35/40

HW # 2

Math 121 B

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(Extension student)

Ch 11

$\boxed{9.1}$ sketch $y = e^{-x^2}$

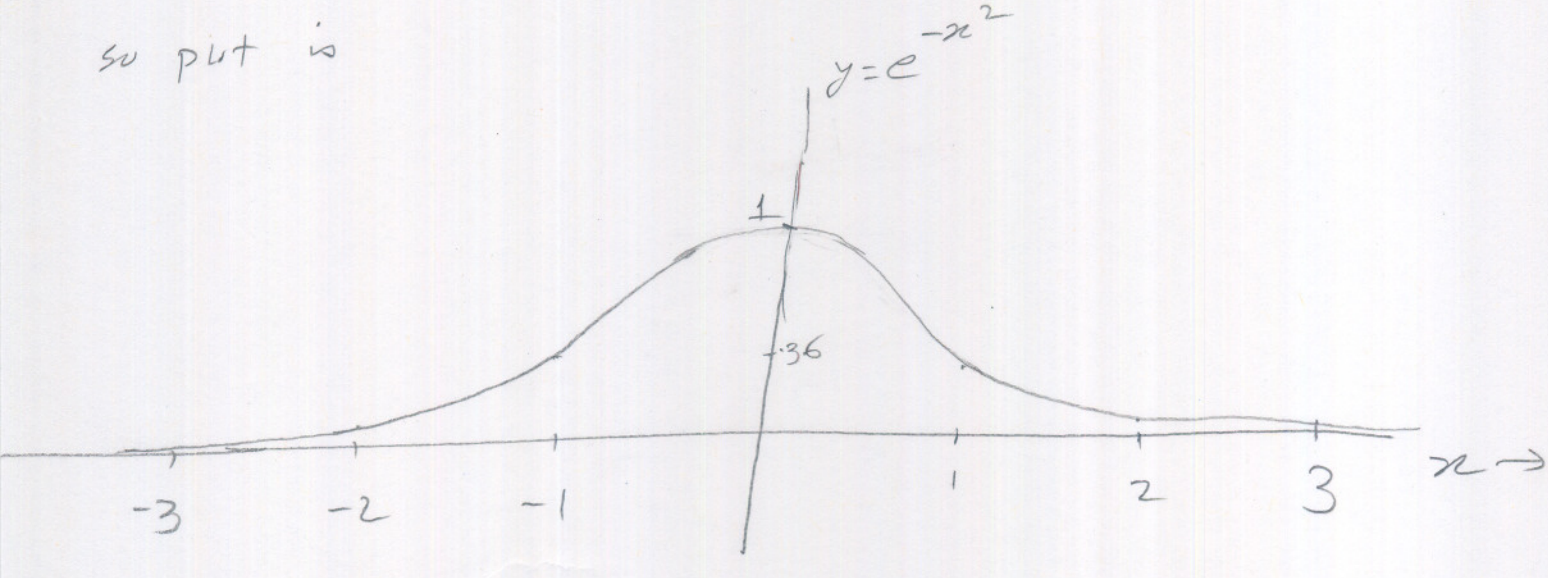
$e = 2.718$

look at few points.

x	y
0	1
+1	$1/e = .367$
+2	$1/e^4 = .018$
+3	$1/e^9 = .00012$
−1	$1/e = .367$
−2	$1/e^4 = .018$
−3	$1/e^9 = .00012$

the same since x^2

so plot is



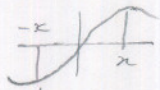
notice how quickly y approaches zero on each side due to the e^{x^2} being in the denominator.

ch 11

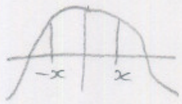
$\boxed{9.3}$ Prove that erf(x) is an odd function of x.

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

a function is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$.



odd



even.

So need to show that $\mathrm{erf}(-x) = -\mathrm{erf}(x)$

$$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt.$$ but $\int_a^b = -\int_b^a$

let $t = -s$

$dt = -ds$

when $t = 0 \Rightarrow s = 0$

when $t = -x \Rightarrow s = x$

So $$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-(-s)^2} (-ds)$$

$$\mathrm{erf}(-x) = \frac{-2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds$$

since 's' is a dummy variable, I can rewrite above as

$$\mathrm{erf}(-x) = \frac{-2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = -\mathrm{erf}(x)$$

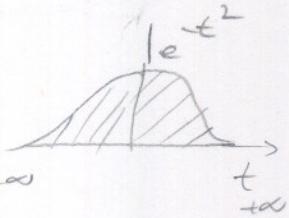
hence odd.

Ch 11

$\boxed{9.4}$ Show that $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}$

(a) by using 9.5 and 9.2 a
(b) by reducing it to a Γ function and using 5.3

Dr I show part (b) first (since easier)



(b)

let $y = \sqrt{2}\, t \Rightarrow dy = \sqrt{2}\, dt$.

$y = -\infty \Rightarrow t = -\infty$

$y = +\infty \Rightarrow t = +\infty$

so $I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\sqrt{2}t)^2} \sqrt{2}\, dt \Rightarrow \sqrt{2} \int_{-\infty}^{\infty} e^{-t^2} dt$

since $\int e^{-t^2}$ is even, then $\int_{-\infty}^{\infty} e^{-t^2} = 2\int_0^{\infty} e^{-t^2}$

so $I = 2\sqrt{2} \int_0^{\infty} e^{-t^2} dt$. but $\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

so $I = 2\sqrt{2}\, \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{2\pi}}$

(a) from 9.2 (a) $P(-\infty, x)$ is given. so I replace x with zero to get

$$P(-\infty, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2} dt = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{0}{\sqrt{2}}\right)$$

$$= \frac{1}{2}$$ since $\operatorname{erf}(0) = 0$

and from 9.2 (b), put $x = +\infty$ to get:

$$P(0, +\infty) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-t^2/2} dt = \frac{1}{2}\operatorname{erf}\left(\frac{\infty}{\sqrt{2}}\right)$$

$$= \frac{1}{2}$$ since $\operatorname{erf}(\infty) = 1$ by definition.

since $P(-\infty, 0) + P(0, +\infty) = P(\infty)$

hence from (9.5) $P(-\infty, 0) + P(0, +\infty) = 1$

hence $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2} dt = 1 \Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}}$

Ch 11

10.3 $\int_0^2 e^{-x^2}\,dx.$

$$\text{erf}(2) = \frac{2}{\sqrt{\pi}} \int_0^2 e^{-x^2}\,dx \quad \text{by definition}$$

hence $$\int_0^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,\text{erf}(2).$$

From Table, $\text{erf}(2) = 0.99532\,22650$

and $\frac{\sqrt{\pi}}{2} = 0.88622\,69255$

hence $$\int_0^2 e^{-x^2}\,dx = \boxed{0.88208\,139079}$$

10.5 Find erfc(5)

$$\text{erfc}(5) = 1 - \text{erf}(5)$$

Table for erf(x) only goes up to 2. however Table 7.3 in handbook of Math. Functions, Abramowitz, Page 316, contain Table for erfc(x) for $x=5$.

using Table entry for $\frac{1}{x^2} = 0.04$, then

for $x=5$: $$x e^{x^2}\,\text{erfc}(x) = 0.5535232$$

hence $$\text{erfc}(5) = \frac{0.5535232}{5\,e^{25}} = \boxed{1.53746 \times 10^{-12}}$$

$\boxed{10.13}$ by repeated integration by parts, find several terms of the asymptotic series for

$$I = \int_x^\infty t^{n-1} e^{-t}\, dt$$

$$t^{n-1} e^{-t} = t^{n-1} \frac{d}{dt}(-e^{-t})$$

so let $u = t^{n-1}$, $dv = -e^{-t}$ and apply $\int u\,dv = uv - \int v\,du$

hence $I = \underbrace{t^{n-1}(-e^{-t})\Big|_x^\infty}$ $- \int_x^\infty (n-1)\, t^{n-2} (-e^{-t})\, dt$

looking at this alone $\displaystyle \lim_{t\to\infty}\left(-\frac{t^{n-1}}{e^t}\right) + \frac{x^{n-1}}{e^x}$

$$\lim_{t\to\infty}\left(\frac{t^{n-1}}{e^t}\right) = 0$$

I can show this by expanding e^t as power series and then dividing by t^{n-1} both numerator and denominator:

$$\frac{t^{n-1}}{1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots} = \frac{1}{\frac{1}{t^{n-1}}+\frac{t}{t^{n-1}}+\frac{t^2}{2!\,t^{n-1}}+\frac{t^3}{3!\,t^{n-1}}+\cdots+\frac{t^{n-2}}{(n-2)!\,t^{n-1}}+\frac{t^{n-1}}{(n-1)!\,t^{n-1}}+\frac{t^n}{n!\,t^{n-1}}+\cdots}$$

$$= \frac{1}{\frac{1}{t^{n-1}}+\frac{1}{t^{n-2}}+\frac{1}{2!\,t^{n-3}}+\frac{1}{3!\,t^{n-4}}+\cdots+\frac{1}{(n-2)!\,t}+\frac{1}{(n-1)!}+\frac{t}{n!}+\frac{t^2}{(n+1)!}+\cdots}$$

now let $t\to\infty$ we get

$$= \frac{1}{\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\cdots+\frac{1}{(n-1)!}\quad \infty \quad \infty \quad \infty \cdots}$$

$$= \frac{1}{0+0+0+0+0+\cdots+\frac{1}{(n-1)}+\infty} = \frac{1}{\infty} = 0 \Rightarrow$$

So this means

$$I = t^{n-1}\left(-e^{-t}\right)\Big|_x^{\infty} - \int_x^{\infty} (n-1)t^{n-2}(-e^{-t})\,dt$$

$$= \left(0 + \frac{x^{n-1}}{e^x}\right) + (n-1)\underbrace{\int_x^{\infty} t^{n-2} e^{-t}\,dt}.$$

apply integration by parts again to ⤴ to get

$$= \frac{x^{n-1}}{e^x} + (n-1)\left[\left(0 + \frac{x^{n-2}}{e^x}\right) + (n-2)\int_x^{\infty} t^{n-3} e^{-t}\,dt\right]$$

$$= \frac{x^{n-1}}{e^x} + (n-1)\left[\frac{x^{n-2}}{e^x} + n-2\int_x^{\infty} t^{n-3}e^{-t}\,dt\right]$$

$$= \frac{x^{n-1}}{e^x} + \frac{(n-1)\,x^{n-2}}{e^x} + (n-1)(n-2)\left[\frac{x^{n-3}}{e^x} + (n-3)\int_x^{\infty} t^{n-4} e^{-t}\,dt\right]$$

$$= \frac{x^{n-1}}{e^x} + \frac{(n-1)\,x^{n-2}}{e^x} + \frac{(n-1)(n-2)\,x^{n-3}}{e^x} + (n-1)(n-2)(n-3)\left[\frac{x^{n-4}}{e^x} + (n-4)\int_x^{\infty} t^{n-5} e^{-t}\,dt\right]$$

$$= \frac{x^{n-1}}{e^x}\left(1 + \frac{(n-1)}{x} + \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)}{x^3} + \cdots\right)$$

$$+ \underbrace{(n-1)(n-2)\cdots\overbrace{(n-(n+1))}^{=1}}_{(n-1)!}\Big[\underbrace{\int_x^{\infty} \overbrace{t^{n-n}}^{=1} e^{-t}\,dt}_{\downarrow\ \int_x^{\infty} e^{-t}\,dt}\Big]$$

$\Longrightarrow$

$$= \frac{x^{n-1}}{e^x}\left(1+\frac{(n-1)}{x}+\frac{(n-1)(n-2)}{x^2}+\cdots+\frac{(n-1)(n-2)\cdots(2)}{x^{n-2}}\right)+(n-1)!\int_x^{\infty} e^{-t}\,dt$$

but $\int_x^{\infty} e^{-t}\,dt = -e^{-t}\Big|_x^{\infty} = -e^{-\infty}+e^{-x} = \frac{1}{e^x}$

hence $$\boxed{I = \frac{x^{(n-1)}}{e^x}\left(1+\frac{(n-1)}{x}+\cdots+\frac{(n-1)(n-2)\cdots(2)}{x^{n-2}}\right)+\frac{(n-1)!}{e^x}}$$

To verify if $x=0$ then we get

$$0+\frac{(n-1)!}{e^0} = \boxed{(n-1)! = \Gamma(n)}$$

as expected. ok.

Dr note on my solution. after I solved this I found in the back of book it gives solution the same but without the $\frac{(n-1)!}{e^x}$ term.

i.e book says $\int_x^{\infty} t^{n-1}e^{-t}\,dt = \frac{x^{n-1}}{e^x}\left[1+\frac{(n-1)}{x}+\cdots \frac{(n-1)(n-2)\cdots}{x^{n-1}}\right]$ (1)

however, book also says that when $x=0$, we should get the Gamma function $\Gamma(n)$. (page 472). however, if I put $x=0$ in (1), I get 0. while in my solution, I do get $\Gamma(n)$ due to the extra term $\frac{(n-1)!}{e^x}$.

did I overlook something?

Ch 11

$\boxed{10.14}$ express error functions as incomplete Γ function.

I'll will start with the sequence for $\Gamma(n,x)$ obtained from 10.13

$$\Gamma(n,x) = \frac{x^{n-1}}{e^x}\left(1 + \frac{(n-1)}{x} + \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)}{x^3} + \cdots\right)$$

put $n=\frac{1}{2}$, and sub $x=y^2$ we get

$$\Gamma(\tfrac{1}{2},y^2) = \frac{y^{2(-\frac{1}{2})}}{e^{y^2}}\left(1 + \frac{-\frac{1}{2}}{y^2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{y^4} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{y^6} + \cdots\right)$$

$$= \frac{1}{ye^{y^2}}\left(1 - \frac{1}{2y^2} + \frac{3}{4y^4} - \frac{3\times 5}{8y^6} + \cdots\right)$$

$$= \frac{1}{ye^{y^2}}\left(1 - \frac{1}{2y^2} + \frac{3}{(2y^2)^2} - \frac{3\times 5}{(2y^2)^3} + \cdots\right)$$

but expression on The R.H.S. is $\boxed{\sqrt{\pi}\ \mathrm{erfc}(y)}$ by looking at 10.4 on page 469. ✓ OK.

hence $\Gamma(\frac{1}{2},y^2) = \sqrt{\pi}\ \mathrm{erfc}(y)$ since y now is a dummy variable I can rewrite as x.

hence $\Gamma(\frac{1}{2},x^2) = \sqrt{\pi}\ \mathrm{erfc}(x) = \sqrt{\pi}\left(1-\mathrm{erf}(x)\right)$

so $$\boxed{\mathrm{erf}(x) = 1 - \frac{\Gamma(\frac{1}{2},x^2)}{\sqrt{\pi}}}$$ ✓

Ch 11

$\boxed{12.1}$ Expand $F(K,\phi)$ and $E(K,\phi)$ in power series in $K^2\sin^2\phi$ for small K and integrate term by term. from these series find series for the Complete elliptic integrals K and E.

$$F(K,\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-K^2\sin^2\phi}}\qquad 0\le K\le 1$$

since $0\le K\le 1$, then $K^2\sin^2\phi\le 1$ (since max $\sin\phi=1$)

hence can expand using binomial theorem $(1-x)^P$, where

here $P=-\frac{1}{2}$ and $x=(K\sin\phi)^2$

i.e. $F(K,\phi)=\int_0^{\phi}\underbrace{\left(1-(K\sin\phi)^2\right)^{-\frac{1}{2}}}_{\text{expand this}}\,d\phi$

$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4+\cdots$$

I will use the first 4 terms, hence

$$=1+\tfrac{1}{2}(K\sin\phi)^2+\frac{1\cdot 3}{2\cdot 4}(K\sin\phi)^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(K\sin\phi)^6+\cdots$$

So $F(K,\phi)=\int_0^{\phi}d\phi+\frac{1}{2}\int_0^{\phi}(K\sin\phi)^2\,d\phi+\frac{1\cdot 3}{2\cdot 4}\int_0^{\phi}(K\sin\phi)^4\,d\phi+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int_0^{\phi}(K\sin\phi)^6\,d\phi$

Since K is a parameter not dependent on ϕ, take outside $\int$,

$$F(K,\phi)=\int_0^{\phi}d\phi+\frac{K^2}{2}\int_0^{\phi}\sin^2\phi\,d\phi+\frac{1\cdot 3}{2\cdot 4}K^4\int_0^{\phi}\sin^4\phi\,d\phi+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}K^6\int_0^{\phi}\sin^6\phi\,d\phi.$$

now integrate each term:

$$\int_0^{\phi}d\phi=\boxed{\phi}$$

$$\frac{K^2}{2}\int_0^{\phi}\sin^2\phi\,d\phi=\frac{K^2}{2}\left[\int_0^{\phi}\frac{1-\cos(2\phi)}{2}\,d\phi=\int_0^{\phi}\frac{1}{2}d\phi-\frac{1}{2}\int_0^{\phi}\cos 2\phi\,d\phi\right]$$

$$=\frac{K^2}{2}\left(\frac{1}{2}(\phi)-\frac{1}{2}(\cos\phi\sin\phi)\right)=\boxed{\frac{K^2}{4}\phi-\frac{K^2}{4}\cos\phi\sin\phi}\longrightarrow$$

Third term

$$\frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\phi}\sin^4\phi\,d\phi = \frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\phi}(\sin^2\phi)^2\,d\phi$$

but $\sin^2\phi = \dfrac{1-\cos 2\phi}{2}$ so $\dfrac{1\cdot 3}{2\cdot 4}k^4\displaystyle\int_0^{\phi}\left(\frac{1-\cos 2\phi}{2}\right)^2 d\phi$

$$= \frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\phi}\frac{1}{4}\left(1-2\cos 2\phi+\cos^2\phi\right)d\phi$$

$$= \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\left(\int_0^{\phi}1\,d\phi - 2\int_0^{\phi}\cos 2\phi + \int_0^{\phi}\cos^2\phi\right)$$

$$= \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\left(\phi - 2\cos\phi\sin\phi + \int_0^{\phi}\frac{1+\cancel{2}\cos 2\phi}{2}d\phi\right)$$

$$= \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\left(\phi - 2\cos\phi\sin\phi + \int_0^{\phi}\frac{1}{2}d\phi + \int_0^{\phi}\cos 2\phi\,d\phi\right)$$

$$= \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\left(\phi - 2\cos\phi\sin\phi + \frac{1}{2}\phi + \cos\phi\sin\phi\right)$$

$$= \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\phi - \frac{1\cdot 3}{2\cdot 4}\frac{2}{4}k^4\cos\phi\sin\phi + \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}\frac{1}{2}k^4\phi + \frac{1\cdot 3}{2\cdot 4}\frac{1}{4}k^4\cos\phi\sin\phi$$

$$= \frac{3}{32}k^4\phi - \frac{3}{16}k^4\cos\phi\sin\phi + \frac{3}{64}k^4\phi + \frac{3}{32}k^4\cos\phi\sin\phi$$

$$= \boxed{\frac{9}{64}k^4\phi - \frac{3}{32}k^4\cos\phi\sin\phi}$$

we can continue as this for $\int \sin^6\phi\,d\phi$. stopping here, I get

$$\boxed{F(k,\phi) = \phi + \frac{k^2}{4}\phi - \frac{k^2}{4}\cos\phi\sin\phi + \frac{9}{64}k^4\phi - \frac{3}{32}k^4\cos\phi\sin\phi}\cdots$$

before I continue to $E(k,\phi)$, I verify the above is OK →

from Tables, $F(K,\phi)$ for $K=0.5$ and $\phi = \frac{\pi}{4}$ is $\boxed{0.804366}$

using series expansion:

$$\frac{\pi}{4} + \frac{.5^2}{4}\frac{\pi}{4} - \frac{.5^2}{4}\cos\frac{\pi}{4}\sin\frac{\pi}{4} + \frac{9}{64}\,.5^4\,\frac{\pi}{4} - \frac{3}{32}\,.5^4\cos\frac{\pi}{4}\sin\frac{\pi}{4}$$

$$= \boxed{\cancel{0.82504}} \quad \boxed{0.77153}$$

This is a good approximation considering I only used 3 terms in the expansion. error is about 2.5%

Now I will do the expansion for $E(K,\phi)$.

$$E(K,\phi) = \int_0^\phi \sqrt{1-K^2\sin^2\phi}\; d\phi$$

Similarly, expand $(1-x)^P$ using binomial theory, where here $P=\frac{1}{2}$ not $-\frac{1}{2}$ as was the case with $F(K,\phi)$.

$$(1-x)^P = (1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots$$

where $x = (K\sin\phi)^2$

so
$$\int_0^\phi (1-x)^{1/2}\,d\phi = \int_0^\phi 1 - \frac{1}{2}\int_0^\phi (K\sin\theta)^2\,d\theta - \frac{1}{8}\int_0^\phi (K^2\sin^2\theta)^2\,d\phi - \cdots$$

$$= \phi - \frac{1}{2}K^2\int_0^\phi \sin^2\phi\,d\phi - \frac{K^4}{8}\int_0^\phi \sin^4\phi\,d\phi - \cdots$$

$$= \phi - \frac{1}{2}K^2\left[\frac{1}{2}\phi - \frac{1}{2}\cos\phi\sin\phi\right] - \frac{K^4}{8}\left[\frac{1}{4}\left(\phi - 2\cos\phi\sin\phi + \frac{1}{2}\phi + \cos\phi\sin\phi\right)\right]$$

$$= \phi - \frac{1}{4}K^2\phi + \frac{K^2}{4}\cos\phi\sin\phi - \frac{K^4}{32}\left(\frac{3}{2}\phi - \cos\phi\sin\phi\right)$$

$\longrightarrow$

$$\boxed{E(k,\phi) = \phi - \tfrac{1}{4}k^2\phi + \frac{k^2}{4}\cos\phi\sin\phi - \frac{3}{64}k^4\phi + \frac{k^4}{32}\cos\phi\sin\phi \cdots}$$

verify: try with $k = .5$ and $\phi = \frac{\pi}{4}$.

from Tables, $E(k,\phi) = \boxed{0.767196}$

From series:

$$= \frac{\pi}{4} - \frac{.5^2}{4}\frac{\pi}{4} + \frac{.5^2}{4}\cos\frac{\pi}{4}\sin\frac{\pi}{4} - \frac{3}{64}.5^4\frac{\pi}{4} + \frac{.5^4}{32}\cos\frac{\pi}{4}\sin\frac{\pi}{4}$$

$$= \boxed{\cancel{0.767198}} \quad \boxed{0.766236}$$

This is a very good approximation with only 3 terms.

error is only 0.1%.

this tells me that $F(k,\phi)$ does not converges as quickly as $E(k,\phi)$ (unless I made a mistake).

Now use these series to find series for complete elliptic integrals K and E:

$K = F(k, \frac{\pi}{2})$, so replace ϕ with $\frac{\pi}{2}$ in the series for $F()$ found.

$$= \frac{\pi}{2} + \frac{k^2}{4}\frac{\pi}{2} - \frac{k^2}{4}\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{9}{64}k^4\frac{\pi}{2} - \frac{3}{32}k^4\cos\cancel{\phi}^{\pi/2}\sin\cancel{\phi}^{\pi/2}$$

$$\boxed{K = \frac{\pi}{2} + \frac{k^2}{8}\pi + \frac{9}{124}k^4\pi + \cdots}$$

(since $\cos\frac{\pi}{2} = 0$ all Trig term disappears)

Similary Find E $\longrightarrow$

$$E\left(k,\tfrac{\pi}{2}\right) = \frac{\pi}{2} - \frac{1}{4}k^2\frac{\pi}{2} - \frac{k^2}{4}\cos\frac{\pi}{2}\sin\frac{\pi}{2} - \frac{3}{64}k^4\frac{\pi}{2} + \frac{k^4}{32}\cos\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$\boxed{E(k) = \frac{\pi}{2} - \frac{k^2}{8}\pi - \frac{3k^4}{124}\pi - \cdots}$$

$\boxed{12.2}$ Find from Tables or (for small k) from power series of problem 1

$$K(0.13) = F\left(0.13, \tfrac{\pi}{2}\right)$$

lower case k

From Tables, (in Abramowitz), page 608,

for $m = k^2 = 0.13^2 = 0.0169$

Table shows $K(0.01) =$ ~~[illegible]~~ 1.57474

and $K(0.02) =$ ~~[illegible]~~ 1.57873

(what is best way to find values for K between entries in Table? use interpolation?).

Take $\boxed{K(0.13) = 1.58}$ From Table used 0.02 as closer!

using Series:

$$K = \frac{\pi}{2} + \frac{k^2}{8}\pi + \frac{9}{124}k^4\pi$$

when $k = .13$, I get $\boxed{K = 1.5775}$ From my series using only 3 terms.

Ch 11

$\boxed{12.7}$

$$\int_0^{\pi/4} \frac{d\phi}{\sqrt{1-0.25\sin^2\phi}}$$

here $k^2 = 0.25 \Rightarrow k = 0.5$

so use ~~$F(\phi,$~~ $F(0.5, \frac{\pi}{4})$

From series expansion of the Legendre form of elliptic integral F:

$$F(k,\phi) = \phi + \frac{k^2}{4}\phi - \frac{k^2}{4}\cos\phi\sin\phi + \frac{9}{64}k^4\phi - \frac{3}{32}k^4\cos\phi\sin\phi$$

$$= \frac{\pi}{4} + \frac{.5^2}{4}\frac{\pi}{4} - \frac{.5^2}{4}\cos\frac{\pi}{4}\sin\frac{\pi}{4} + \frac{9}{64}.5^4\frac{\pi}{4} - \frac{3}{32}.5^4\cos\frac{\pi}{4}\sin\frac{\pi}{4}$$

$$= 0.807209$$

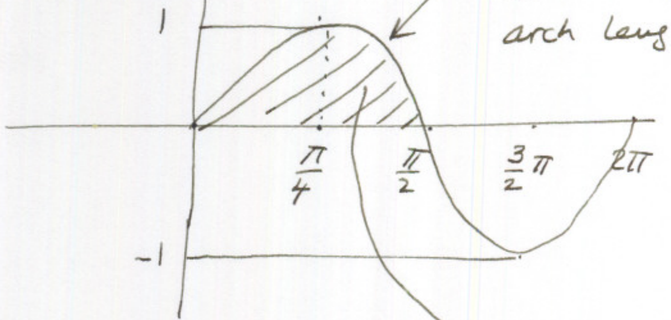
From Mathematica I get $\boxed{0.804366}$

12.9 ?

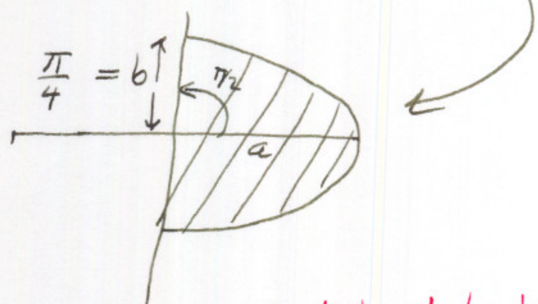
ch 11

$\boxed{12.16}$ Find the length of one arch of $y = \sin x$

need to find this arch length.



looking at the arch, after rotating it sideways by 90°, it becomes an ellipse as



so now I can use E function to find length of arc for 0 to $\frac{\pi}{2}$, then arch length will be twice that.

I don't understand your method.

Arclength $= \int \sqrt{1+\cos^2 t}\, dt$

$\left(\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dt \right)$

i.e. $\boxed{\text{arch length} = 2\, a E\left(k, \frac{\pi}{2}\right)}$

must use $a > b$. , here $a = 1$ (which is max value for sin).

1/5

so since $k^2 = \frac{a^2 - b^2}{a^2} = \frac{1 - \left(\frac{\pi}{4}\right)^2}{1} = 0.38315$

so $\boxed{k = 0.618991}$

so arch length $= 2\, E\left(0.618991, \frac{\pi}{2}\right)$ → here E was obtained from mathematica not tables.

$= 2\ (1.4)$

$= \boxed{2.8}$ → Please note: book gives answer as 3.8. I went over this few times and don't see where I am making an error?

ch 11

$\boxed{12.18}$ sketch graph of $sn(u)$ as a function of u for $k=\frac{1}{2}$. Use table for the elliptic integral

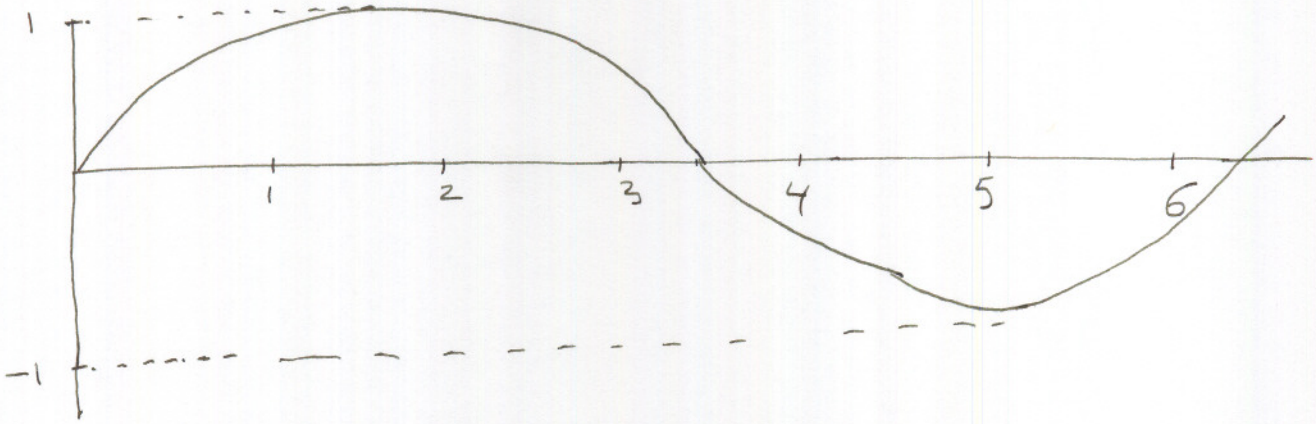
$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

and remember that $sn(u) = \sin\phi$.

I generate this Table (use $k=1/2$)

φ	$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = F(k,\varphi)$	$sn(u) = \sin\varphi$
0	0	0
$\pi/8$	$F(1/2, \frac{\pi}{8}) = 0.395$	0.382
$\frac{\pi}{4}$	$F(1/2, \pi/4) = 0.804$	0.707
$\frac{3}{8}\pi$	$F(1/2, \frac{3}{8}\pi) =$ ~~[illegible]~~ 1.235	0.92
$\frac{\pi}{2}$	$F(1/2, \frac{\pi}{2}) = 1.685$	1
$\frac{5}{8}\pi$	$F(1/2, \frac{5}{8}\pi) = 2.13$	0.92
$\frac{3}{4}\pi$	$F(1/2, \frac{3}{4}\pi) = 2.56$	0.707
$7/8\,\pi$	$F(1/2, 7/8\,\pi) = 2.97$	0.38
π	$F(1/2, \pi) = 3.37$	0
$\frac{3}{2}\pi$	$F(1/2, \frac{3}{2}\pi) = 5.05$	-1
2π	$F(1/2, 2\pi) = 6.74$	0

$\longrightarrow$ plot

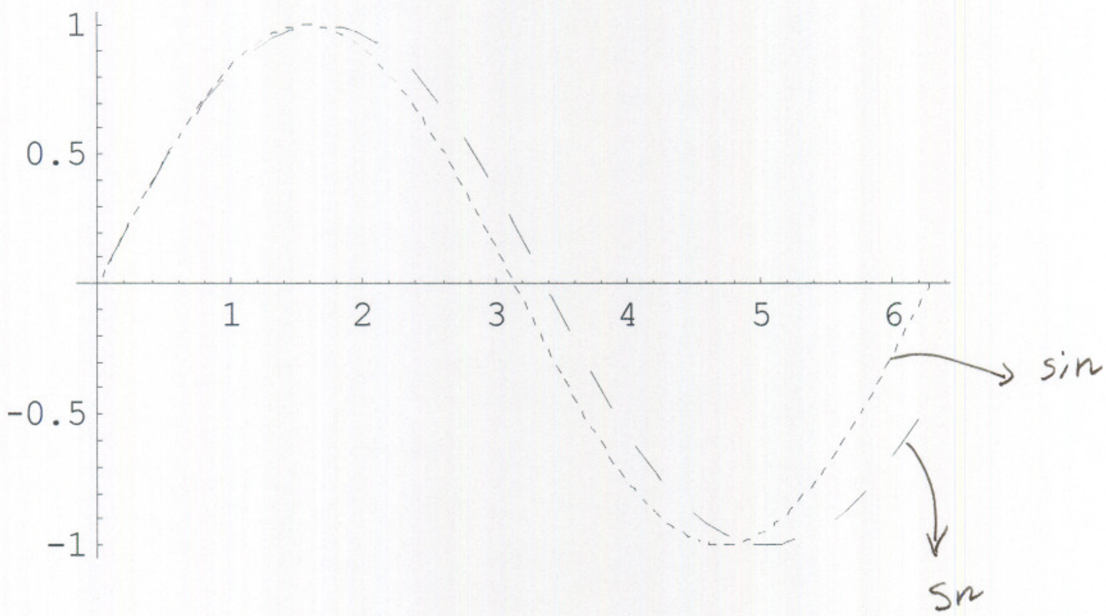


it looks very much like a clear sin function.

please see plot next page
using mathematica.

this is a plot of sn function using mathematica, along with the sin function to compare the two.

```
Plot[{JacobiSN[x, .5^2], Sin[x]}, {x, 0, 2 Pi},
 PlotStyle → {Dashing [{0.05 , 0.05 }], Dashing [{0.01 , 0.01 }]}]
```



Ch 11 (19)

$\boxed{12.21}$ by transforming $\int_0^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}}$ to one of the standard forms for an elliptic integral of first kind, show that

Beta function

$$B\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = 2\sqrt{2}\; F\left(\tfrac{1}{\sqrt{2}}, \tfrac{\pi}{2}\right) = 2\sqrt{2}\; K\left(\tfrac{1}{\sqrt{2}}\right) \quad \text{and}$$

so $$K\left(\tfrac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\tfrac{1}{4}\right).$$

Evaluate these expressions from tables to check result.

The First kind is either the Legendre form $\int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ or the Jacobi form $\int_0^{x} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$.

I will use the Legendre form.

let $\quad \phi = 2\alpha \quad \Rightarrow \quad d\phi = 2\, d\alpha$

then $$\int_0^{\pi/2} \frac{d\phi}{\sqrt{\cos\phi}} = \int_0^{\pi/4} \frac{2\, d\alpha}{\sqrt{\cos 2\alpha}}$$

now since $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ then $\cos 2\alpha = 1 - 2\sin^2\alpha$

so integral becomes $$2\int_0^{\pi/4} \frac{d\alpha}{\sqrt{1-2\sin^2\alpha}}$$

this is in the form needed, however $k^2 = 2$ here, i.e. $k = \sqrt{2}$ which is irrational and also k is supposed to be $(0,1)$ range? since $k^2 = \frac{a^2-b^2}{a^2}$, and in ellipse $a > b$. still, not knowing what else to do, I used above form, and evaluated →

$$2\int_0^{\pi/4} \frac{d\alpha}{\sqrt{1-2\sin^2\alpha}} = 2\,F\left(\sqrt{2}, \frac{\pi}{4}\right)$$

$$= 2\,\Big(\underbrace{1.31 - 1.8\times 10^{-16}\, i}_{\text{complex value obtained from mathematica}}\Big)$$

since complex part is so small, I drop it.

so integral is $2(1.31) = \boxed{2.622}$

But if I keep the complex part I get

$$\boxed{2.622 - 3.7\times 10^{-16}\, i}$$