HW 11

MATH 121B

Spring 2004
UC BERKELEY

Nasser M. Abbasi

Spring 2004 Compiled on April 17, 2021 at 7:16pm

Contents

1 chapter 16, problem 1.1 Mary Boas, second edition

Find the probability that a single throw of die will give a number less than 3; an even number; a 6.

Solution

Sample space $S = \{1, 2, 3, 4, 5, 6\}$, hence $n_s = 6$

Let A the event that a number is less than 3

Hence $\Pr(A) = \frac{n_A}{n_S} = \frac{2}{6} = \frac{1}{3}$

Let B the event that a number is even

Hence $\Pr(B) = \frac{n_B}{n_S} = \frac{3}{6} = \frac{1}{2}$

Let C the event that a number is 6

Hence $\Pr(C) = \frac{n_c}{n_S} = \frac{1}{6}$

2 chapter 16, problem 1.2 Mary Boas, second edition

3 coins are tossed; what is the probability that 2 are heads and one is tail? that the first 2 are heads and the third tail? if at least 2 are heads, what is the probability that all are heads?

Solution

Sample space $S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$, hence $n_s = 8$

Let A the event that 2 are heads and one is tail

Hence $\Pr(A) = \frac{n_A}{n_S} = \frac{3}{8}$

Let B the event that the first 2 are heads and the third tail

Hence $\Pr(B) = \frac{n_B}{n_S} = \frac{1}{8}$

For the third case, since at least 2 are heads, hence our sample space now is different, it is a subset of the original sample space and is the following: $S_1 = \{hhh, hht, hth, thh\}$, and $n_{S_1=4}$

Let C the event that all are heads out of the above same space S_1

Hence $\Pr(C) = \frac{n_C}{n_{S_1}} = \frac{1}{4}$

3 chapter 16, problem 1.3, Mary Boas, second edition

In a box there are 2 whites, 3 blacks and 4 red balls. If a ball is drawn at random, what is the probability that it is black? that it is not red?

Solution

Sample space $S = \{W, W, B, B, B, R, R, R, R\}$, hence $n_s = 9$

Let A be the event that ball thrown is black

Hence

$$\Pr(A) = \frac{n_A}{n_S} = \frac{3}{9}$$

Let B the event that it is not red Hence $\Pr(B) = 1 - \Pr(C)$ Where C is the event that ball drawn is red. Hence

$$\Pr(B) = 1 - \frac{n_C}{n_S} = 1 - \frac{4}{9} = \frac{5}{9}$$

4 chapter 16, problem 1.4, Mary Boas, second edition

A single card is drawn at random from a shuffled deck. What is the probability that it is red? that it is the ace of hearts? that it is either a 3 or a 5? that it is either an ace or red or both?

Solution

Sample space $S = \{...\}$ the 52 cards, hence $n_s = 9$

For first part, let A be the event that card is red, Hence

$$\Pr(A) = \frac{n_A}{n_S} = \frac{13}{52}$$

For second part, let A the event that it is the ace of hearts Hence,

$$\Pr(A) = \frac{n_A}{n_S} = frac152$$

For third part, let A be the event it is a 5, and let B be the event it is a 3. Hence

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

But $\Pr(A \cap B) = 0$ since events are mutually exclusive. Hence

$$\Pr(A \cup B) = \frac{n_A}{n_S} + \frac{n_B}{n_S} = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$$

For last part, let A event that it is an Ace. Let B event that it is a red. Hence

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}\\ &= \frac{16}{52}\end{aligned}$$

In the above, $\Pr(A \cap B) = \frac{1}{52}$ since there is only one card that is both an Ace and a red.

5 chapter 16, problem 1.5, Mary Boas, second edition

Given a family of 2 children, what is the probability that both are boys? that at least one is girl? Given that at least one is girl, what is the probability that both are girls? Given that the first 2 are girls, what is the probability that an expected 3rd child will be a boy? (assume boys and girls are equally likely).

Solution

What is the probability that both are boys?

Let A be event that both are boys. The family could have had a boy followed by a girl or a boy followed by a boy or a girl followed by a boy or a girl followed by a girl. Hence the sample space $S = \{BG, BB, GB, GG\}$ hence $n_s = 4$

So $\Pr(A) = \frac{n_A}{n_s} = \frac{1}{4}$

At least one is girl?

Let A be event that at least one is girl So $\Pr(A) = \frac{n_A}{n_s} = \frac{3}{4}$

Given that at least one is girl, what is the probability that both are girls?

Since we are given that at least one is girl, then the sample space now is

$$S = \{BG, GB, GG\}$$

hence $n_s = 3$

Let A event that both are girls. So $\Pr(A) = \frac{n_A}{n_s} = \frac{1}{3}$

Given that the first 2 are girls, what is the probability. that an expected 3rd child will be a boy?

Since the event of having a new child is independent of gender of previous children, the expected 3rd child being a boy is $\frac{1}{2}$ regardless of what gender or number of children already born. (This is like tossing a coin, getting a head will have Probability of 50% regardless of the history of tosses.).

6 chapter 16, problem 1.8, Mary Boas , second edition

An integer N is chosen at random with $1 \le N \le 100$. What is the probability that N is divisible by 11? That $N > 90$? That $N \le 3$? That N is a perfect square?

Solution

What is the probability. that N is divisible by 11?

Let A be event that N is divisible by 11. Sample space $S = \{1, 2, \cdots, 100\}$ hence $n_s = 100$

Numbers that are divisible by 11 are $11, 22, 33, 44, 55, 66, 77, 88, 99$ so $n_A = 9$. So $\Pr(A) = \frac{n_A}{n_s} = \frac{9}{100}$

That $N > 90$?

Let A be event that $N > 90$. Numbers that are > 90 are $91, 92, 93, 94, 95, 96, 97, 98, 99, 100$ so $n_A = 10$ So $\Pr(A) = \frac{n_A}{n_s} = \frac{10}{100} = \frac{1}{10}$

That $N \le 3$? Let A be event that $N \le 3$

Numbers that $N \le 3$ are $1, 2, 3$ so $n_A = 3$. So $\Pr(A) = \frac{n_A}{n_s} = \frac{3}{100}$

That N is a perfect number? Let A be event that N is perfect square.

Numbers that are perfect squares are $1, 4, 9, 16, 25, 36, 49, 64, 91, 100$

$\Pr(A) = \frac{n_A}{n_s} = \frac{10}{100} = \frac{1}{10}$

7 chapter 16, problem 1.10, Mary Boas, second edition

A shopping mall has 4 entrances, one in North, one in south, and 2 on the east. If you enter at random, shop and then exit at random, what is the probability that you enter and exist on the same side of the mall?

Solution

Let N_x represent the event of entering using entrance x where x is north,south, or east.

So, N_n the event of entering from the north entrance, N_s means enter from the south entrance, N_{e1} means enter from the first entrance on the east side, and N_{e2} means enter from the second entrance on the east side.

Let

Hence here $S = \{N_n, N_s, N_{e1}, N_{e2}\}$, $n_s = 4$

Then $n_e = 2$ (since there are 2 doors on the east side, and each is equally likely to use to enter).

Hence, $\Pr(N_{e1} \cup N_{e2})$ =Probability of entering from the east side is $\frac{n_e}{n_s} = \frac{2}{4} = \frac{1}{2}$

Hence, $\Pr(N_n)$ =Probability of entering from the north side is $\frac{1}{n_s} = \frac{1}{4}$

Hence, $\Pr(N_s)$ =Probability of entering from the south side is $\frac{1}{n_s} = \frac{1}{4}$.

Similarly let E_x represent the event of leaving the mall using exist x

Hence here $S = \{E_n, E_s, E_{e1}, E_{e2}\}$, $n_s = 4$

Then $n_e = 2$ (since there are 2 doors on the east side, and each is equally likely to use to leave).

Hence, $\Pr(E_{e1} \cup E_{e2})$ = Probability of leaving from the east side is $\frac{2}{4} = \frac{1}{2}$

Hence, $\Pr(E_n)$ = Probability of leaving from the north side is $\frac{1}{4}$

Hence, $\Pr(E_s)$ = Probability of leaving from the south side is $\frac{1}{4}$

Let X be the event of entering and exiting from the same side. So this is the probability of leaving from the east side given that we entered from the east side, or leaving from the north side given we entered from the north side, or leaving from the south side given we entered from the south side. Using conditional probability, Hence we write:

$$\begin{aligned}\Pr(X) &= \Pr\left(\ [\ (E_{e1} \cup E_{e2}) \mid (N_{e1} \cup N_{e2})\ ]\ \cup (E_n|N_n)\ \ \cup\ (E_s|N_s)\ \right) \\ &= \Pr((E_{e1} \cup E_{e2})|(N_{e1} \cup N_{e2})) + \Pr(E_n|N_n) + \Pr(E_s|N_s) \end{aligned} \tag{1}$$

But

$$\begin{aligned}\Pr((E_{e1} \cup E_{e2}) \mid (N_{e1} \cup N_{e2})) &= \Pr(E_{e1} \cup E_{e2}) \times \Pr(N_{e1} \cup N_{e2}) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\end{aligned}$$

And

$$\begin{aligned}\Pr(E_n|N_n) &= \Pr(E_n) \times \Pr(N_n) \\ &= \frac{1}{4} \times \frac{1}{4} \\ &= \frac{1}{16}\end{aligned}$$

And

$$\begin{aligned}\Pr(E_s|N_s) &= \Pr(E_s) \times \Pr(N_s)\\ &= \frac{1}{4} \times \frac{1}{4}\\ &= \frac{1}{16}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\Pr(X) = &\ \Pr((E_{e1} \cup E_{e2})|(N_{e1} \cup N_{e2})) + \Pr(E_n|N_n) + \Pr(E_s|N_s)\\ &= \frac{1}{4} + \frac{1}{16} + \frac{1}{16}\\ &= \frac{3}{8}\end{aligned}$$

There is a much faster method to solve this.

Label each entrance as N, S, E_1, E_2. Set up the sample space (all possible events) as $S = \{NN, NS, NE_1, NE_2,\quad SN, SS, SE_1, SE_2,\quad E_1E_1, E_1E_2,\ E_1S,\ E_1N,\quad E_2E_1, E_2E_2,\ E_2S,\ E_2N, \}$

Where the first letter is the entrance, and the second letter is the exit.

By counting we count all those with BOTH E as the first letter and the second letter. We see there are 6 of these, and number of possible events is 16, hence the answer is $\frac{3}{8}$ as above

8 chapter 16, problem 2.12, Mary Boas, second edition

Use sample space of example 1 to answer the following questions.

Solution

Sample space of example 1 is

$S = \{hhh, hth, ttt, tht, hht, thh, tth, htt\}$

(a) If there are more heads than tails, what is the probability of one tail?

The sample space here is

$$S = \left\{hhh, \overparen{hth}, \overparen{hht}, \overparen{thh}\right\}$$

$n_s = 4$. So from this sample space, $\Pr(\textit{one tail}) = \frac{3}{4}$

(b) If two heads did not appear in succession, what is the probability of all tails?

The sample space here is

$$S = \left\{hth, \overparen{ttt}, tht, tth, htt\right\}$$

$n_s = 5$. So from this sample space, $\Pr(\textit{all tails}) = \frac{1}{5}$

(c) if the coins did not all fall alike, what is the probability that 2 succession were alike?

$$S = \left\{hth, tht, \overparen{hht}, \overparen{thh}, \overparen{tth}, \overparen{htt}\right\}$$

$n_s = 5$ So from this sample space,

$$\Pr(2 \text{ succession were alike}) = \frac{4}{6} = \frac{2}{3}$$

(d) if N_t =number of tails, N_h =number of heads, what is the probability. that $|N_h - N_t| = 1$?

From $S = \{hhh, hth, ttt, tht, hht, thh, tth, htt\}$, we see that $|N_h - N_t|$ for each sample point is

$$\{3, 1, 3, 1, 1, 1, 1, 1\}$$

$n_s = 8$. Hence $\Pr(|N_h - N_t| = 1) = \frac{6}{8} = \frac{3}{4}$

(e) If there is at least one head, what is the probability. of exactly two heads?

Since we are told there is at least one head, then we remove the sample points that has no head in them, then our new sample space is

$S = \left\{hhh, \overparen{hth}, tht, \overparen{hht}, \overparen{thh}, tth, htt\right\}, n_s = 7$

So $\Pr\left(\text{exactly two heads}\right) = \frac{3}{7}$

9 chapter 16, problem 2.13, Mary Boas, second edition

A student claims in problem 1.5 that if one child is a girl, the probability that both are girls is $\frac{1}{2}$. Use appropriate sample spaces to show what is wrong with the following argument: It does not matter whether the girl is the older child or the younger; in either case the probability is $\frac{1}{2}$ that the other child is a girl.

This is problem 1.5 for reference:

Given a family of 2 children, what is the probability that both are boys? that at least one is girl? Given that at least one is girl, what is the probability that both are girls? Given that the first 2 are girls, what is the probability. that an expected 3rd child will be a boy? (assume boys and girls are equally likely).

Solution

Need to distinguish between the older and the younger child.

Let a subscript o means older child, and subscript y means younger child.

Then the sample space is written as

$S = \left\{B_oB_y, B_oG_y, G_oB_y, G_oG_y\right\}$

Here we see that if one child is a girl, then the probability that the other child is a girl is taken from this sample space $S = \left\{B_oG_y, G_oB_y, G_oG_y\right\}$ which is then $\frac{1}{3}$ and not $\frac{1}{2}$

If the **older child is a girl** G_o, then the sample space is $S = \left\{ G_oB_y, G_oG_y\right\}$, and from this the probability that the other child is a girl is $\frac{1}{2}$

If the **younger child is a girl** G_y, then the sample space is $S = \left\{ B_oG_y,\ G_oG_y\right\}$, and from this the probability that the other child is a girl is $\frac{1}{2}$

We see that the student is wrong, since we do get a different probability for the other child being a girl is we know that the first child is the older or the younger girl compared to if we know only that the first child is a girl. The reason this happens is because in each case we have different sample space to use.

10 chapter 16, problem 2.14, Mary Boas, second edition

Problem

2 dice are thrown, use the sample space in 2.4 to answer the following questions.

Solution

Sample space 2.4 is

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Here $n_s = 36$.

The entry in the above same space shows the number from the first die throw, followed by the number from the second die throw.

(a) What is the probability of being able to form a 2 digit number greater than 33 with the 2 numbers of the dice?

Looking at the sample space above, we see that these numbers in bold are all greater than 33:

1,1	1,2	1,3	**1,4**	**1,5**	**1,6**
2,1	2,2	2,3	**2,4**	**2,5**	**2,6**
3,1	3,2	3,3	**3,4**	**3,5**	**3,6**
4,1	**4,2**	**4,3**	**4,4**	**4,5**	**4,6**
5,1	**5,2**	**5,3**	**5,4**	**5,5**	**5,6**
6,1	**6,2**	**6,3**	**6,4**	**6,5**	**6,6**

Hence the Probability is $\frac{27}{36} = \frac{3}{4}$

(b) Repeat part (a) for the probability. of being able to form a 2 digit number greater than or equal to 42.

1,1	1,2	1,3	1,4	**1,5**	**1,6**
2,1	2,2	2,3	**2,4**	**2,5**	**2,6**
3,1	3,2	3,3	**3,4**	**3,5**	**3,6**
4,1	**4,2**	**4,3**	**4,4**	**4,5**	**4,6**
5,1	**5,2**	**5,3**	**5,4**	**5,5**	**5,6**
6,1	**6,2**	**6,3**	**6,4**	**6,5**	**6,6**

Here the numbers in bold meet the condition. So Probability is $\frac{25}{36}$

(c) Can you find a 2 digit number (or numbers) such that the probability. of being able to form a larger number is the same as the probability. of being able to form a small number?

Let me write all the numbers that can occur in sequence.

$$11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53,$$

$$54, 55, 56, 61, 62, 63, 64, 65, 66$$

Since there are 36 numbers, we want to find the middle of the above sequence such that there are as many numbers above as below.

We see that the numbers after the 18th indexed number and before the 19th indexed number will meet this criteria. The 18th number is 36 and the 19th number is 40.

Hence the numbers with the probability that to form a larger number is the same as the probability. of being able to form a small number are

$37, 38, 39, 40$

11 chapter 16, problem 2.15, Mary Boas, second edition

Use sample space in 2.4 and sample space 2.5 to answer the following questions about a toss of 2 dice.

Solution

Sample space 2.4 is

$$\begin{matrix} 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{matrix}$$

Here $n_s = 36$.

Sample space 2.5 is

sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Sum means the sum of one die throw and the second die throw. So Max sum 12 means 6 + 6, etc...

(a)What is the probability. that the sum is ≤ 4?

Here the probability. is $\frac{1}{36}+\frac{2}{36}+\frac{3}{36}=\frac{6}{36}=\frac{1}{6}$

(b) What is the probability. that the sum is even.

These are the sample points $\{2,4,6,8,10,12\}$

Here the probability. is the sum of the probability of each of these sample points, which is

$\frac{1}{36}+\frac{3}{36}+\frac{5}{36}+\frac{5}{36}+\frac{3}{36}+\frac{1}{36}=\frac{18}{36}=\boxed{\frac{1}{2}}$

To verify, I can use the 2.4 sample space to mark those points which sum to even number

$$\begin{matrix} 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{matrix}$$

We see that these are half the points. Which agrees with the above.

(c) What is the probability. that the sum is divisible by 3?

The sums that are divisible by 3 are: $\{3,6,9,12\}$

Here the probability. is the sum of the probability of each of these sample points, which is
$\frac{2}{36}+\frac{5}{36}+\frac{4}{36}+\frac{1}{36}=\frac{12}{36}=\frac{1}{3}$

(d) If the sum is odd, what is the probability. that it is equal to 7?

Here the sample space is $\{3,5,7,9,11\}$

Since here the events are not equally likely, *I can not say* that probability of it being a 7 is $\frac{1}{5}$, instead, the probability. is given by

$$
\begin{aligned}
&= \frac{\Pr(7)}{\Pr(3) + \Pr(5) + \Pr(7) + \Pr(9) + \Pr(11)} \\
&= \frac{\frac{6}{36}}{\frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36}} \\
&= \frac{6}{2 + 4 + 6 + 4 + 2} \\
&= \frac{6}{18} \\
&= \frac{1}{3}
\end{aligned}
$$

(e) What is the probability that the product of the numbers on the 2 dice is 12?

Using sample space 2.4, the numbers marked in italic have product that is 12

$$
\begin{array}{cccccc}
1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\
2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\
3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\
4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\
5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\
6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6
\end{array}
$$

So there are 4 tosses that can result in number whose product is 12. Hence the probability is $\frac{4}{36} = \frac{1}{9}$

12 chapter 16, problem 2.17, Mary Boas, second edition

Two dice are thrown. Given the information that the number on the first die is even and the number on the second is < 4, set up an appropriate sample space and answer the following questions

Solution

Sample space is $S = \{(2,1),(2,2),(2,3),(4,1),(4,2),(4,3),(6,1),\ (6,2),(6,3)\}$

Where in (a,b), a, is the first die number (which must be even), and b is the second die number (which must be < 4). hence $n_s = 9$, i.e. 9 sample points.

(a) what are the probable sums and their probabilities?

Possible sums are (1:1) corresponding with the sample space above :

$Sum = \{3,4,5,5,6,7,7,\ 8,9\}$

Hence $\Pr(3) = \frac{1}{9}; \Pr(4) = \frac{1}{9}; \Pr(5) = \frac{2}{9}; \Pr(6) = \frac{1}{9}; \Pr(7) = \frac{2}{9}; \Pr(8) = \frac{1}{9}; \Pr(9) = \frac{1}{9}$

(b) What are the most probable sums?

From above we see it is 5 and 7

(c) What is the probability that the sum is even?

From the sum sample space $Sum = \{3,\overbrace{4},5,5,\overbrace{6},7,7,\ \overbrace{8},9\}$ we see that the probability of the sum is even $= \frac{3}{9} = \frac{3}{9}$