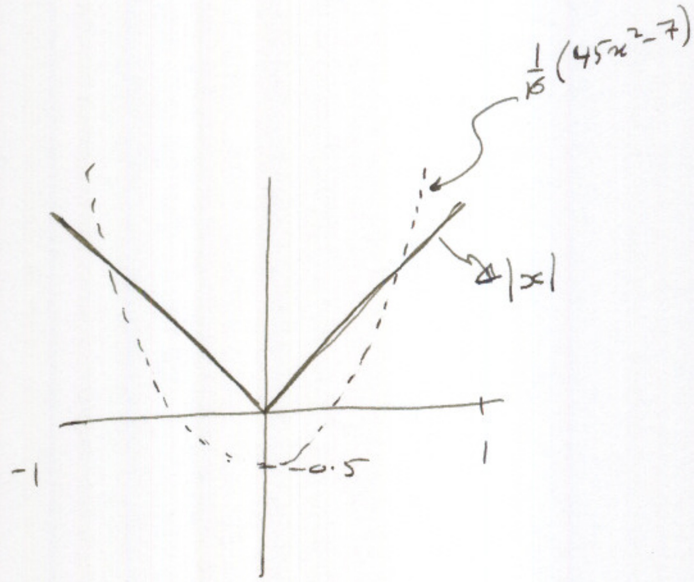
$\rightarrow$

so $\hat{f}(x) = a_0 P_0 + a_1 P_1 + a_2 P_2$

$= \frac{1}{2} P_0 + 0 P_1 + \frac{15}{8} P_2$

$= \frac{1}{2} + \frac{15}{8}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$= \frac{1}{2} + \frac{45}{16}x^2 - \frac{15}{16} = \frac{45}{16}x^2 - \frac{7}{16} = \boxed{\frac{1}{16}\left(45x^2 - 7\right)}$



ch 12

$\boxed{10.1}$ verify equations (10.3) and (10.4)

(10.3) $(1-x^2)u'' - 2(m+1)xu' + [l(l+1) - m(m+1)]u = 0$

(10.4) $(1-x^2)(u')'' - 2[(m+1)+1]x(u')' + [l(l+1) - m(m+1)(m+2)]u' = 0$

To verify (10.3) start with the associated Legendre DE:

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0 \qquad (1)$$

subtitute $y = (1-x^2)^{\frac{m}{2}} u$

$$\text{so } \frac{dy}{dx} = \frac{d}{dx}\left((1-x^2)^{\frac{m}{2}}u\right) = \frac{d}{dx}u\,(1-x^2)^{\frac{m}{2}} + u\frac{d}{dx}\left((1-x^2)^{\frac{m}{2}}\right)$$

$$= \frac{du}{dx}(1-x^2)^{\frac{m}{2}} + u\left(\frac{m}{2}(1-x^2)^{\frac{m}{2}-1}\cdot(-2x)\right)$$

$$\boxed{y' = \frac{du}{dx}(1-x^2)^{\frac{m}{2}} - umx(1-x^2)^{\frac{m}{2}-1}}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}(1-x^2)^{\frac{m}{2}}\right) - \frac{d}{dx}\left(um\,x(1-x^2)^{\frac{m}{2}-1}\right)$$

$$= \frac{d^2u}{dx^2}(1-x^2)^{\frac{m}{2}} + \frac{du}{dx}\left(\frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)\right) - \left(\frac{du}{dx}m(1-x^2)^{\frac{m}{2}-1}x + um\frac{d}{dx}\left(x(1-x^2)^{\frac{m}{2}-1}\right)\right)$$

$$= \frac{d^2u}{dx^2}(1-x^2)^{\frac{m}{2}} - \frac{du}{dx}mx(1-x^2)^{\frac{m}{2}-1} - \left(\frac{du}{dx}mx(1-x^2)^{\frac{m}{2}-1} + um\left((1-x^2)^{\frac{m}{2}-1} + x\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}(-2x)\right.\right.$$

$$= u''(1-x^2)^{\frac{m}{2}} - u'mx(1-x^2)^{\frac{m}{2}-1} - u'mx(1-x^2)^{\frac{m}{2}-1} - um(1-x^2)^{\frac{m}{2}-1} - umx\left(\frac{m}{2}-1\right)(-2x)(1-x^2)^{\frac{m}{2}-2}$$

$$y'' = u''(1-x^2)^{\frac{m}{2}} - u'mx(1-x^2)^{\frac{m}{2}-1} - u'mx(1-x^2)^{\frac{m}{2}-1} - um(1-x^2)^{\frac{m}{2}-1} + 2umx^2\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}$$

Subtitute y', y'' above into (1)

$\longrightarrow$

$$(1-x^2)\left[u''(1-x^2)^{\frac{m}{2}} - 2u'mx(1-x^2)^{\frac{m}{2}-1} - um(1-x^2)^{\frac{m}{2}-1} + 2umx^2\left(\tfrac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}\right]$$

$$-2x\left[u'(1-x^2)^{\frac{m}{2}} - umx(1-x^2)^{\frac{m}{2}-1}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^{\frac{m}{2}}u = 0$$

divide by $(1-x^2)^{\frac{m}{2}-2}$ both sides of the equation. $\Rightarrow$

$$(1-x^2)\left[u''(1-x^2)^2 - 2u'mx(1-x^2) - um(1-x^2) + 2umx^2\left(\tfrac{m}{2}-1\right)\right]$$

$$-2\left[u'x(1-x^2)^2 - umx^2(1-x^2)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^2u = 0$$

$\Rightarrow$

$$u''\left[(1-x^2)^3\right] + u'\left[-2mx(1-x^2)^2 - 2x(1-x^2)^2\right]$$

$$+u\left[-m(1-x^2)^2 + 2mx^2\left(\tfrac{m}{2}-1\right)(1-x^2) + 2mx^2(1-x^2)\right.$$

$$\left. + \left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^2\right] = 0$$

divide by $(1-x^2)^2 \Rightarrow$

$$u''(1-x^2) + u'(-2mx - 2x)$$

$$+u\left(-m + 2mx^2\left(\tfrac{m}{2}-1\right)\frac{1}{(1-x^2)} + 2mx^2\frac{1}{(1-x^2)} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\right) = 0$$

$$(1-x^2)u'' - 2(m+1)xu' + u\left(-m + \frac{\left(\cancel{2}mx^2\frac{m}{\cancel{2}} - 2mx^2\right)}{(1-x^2)} + \frac{2mx^2}{1-x^2} + [\ \ ]\right) = 0$$

$$(1-x^2)u'' - 2(m+1)xu' + u\left(-m + \frac{\cancel{2}mx}{1-x^2}(mx - 2x) + \frac{2mx^2}{1-x^2} + [\ \ ]\right) = 0$$

$$(1-x^2)u'' - 2(m+1)xu' + u\left(-m + \frac{mx^2}{1-x^2}(m\cancel{x} - 2\cancel{x} + 2) + l(l+1) - \frac{m^2}{1-x^2}\right) = 0$$

$\rightarrow$

$$\underbrace{(1-x^2)u'' - 2(m+1)xu'} + u\left(-m + \frac{m^2x^2}{1-x^2} - \frac{m^2}{1-x^2} + l(l+1)\right) = 0$$

$$\downarrow$$

$$+u\left(-m + \frac{m^2(x^2-1)}{1-x^2} + l(l+1)\right) = 0$$

$$\downarrow$$

$$+u\left(-m - \frac{m^2(1-x^2)}{1-x^2} + l(l+1)\right) = 0$$

$$\downarrow$$

$$+u\left(-m - m^2 + l(l+1)\right) = 0$$

$$\downarrow$$

$$+u\left(l(l+1) - m(m+1)\right) = 0$$

hence we get

$$\boxed{(1-x^2)u'' - 2(m+1)xu' + \left[l(l+1) - m(m+1)\right]u = 0}$$

$\longrightarrow$

now verify (10·4)

differentiate $(1-x^2)u'' - 2(m+1)xu' + [l(l+1) - m(m+1)]u = 0$

$$-2xu'' + (1-x^2)(u'')' - 2(m+1)[xu'' + u'] + [l(l+1) - m(m+1)]u' = 0$$

$$(1-x^2)(u'')' + u''(-2x - 2(m+1)x) + u'(-2(m+1) - m(m+1) + l(l+1)] = 0$$

$$(1-x^2)(u')'' + u''(-2x - 2mx - 2x) + u'(\cancel{m(+1)}\,(m+1)(-2-m) + l(l+1)] = 0$$

$$(1-x^2)(u')'' + u''x(-4 - 2m) + u'(-(m+1)(m+2) + l(l+1)] = 0$$

$$(1-x^2)(u')'' - 2[(m+1)+1]x(u')' + [l(l+1) - (m+1)(m+2)]u' = 0$$

which is 10·4.

ch 12

$\boxed{10.2}$ the equation for associated Legendre functions (and for Legendre functions when $m=0$) usually arises in the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]y=0 \quad (1)$$

make the change of variable $x=\cos\theta$ and obtain

$$(1-x^2)y''-2xy'+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0 \quad (2)$$

Solution

let $x=\cos\theta$ ~~in (1)~~

$$\frac{dx}{d\theta}=-\sin\theta \implies d\theta=-\frac{dx}{\sin\theta}$$

so also $\frac{dy}{d\theta}$ becomes $-\frac{dy}{dx}\sin\theta = -\sin\theta\frac{dy}{dx}$.

hence using these, we substitute in (1)

$$\frac{1}{\sin\theta}\frac{d}{-\frac{dx}{\sin\theta}}\left(\sin\theta\left(-\sin\theta\frac{dy}{dx}\right)\right)+\left[l(l+1)-\frac{m^2}{1-\cos^2\theta}\right]y=0$$

$$-\frac{\sin\theta}{\sin\theta}\frac{d}{dx}\left(-\sin^2\theta\frac{dy}{dx}\right)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0$$

$$\frac{d}{dx}\left((1-\cos^2\theta)\frac{dy}{dx}\right)+ \quad \downarrow \quad =0$$

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right)+ \quad \downarrow \quad =0$$

$$-2x\frac{dy}{dx}+(1-x^2)\frac{d^2y}{dx^2}+ \quad \downarrow \quad =0$$

$$(1-x^2)y''-2xy'+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0$$

QED

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$\boxed{10.5}$

Find $P_4^1 \cos\theta$ by substituting $P_4(x)$ in

$$P_l^m = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l$$

$$P_4(x) = \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)$$ found in problem 4.3.

so $\boxed{l = 4 \quad \text{and} \quad m = 1}$

hence

$$P_4^1(x) = (1-x^2)^{\frac{1}{2}} \frac{1}{8} \frac{d^1}{dx^1}\left(3 - 30x^2 + 35x^4\right)$$

$$= \sqrt{1-x^2}\ \frac{1}{8}\left(-60x + 140x^3\right)$$

$$= \sqrt{1-x^2}\ \frac{1}{2}\left(35x^3 - 15x\right)$$

let $x = \cos\theta$.

$$P_4^1(\cos\theta) = \sqrt{1-\cos^2\theta}\ \frac{1}{2}\left(35\cos^3\theta - 15\cos\theta\right)$$

$$\boxed{P_4^1(\cos\theta) = \frac{1}{2}\sin\theta\left(35\cos^3\theta - 15\cos\theta\right)}$$