Math 121a, Spring 2004, F.Rezakhanlou

1. (5 points) Define

$$f(z) = \frac{(z+2)^2(z+3)}{1-\cos \pi z}.$$

Find all the isolated singularitis of f and identify each as a removable, a pole (give the order) or an essential singularity. What is the radius of convergence of the Taylor expansion of f at $z = 1/2$?

The denominator vanishes when $\cos \pi z = 1$ or $z = 2k$ for some integer k. Since for $\varphi(z) = 1 - \cos \pi z$ we have $\begin{cases} \varphi'(z) = -\pi \sin \pi z, \\ \varphi''(z) = -\pi^2 \cos \pi z \end{cases}$ we have $\varphi'(2k) = 0$, $\varphi''(2k) \neq 0$, so each $2k$ is a zero of order 2. This means $z = -2$ is a removable singularity but $2k$ for $k \neq -1$ is a pole of order 2.

The radius of convergence at $z = 1/2$ is $\frac{1}{2}$ because a disk of center $1/2$, radius $1/2$ is the largest to avoid singularities.

2. (4 point) Find the Laurent expansion of $f(z) = (1+z^2)^{-1} + (z+3)^{-1}$ in the set $\{z : 1 < |z| < 3\}$.

First $\frac{1}{1+z^2} = \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} = \sum_0^\infty (-1)^n z^{-2n-2}$ is convergent for $|z| > 1$. Also

$$\frac{1}{z+3} = \frac{1}{3} \frac{1}{1+z/3} = \frac{1}{3} \sum_{n=0}^\infty \left(\frac{z}{3}\right)^n (-1)^n \text{ is convergent for}$$

$|z| < 3$. Hence for $1 < |z| < 3$,

$$f(z) = \sum_0^\infty (-1)^n z^{-2-2n} + \sum_0^\infty (-1)^n z^n 3^{-n-1}.$$

3. (6 points) Evaluate $\int_0^{2\pi} \frac{1}{\sin\theta - 3i} d\theta$ and $\int_0^{\infty} \frac{\cos 2x}{x^2} \ dx$.

① $\int_0^{2\pi} \frac{d\theta}{\sin\theta - 3i} = \int_C \frac{1}{(z-\frac{1}{z})\frac{1}{2i} - 3i} \frac{dz}{iz} = \int_C \frac{2\,dz}{6z + z^2 - 1}$ C unit circle

$z^2 + 6z - 1 = 0 \Rightarrow z = -3 \pm \sqrt{10}$, $z_0 = -3 + \sqrt{10}$ inside C, so

$\int_0^{2\pi} \frac{d\theta}{\sin\theta - 3i} = 2\pi i \operatorname{Res}_{z_0} F$ with $F(z) = \frac{1}{z - z_0} \frac{2}{z - z_1}$ with $z_1 = -3 - \sqrt{10}$,

$\operatorname{Res}_{z_0} F = \frac{2}{z_0 - z_1} = \frac{1}{\sqrt{10}}$. Hence integral $= \frac{2\pi i}{\sqrt{10}}$

② $\int_0^{\infty} \frac{\cos 2x}{x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2x}{x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2} dx$

The function $F(z) = \frac{e^{i2z}}{z^2}$ has a simple pole at $z = 0$. So

4. (2 points) Find
$$\frac{d}{dx} \int_{x^2}^{\cos x} e^{xt^2} dt.$$

$$= \int_{x^2}^{\cos x} t^2 e^{xt^2} dt - e^{x\cos^2 x} \sin x - 2x e^{x^5}$$

$\int_{C_R} F + \int_{C_\epsilon} F + \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right) F = 0.$

First $\int_{C_R} F = 0$ because $|e^{2iz}| = e^{2\operatorname{Im} z} \leq 1$, so

$\left| \int_{C_R} \frac{F(z)}{z^2} \right| \leq \pi R \frac{1}{R^2} \to 0$ as $R \to \infty$. Also

$\int_{C_\epsilon} F = \int_{C_\epsilon} \frac{e^{2iz} - 2iz}{z^2} dz + \int_{C_\epsilon} \frac{2iz}{z^2}$, first integral goes to

zero because the integrand is bounded. The second one is $2 \int_{C_\epsilon} \frac{i}{z} = -2\pi$.

Thus $\int^{\infty} \frac{\cos 2x}{} = \pi$.

5. (4 points) Assume

$$xz_x + yz_y + z = 0 \qquad (1)$$

where $z = z(x, y)$ and z_x and z_y denote the partial derivatives of z with respect to x and y. Make the change of variables $r = x$, $t = y/x$. Derive an equation for z as a function of r and t. Use this equation to solve the equation (1).

$$z_x = z_r \frac{\partial r}{\partial x} + z_t \frac{\partial t}{\partial x} = z_r + z_t\left(\frac{-y}{x^2}\right) = z_r - z_t \frac{t}{r}$$

$$z_y = z_r \frac{\partial r}{\partial y} + z_t \frac{\partial t}{\partial y} = z_t \frac{1}{x} = z_t \frac{1}{r}$$

So $xz_x + yz_y + z = r\left(z_r - \frac{t}{r} z_t\right) + tr\, z_t \frac{1}{r} + z = rz_r + z = 0.$

Hence $\frac{\partial z}{\partial r} = -\frac{z}{r}$, $\frac{dz}{z} = -\frac{dr}{r} \Rightarrow \ln z + c = -\ln r$,

$z = \frac{c}{r}$. But C could depend on t. Thus

$$z = \frac{1}{r} C(t) = \frac{1}{x} C\left(\frac{y}{x}\right).$$

6. (5 points)Find the smallest and largest value of $F = yz + x^2 + z$ on the sphere $x^2 + y^2 + z^2 = 1$.

Using Lagrange multiplier, $\nabla F = \lambda \nabla G$, hence at a critical point $(2x, z, y+1) = \lambda(2x, 2y, 2z)$. If $x \neq 0$ we get $\lambda = 1$, $\begin{cases} z = 2y \\ y+1 = 2z \end{cases} \Rightarrow y+1 = 4y \Rightarrow y = \frac{1}{3}, z = \frac{2}{3}$, $x = \sqrt{1 - \frac{5}{9}} = \pm\frac{2}{3}$

The corresponding F is $\frac{2}{9} + \frac{4}{9} + \frac{6}{9} = \frac{12}{9} = \frac{4}{3}$. If $x = 0$ we get

$\begin{matrix} y+1 = 2\lambda z \\ z = 2\lambda y \end{matrix}$ $\quad \frac{y+1}{z} = \frac{z}{y}$, $y^2 + y = z^2$, also $y^2 + z^2 = 1$. So

$y^2 + y = 1 - y^2$, $2y^2 + y - 1 = 0$, $y = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} -1 & \Rightarrow z = 0 \\ 1/2 & \Rightarrow z = \pm\frac{\sqrt{3}}{2} \end{cases}$

The corresponding F are $\begin{cases} 0 & x = 0, z = 0, y = -1 \\ \frac{3\sqrt{3}}{4} & x = 0, y = 1/2, z = \frac{\sqrt{3}}{2} \end{cases}$

Max $= 4/3$