30/40

HW # 1

Math 121 B

UC Berkeley.

spring 2004

by NASSER ABBASI

problem ch 11, section 3, number 4

Q) Evaluate $\Gamma(5.7)$ using tables and recursion relation $\Gamma(P+1)=P\,\Gamma(P)$

A) Table for $\Gamma(x)$ is given for $1 \le x \le 2$ (using Handbook of math. functions by Abramowitz). page 267.

so $\Gamma(5.7) = 4.7\,\Gamma(4.7) = (4.7)(3.7)\,\Gamma(3.7) = (4.7)(3.7)(2.7)\,\Gamma(2.7)$

$= (4.7)(3.7)(2.7)(1.7)\,\Gamma(1.7)$.

from Table, $\Gamma(1.7) = 0.9086387329$

hence $\boxed{\Gamma(5.7) = 72.52763452395129}$

Problem ch 11, section 3, number 8

Q) express the following integral as Γ function and evaluate using table of Γ function

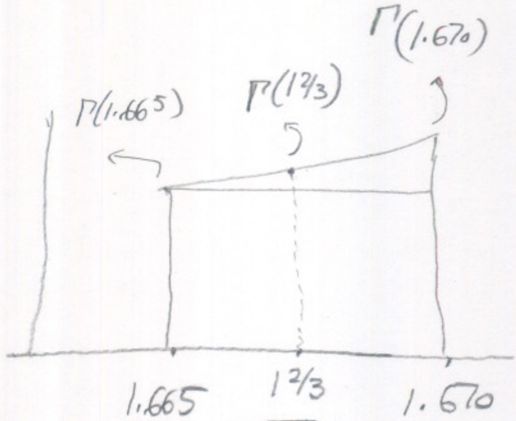
$$\int_0^\infty x^{2/3} e^{-x}\, dx$$

A) since by definition, $\Gamma(P) = \int_0^\infty x^{P-1} e^{-x}\, dx \qquad P>0$

hence here $P-1 = 2/3$ or $\boxed{P = 1\frac{2}{3}}$

From Table, There is no value for $\Gamma(1\frac{2}{3})$, but There is a value for $\Gamma(1.665)$ and $\Gamma(1.670)$. and $\Gamma(1\frac{2}{3})$ is between These two values. so use interpolation to find $\Longrightarrow$

$$\Gamma(1.665) = 0.9024728748$$
$$\Gamma(1.670) = 0.9032964995$$



PS. I am assuming constant slope, which is not accurate, but better than using closest value $\Gamma(1.665)$.

$$\frac{\Gamma(1\frac{2}{3}) - \Gamma(1.665)}{1\frac{2}{3} - 1.665} = \frac{\Gamma(1.670) - \Gamma(1.665)}{1.670 - 1.665}$$

$$\Gamma(1\tfrac{2}{3}) = \left(\frac{\Gamma(1.670) - \Gamma(1.665)}{1.670 - 1.665}\right)\left(1\tfrac{2}{3} - 1.665\right) + \Gamma(1.665)$$

$$\boxed{\Gamma\left(\tfrac{5}{3}\right) = 0.902745292950933 6}$$

problem ch 11, section 3, number 13

Q) express following integral as Γ function and evaluate using Tables.

$$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx$$

A) let $x = e^{-u}$

so $dx = -e^{-u}\,du$

when $x=0 \Rightarrow u=\infty$

when $x=1 \Rightarrow u=0$

hence integral becomes $I = \int_\infty^0 e^{-2u}\left(\ln\frac{1}{e^{-u}}\right)^3 (-e^{-u}du)$

but $\int_\infty^0 = -\int_0^\infty$, hence

$$I = \int_0^\infty e^{-3u}\left(\ln e^{u}\right)^3 du$$

but $\ln e^u = u$

so integral becomes $I = \int_0^\infty e^{-3u}u^3\,du = \int_0^\infty u^3 e^{-3u}\,du$

to convert to form $\int_0^\infty x^{P-1}e^{-x}dx$, let $3u = x$

hence $3du = dx$

when $u=0 \Rightarrow x=0$

when $u=\infty \Rightarrow x=\infty$

$$I = \int_0^\infty \left(\frac{x}{3}\right)^3 e^{-x}\frac{dx}{3} = \left(\frac{1}{3}\right)\left(\frac{1}{27}\right)\int_0^\infty x^3 e^{-x}dx$$

i.e $P-1=3$, hence $\Gamma(4) = \int_0^\infty x^3 e^{-x}dx \Rightarrow \boxed{I = \left(\frac{1}{3}\right)\left(\frac{1}{27}\right)\Gamma(4)} \Rightarrow$

To find $\Gamma(4)$

$$\Gamma(4) = 3\Gamma(3) = (3)(2)\,\Gamma(2) = (3)(2)\overbrace{(1.0)}^{\Gamma(2)} = 6$$

So $$I = \left(\frac{1}{3}\right)\left(\frac{1}{27}\right)\cancelto{2}{6} = \boxed{\frac{2}{27}}$$

$$\boxed{= 0.074074074\ldots}$$

chapter 11, problem 4.5

a) Evaluate the following Γ function using $\Gamma(P) = \frac{1}{P}\Gamma(P+1)$ and tables:

$$\Gamma(-2.3)$$

A) $$\Gamma(-2.3) = \frac{1}{-2.3}\Gamma(-1.3)$$

$$= \left(-\frac{1}{2.3}\right)\left(-\frac{1}{1.3}\right)\Gamma(-.3)$$

$$= \left(-\frac{1}{2.3}\right)\left(-\frac{1}{1.3}\right)\left(-\frac{1}{.3}\right)\Gamma(0.7)$$

$$= \left(-\frac{1}{2.3}\right)\left(-\frac{1}{1.3}\right)\left(-\frac{1}{.3}\right)\left(\frac{1}{0.7}\right)\underbrace{\Gamma(1.7)}_{= 0.908638732 9}$$

From Table.

so $$\boxed{\Gamma(-2.3) = -1.4471073943303074}$$

Chap 11, problem 4.7

Q) using table of Γ, sketch Γ between 1 and 2; then compute few points and sketch it from -4 to $+4$.

A) From Table (look at 1, 1.25, 1.5, 1.75, 2.0)

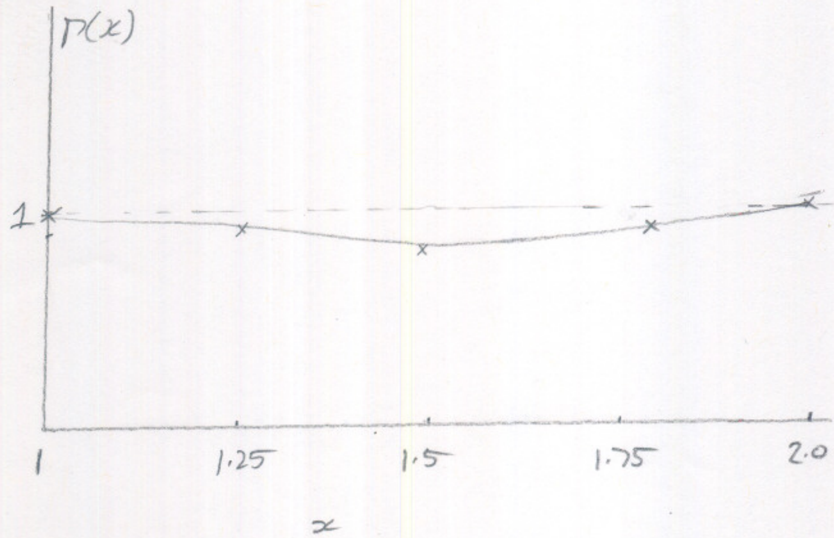
$\Gamma(1) = 1$

$\Gamma(1.25) = 0.9064$

$\Gamma(1.5) = 0.88622$

$\Gamma(1.75) = 0.91906$

$\Gamma(2.0) = 1$



To sketch from -4 to $+4$, find Γ at (.5) intervals

$\Gamma(4) = 3\Gamma(3) = (3)(2)\overbrace{\Gamma(2)}^{=1} = 6$

$\Gamma(3.5) = 2.5\,\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.88622) = 3.32335$

$\Gamma(3) = 2\Gamma(2) = 2$

$\Gamma(2) = 1$

$\Gamma(1.5) = 0.88622$

$\Gamma(1) = 1$

$\Gamma(.5) = \frac{1}{.5}\Gamma(1.5) = \frac{1}{.5}\,0.88622 = 1.77245$

$\Gamma(0) = \infty$

$\Gamma(-.5) = \frac{1}{-.5}\Gamma(.5) = \frac{1}{-.5}\frac{1}{.5}\Gamma(1.5) = \frac{1}{-.5}\frac{1}{.5}\,0.88622 = -3.54491$

$\Gamma(-1.0) = \infty$ (since Γ has singularity at all negative integers)

$\Gamma(-1.5) = \frac{1}{-1.5}\Gamma(-.5) = \frac{1}{-1.5}\frac{1}{-.5}\Gamma(.5) = \frac{1}{-1.5}\frac{1}{-.5}\frac{1}{.5}\Gamma(1.5) = 2.36327$

$\Gamma(-2) = \infty$

$\Gamma(-2.5) = \frac{1}{-2.5}\Gamma(-1.5) = \frac{1}{-2.5}\frac{1}{-1.5}\Gamma(-.5) = \frac{1}{-2.5}\frac{1}{-1.5}\frac{1}{-.5}\Gamma(.5)$

$= \frac{1}{-2.5}\frac{1}{-1.5}\frac{1}{-.5}\frac{1}{.5}\Gamma(1.5)$

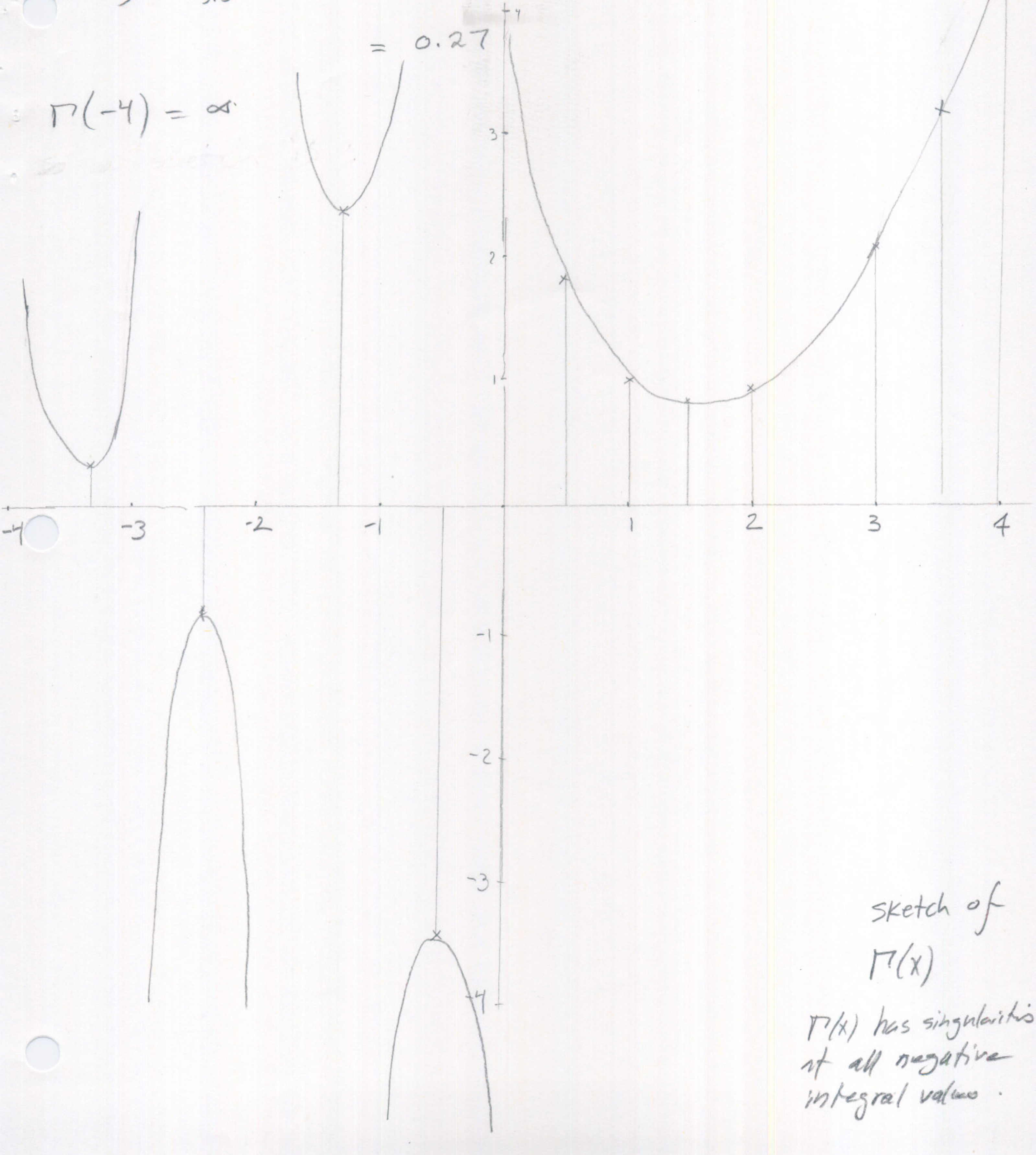
$= -0.945309$

$\Gamma(-3) = \infty$

$$\Gamma(-3.5) = \frac{1}{-3.5}\,\Gamma(-2.5) = \frac{1}{-3.5}\,\frac{1}{-2.5}\,\underbrace{\Gamma(-1.5)}_{=2.36327 \text{ from above}}$$

$$= 0.27$$

$\Gamma(-4) = \infty$



sketch of $\Gamma(x)$

$\Gamma(x)$ has singularities at all negative integral values.

chapter 11, 5.1

prove that for positive integral n

$$\Gamma(n+\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

since n is positive integral, then use

$\Gamma(P+1) = P\,\Gamma(P)$ to expand $\Gamma(n+\frac{1}{2})$.

so $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})\,\Gamma(n-\frac{1}{2})$

apply again to expand $\Gamma(n-\frac{1}{2})$

$$\Gamma(n-\tfrac{1}{2}) = (n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$$

$$\Gamma(n-\tfrac{3}{2}) = (n-\tfrac{5}{2})\,\Gamma(n-\tfrac{5}{2})$$

continue untill we get to $\Gamma(1.5)$ which is $\frac{1}{2}\Gamma(\frac{1}{2})$

hence

$$\Gamma(n+\tfrac{1}{2}) = \overbrace{(n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\cdots(\tfrac{1}{2})}^{n \text{ times}}\,\Gamma(\tfrac{1}{2})$$

for example, for $n=4$

$$\Gamma(4+\tfrac{1}{2}) = \overbrace{(3\tfrac{1}{2})(2\tfrac{1}{2})(1\tfrac{1}{2})(\tfrac{1}{2})}^{4 \text{ terms}}\,\Gamma(\tfrac{1}{2})$$

so $$\Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)}{2}\,\frac{(2n-3)}{2}\,\frac{(2n-5)}{2}\cdots\frac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots(1)}{2^n}\,\Gamma(\tfrac{1}{2})$$

$$= \frac{(1)(3)\cdots(2n-5)(2n-3)(2n-1)}{2^n}\,\Gamma(\tfrac{1}{2}) \Longrightarrow$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-3)\cdot(2n-1)}{2^n}\sqrt{\pi} \quad \text{—— (1)}$$

now need to show that $\dfrac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n} \Rightarrow \dfrac{(2n)!}{4^n\, n!}$

now $1\cdot 2\cdot 3\cdot 4\cdot 5 \cdots (2n-3)(2n-2)(2n-1)(2n) = (2n)!$

so $$\frac{(2n)!}{2\cdot 4\cdot 6\cdot 8 \cdots (2n-2)(2n)} = 1\cdot 3\cdot 5 \cdots (2n-1) \quad \text{—— (2)}$$

but $2\cdot 4\cdot 6\cdot 8 \cdots (2n-2)(2n) = 2[1\cdot 2\cdot 3\cdot 4 \cdots n]$ by factoring 2 out.

$$= 2\, n!$$

hence from (2)

$$\frac{(2n)!}{2\, n!} = 1\cdot 3\cdot 5 \cdots (2n-1)$$

sub the above back into equation (1), I get

$$\frac{\frac{(2n)!}{2\, n!}}{2^n}\sqrt{\pi}$$

or $$\boxed{\frac{(2n)!}{4^n\, n!}\sqrt{\pi} = \Gamma\left(n+\tfrac{1}{2}\right)}$$

QED

Chapter 11, problem ~~7.1~~ 6.1

1) Prove that $\beta(p,q) = \beta(q,p)$ hint: put $x = 1-y$

by definition

$$\beta(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx \qquad p>0 \quad q>0 \quad \text{— (1)}$$

so need to show that $\int_0^1 x^{p-1} (1-x)^{q-1} dx \equiv \int_0^1 x^{q-1} (1-x)^{p-1} dx$.

let $x = 1-y$ in (1)

$dx = -dy$

when $x = 0 \Rightarrow y = 1$
when $x = 1 \Rightarrow y = 0$

so (1) becomes $\int_1^0 (1-y)^{p-1} (1-(1-y))^{q-1} (-dy)$

$$= \int_1^0 (1-y)^{p-1} (y)^{q-1} (-dy) = \int_0^1 (1-y)^{p-1} y^{q-1} dy \text{ — (2)}$$

but y is a dummy variable, so in (2), rewrite 'y' as 'x'

$$= \int_0^1 x^{q-1} (1-x)^{p-1} dx \text{ ——— (3)}$$

but (3) is definition of $\beta(q,p)$. hence

$$\beta(p,q) = \beta(q,p)$$

QED

problem chapter 11, 6.2

Q) prove $B(P,q) = \int_0^\infty \frac{y^{P-1}}{(1+y)^{P+q}} dy$. ——— 1

from definition $\beta(P,q) = \int_0^1 x^{P-1} (1-x)^{q-1} dx$ $p>0$ $q>0$.

let $x = \frac{y}{1+y}$ into above equation:

when $x=0 \Rightarrow y=0$

when $x=1 \Rightarrow \frac{y}{1+y} = 1$ i.e $y=\infty$

$$\frac{dx}{dy} = \frac{d}{dy}(y)(1+y)^{-1} = 1(1+y)^{-1} + y(-1)(1+y)^{-2} = \frac{1}{(1+y)} - \frac{y}{(1+y)^2}$$

$$= \frac{1+y-y}{(1+y)^2} = \frac{1}{(1+y)^2}$$

so $dx = \frac{1}{(1+y)^2} dy$

$$\beta(P,q) = \int_0^\infty \left(\frac{y}{1+y}\right)^{P-1} \left(1-\frac{y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{y^{P-1}}{(1+y)^{P-1}} \left(\frac{1+y-y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2} dy = \int_0^\infty \frac{y^{P-1}}{(1+y)^{P-1}} \frac{1}{(1+y)^{q-1}} \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{y^{P-1}}{(1+y)^{P-1+q-1+2}} dy = \int_0^\infty \frac{y^{P-1}}{(1+y)^{P+q}} dy$$

QED

problem ch 11, 7.3

express following integral as β function, hence in terms of Γ functions, and evaluate from Tables.

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 \frac{dx}{(1-x^3)^{1/2}} = \int_0^1 (1-x^3)^{-\frac{1}{2}} dx$$

Compare to $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$ ——— (1)

let $x^3 = y$

so $x = y^{1/3} \Rightarrow dx = \frac{1}{3} y^{-2/3} dy$

when $x=0 \Rightarrow y=0$

when $x=1 \Rightarrow y=1$

so $\int_0^1 (1-x^3)^{-1/2} dx = \int_0^1 (1-y)^{-1/2} \frac{1}{3} y^{-2/3} dy = \frac{1}{3}\int_0^1 y^{-\frac{2}{3}} (1-y)^{-\frac{1}{2}} dy.$

since y is dummy variable, rewrite as

$I = \frac{1}{3}\int_0^1 x^{-\frac{2}{3}} (1-x)^{-\frac{1}{2}} dx$. Compare to (1)

hence need $p-1 = -\frac{2}{3}$ and $q-1 = -\frac{1}{2}$

i.e $p = \frac{1}{3}$ and $q = \frac{1}{2}$

hence $I = \frac{1}{3}\beta(p,q) = \frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)}$

$= \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)}$, but $\Gamma\left(\frac{1}{3}\right) = \frac{1}{1/3}\Gamma\left(\frac{4}{3}\right)$ so $\Longrightarrow$

$\frac{1}{1/3}\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right)$ $\Gamma\left(\frac{4}{3}\right)\sqrt{\pi}$

From Tables, $\Gamma\left(\frac{4}{3}\right) \cong \Gamma(1.335) = 0.89278$

$\Gamma\left(\frac{5}{6}\right) = \frac{1}{5/6}\Gamma\left(\frac{11}{6}\right) = \frac{6}{5}\underbrace{\Gamma(1.8333)}_{\cong 0.93969}$

so $\boxed{I = 1.403}$ ✓

P.S. To get more accurate result need to use interpolation to find $\Gamma(x)$ for x values not in table. here I used closest value in Table instead.

problem chapter 11, 7.7 (14)

a) Express following integral as β function, hence in terms of Γ functions and evaluate using Table

$$I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} (\sin\theta)^{-1/2}\, d\theta \quad \text{——— (1)}$$

The trig. Form of Beta function is

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta$$

Compare with (1).

need $2p-1 = -\frac{1}{2}$, $2q-1 = 0$

hence $p = \frac{-\frac{1}{2}+1}{2} = .25$, $q = \frac{1}{2}$

so $I = \frac{1}{2}\beta(p,q) = \frac{1}{2}\beta(.25, .5)$

$$I = \frac{1}{2}\,\frac{\Gamma(.25)\,\Gamma(.5)}{\Gamma(0.75)}$$

$\Gamma(.25) = \frac{1}{.25}\Gamma(1.25)$, $= 4\,\Gamma(1.25) = 4\,(0.9064)$

$\Gamma(0.75) = \frac{1}{0.75}\Gamma(1.75) = \frac{1}{.75}(0.91906)$

hence $\boxed{I = 2.62206}$

7.9? 11.8?
11.2? 11.9?
11.5?