HW 7, Math 121 A
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Contents

1 chapter 14, problem 1.6

Problem Find real and imaginary parts u, v of e^z
Solution
Let $z = x + iy$, then

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

Hence $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

2 chapter 14, problem 1.12

Problem Find real and imaginary parts u, v of $f(z) = \frac{z}{z^2+1}$
Solution
Let $z = x + iy$ then

$$\begin{aligned} z^2 + 1 &= (x + iy)^2 + 1 \\ &= \left(x^2 - y^2 + 1\right) + i(2xy) \end{aligned}$$

Hence

$$f(z) = \frac{x + iy}{(x^2 - y^2 + 1) + i(2xy)}$$

Multiplying numerator and denominator by conjugate of denominator gives

$$\begin{aligned} f(z) &= \frac{(x + iy)\left(\left(x^2 - y^2 + 1\right) - i(2xy)\right)}{((x^2 - y^2 + 1) + i(2xy))((x^2 - y^2 + 1) - i(2xy))} \\ &= \frac{\left(x\left(x^2 - y^2 + 1\right) + y(2xy)\right) + i\left(y\left(x^2 - y^2 + 1\right)(y(2xy))\right)}{(x^2 - y^2 + 1)^2 + (2xy)^2} \\ &= \frac{x\left(x^2 - y^2 + 1\right) + 2xy^2}{(x^2 - y^2 + 1)^2 + (2xy)^2} + i\frac{y\left(x^2 - y^2 + 1\right) - 2x^2 y}{(x^2 - y^2 + 1)^2 + (2xy)^2} \end{aligned}$$

Hence

$$\begin{aligned} u(x, y) &= \frac{x\left(x^2 - y^2 + 1\right) + 2xy^2}{(x^2 - y^2 + 1)^2 + 2xy} \\ v(x, y) &= \frac{y\left(x^2 - y^2 + 1\right) - 2x^2 y}{(x^2 - y^2 + 1)^2 + (2xy)^2} \end{aligned}$$

3 chapter 14, problem 2.22

Problem Use Cauchy-Riemann conditions to find if $f(z) = y + ix$ is analytic.
Solution
CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Here $u = y$ and $v = x$, since $f(z) = z = x + iy$. Therefore $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$ and (1) is satisfied. And $\frac{\partial u}{\partial y} = 1$ and $\frac{\partial v}{\partial x} = 1$, hence (2) is NOT satisfied. Therefore not analytic.

4 chapter 14, problem 2.23

Problem Use Cauchy-Riemann conditions to find if $f(z) = \frac{x-iy}{x^2+y^2}$ is analytic.
Solution
CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Here $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, hence

$$u = \frac{x}{x^2 + y^2}$$

$$v = \frac{-y}{x^2 + y^2}$$

Therefore

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} - \frac{x}{(x^2 + y^2)^2}(2x) = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

And

$$\frac{\partial u}{\partial y} = \frac{-1}{x^2 + y^2} + \frac{y}{(x^2 + y^2)^2}(2y) = \frac{-\left(x^2 + y^2\right) + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Hence (1) is satisfied. And

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

And

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}$$

Hence (2) is satisfied also. Therefore $f(z)$ is analytic.

5 chapter 14, problem 2.34

Problem Write power series about origin for $f(z) = \ln(1-z)$. Use theorem 3 to find circle of convergence for each series.
Solution
From page 34, for $-1 < x \leq 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence

$$\begin{aligned}\ln(1-z) &= (-z) - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \frac{(-z)^4}{4} + \cdots \\ &= -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots \\ &= -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right) \\ &= -\sum_{n=1} \frac{1}{n} z^n\end{aligned}$$

To find radius of convergence, use ratio test.

$$\begin{aligned}L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\ &= \lim_{n\to\infty} \frac{\left|\frac{1}{n+1}\right|}{\left|\frac{1}{n}\right|} \\ &= \lim_{n\to\infty} \frac{n}{n+1} \\ &= 1\end{aligned}$$

Hence $R = \frac{1}{L} = 1$. Therefore converges for $|z| < 1$.

6 chapter 14, problem 2.37

Problem Find circle of convergence for $\tanh(z)$
Solution

$$\tanh(z) = -i\tan(iz)$$

But $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{325}x^7 + \cdots$, therefore

$$\begin{aligned}\tanh(z) &= -i\left(iz + \frac{(iz)^3}{3} + \frac{2}{15}(iz)^5 + \frac{17}{325}(iz)^7 + \cdots\right) \\ &= -i\left(iz - \frac{iz^3}{3} + \frac{2}{15}iz^5 + \cdots\right) \\ &= z - \frac{z^3}{3} + \frac{2}{15}z^5 + \cdots\end{aligned}$$

This is the power series of $\tanh(z)$ about $z = 0$. Since $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{\sinh(z)}{\cos(iz)}$ and $\cos(iz) = 0$ at $iz = \pm\frac{\pi}{2}$ then $|z| < \frac{\pi}{2}$ to avoid hitting a singularity. So radius of convergence is $R = \frac{\pi}{2}$.

7 chapter 14, problem 2.40

Problem Find series and circle of convergence for $\frac{1}{1-z}$
Solution
From Binomial expansion

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

For $|z| < 1$. Hence $R = 1$.

8 chapter 14, problem 2.55

Problem Show that $3x^2y - y^3$ is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.
Solution
The given function is the real part of $f(z)$. Hence $u(x,y) = 3x^2y - y^3$. To show this is harmonic, means it satisfies $\nabla^2 u = 0$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. But

$$\begin{aligned}\frac{\partial u}{\partial x} &= 6xy \\ \frac{\partial^2 u}{\partial x^2} &= 6y \\ \frac{\partial u}{\partial y} &= 3x^2 - 3y^2 \\ \frac{\partial^2 u}{\partial y^2} &= -6y\end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence $u(x,y)$ is harmonic. Now, we want to find $f(z) = u(x,y) + iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since $f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x} = 6xy$, so (1) gives

$$\begin{aligned}6xy &= \frac{\partial v}{\partial y} \\ v(x,y) &= \int 6xydy \\ &= 3xy^2 + g(x) \qquad (3)\end{aligned}$$

From (2) we obtain

$$-3x^2 + 3y^2 = \frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x}=3y^2+g'(x)$, hence the above becomes

$$\begin{aligned}-3x^2+3y^2 &= 3y^2+g'(x)\\ g'(x) &= -3x^2\\ g(x) &= \int -3x^2 dx\\ &= -x^3+C\end{aligned}$$

Therefore from (3), we find that

$$v(x,y)=3xy^2-x^3+C$$

We can set any value to C. Let $C=0$ to simplify things. Hence

$$\begin{aligned}f(z) &= u+iv\\ &= \left(3x^2y-y^3\right)+i\left(3xy^2-x^3\right)\end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace.

$$\begin{aligned}\frac{\partial v}{\partial x} &= 3y^2-3x^2\\ \frac{\partial^2 v}{\partial x^2} &= -6x\\ \frac{\partial v}{\partial y} &= 6xy\\ \frac{\partial^2 v}{\partial y^2} &= 6x\end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$. QED.

9 chapter 14, problem 2.55

Problem Show that xy is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.

Solution

The given function is the real part of $f(z)$. Hence $u(x,y)=xy$. To show this is harmonic, means it satisfies $\nabla^2 u=0$ or $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. But

$$\begin{aligned}\frac{\partial u}{\partial x} &= y\\ \frac{\partial^2 u}{\partial x^2} &= 0\\ \frac{\partial u}{\partial y} &= x\\ \frac{\partial^2 u}{\partial y^2} &= 0\end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, hence $u(x,y)$ is harmonic. Now, we want to find $f(z)=u(x,y)+iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since

$f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z) = u + iv$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x} = y$, so (1) gives

$$\begin{aligned} y &= \frac{\partial v}{\partial y} \\ v(x,y) &= \int y dy \\ &= \frac{y^2}{2} + g(x) \end{aligned} \tag{3}$$

From (2) we obtain

$$-x = \frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x} = g'(x)$, hence the above becomes

$$\begin{aligned} -x &= g'(x) \\ g(x) &= \int -x dx \\ &= -\frac{x^2}{2} + C \end{aligned}$$

Therefore from (3), we find that

$$v(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + C$$

We can set any value to C. Let $C = 0$ to simplify things. Hence

$$\begin{aligned} f(z) &= u + iv \\ &= (xy) + i\left(\frac{y^2 - x^2}{2}\right) \end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace.

$$\begin{aligned} \frac{\partial v}{\partial x} &= -x \\ \frac{\partial^2 v}{\partial x^2} &= -1 \\ \frac{\partial v}{\partial y} &= y \\ \frac{\partial^2 v}{\partial y^2} &= 1 \end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. QED.

10 chapter 14, problem 2.60

Problem Show that $\ln\left(x^2+y^2\right)$ is harmonic, that is, it satisfies Laplace equation, and find a function $f(z)$ of which this function is the real part. Show that the function $v(x,y)$ which you also find also satisfies Laplace equation.

Solution

The given function is the real part of $f(z)$. Hence $u(x,y)=xy$. To show this is harmonic, means it satisfies $\nabla^2 u=0$ or $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. But

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{2x}{x^2+y^2}\\
\frac{\partial^2 u}{\partial x^2} &= 2\left(\frac{1}{x^2+y^2}\right)+2x\left(\frac{-1}{\left(x^2+y^2\right)^2}(2x)\right)\\
&= \frac{2}{x^2+y^2}-\frac{4x^2}{\left(x^2+y^2\right)^2}\\
&= \frac{2\left(x^2+y^2\right)-4x^2}{\left(x^2+y\right)^2}\\
&= \frac{-2x^2+2y^2}{\left(x^2+y\right)^2}\\
\frac{\partial u}{\partial y} &= \frac{2y}{x^2+y^2}\\
\frac{\partial^2 u}{\partial y^2} &= 2\left(\frac{1}{x^2+y^2}\right)+2y\left(\frac{-1}{\left(x^2+y^2\right)^2}(2y)\right)\\
&= \frac{2}{x^2+y^2}-\frac{4y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{2\left(x^2+y^2\right)-4y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} &= \frac{-2x^2+2y^2}{\left(x^2+y\right)^2}+\frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}\\
&= 0
\end{aligned}$$

Hence $u(x,y)$ is harmonic. Now, we want to find $f(z)=u(x,y)+iv(x,y)$ and analytic function, where its real part is what we are given above. So we need to find $v(x,y)$. Since $f(z)$ is analytic, then we apply Cauchy-Riemann equations to find $v(x,y)$ CR says a complex function $f(z)=u+iv$ is analytic if

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{2}$$

But $\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}$, so (1) gives

$$\begin{aligned}\frac{2x}{x^2+y^2} &= \frac{\partial v}{\partial y}\\ v(x,y) &= \int \frac{2x}{x^2+y^2}dy\\ &= 2\arctan\left(\frac{y}{x}\right)+g(x)\end{aligned} \tag{3}$$

From (2) we obtain

$$-\frac{2y}{x^2+y^2}=\frac{\partial v}{\partial x}$$

But from (3), we see that $\frac{\partial v}{\partial x}=-\frac{2y}{y^2+x^2}+g'(x)$, hence the above becomes

$$\begin{aligned}-\frac{2y}{x^2+y^2} &= -\frac{2y}{y^2+x^2}+g'(x)\\ g'(x) &= 0\\ g(x) &= C\end{aligned}$$

Therefore from (3), we find that

$$v(x,y)=2\arctan\left(\frac{y}{x}\right)+C$$

We can set any value to C. Let $C=0$ to simplify things. Hence

$$v(x,y)=2\arctan\left(\frac{y}{x}\right)$$

And therefore

$$\begin{aligned}f(z) &= u+iv\\ &= \ln\left(x^2+y^2\right)+i\left(2\arctan\left(\frac{y}{x}\right)\right)\end{aligned}$$

Now we show that $v(x,y)$ is also harmonic. i.e. it satisfies Laplace. We find that

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{4xy}{\left(x^2+y^2\right)^2}\\ \frac{\partial^2 v}{\partial y^2} &= -\frac{4xy}{\left(x^2+y^2\right)^2}\end{aligned}$$

Hence we see that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$. QED.

11 chapter 14, problem 3.3(b)

Problem Find $\oint_C z^2dz$ over the half unit circle arc shown.

Solution
Since $f(z)=z^2$ is clearly analytic on and inside C and no poles are inside, then by Cauchy's theorem $\oint_C z^2dz=0$

12 chapter 14, problem 3.5

Problem Find $\int e^{-z}dz$ along positive part of the line $y=\pi$. This is frequently written as $\int_{i\pi}^{\infty+i\pi} e^{-z}dz$
Solution
Let $z=x+iy$, then

$$\begin{aligned} I &= \int_{i\pi}^{\infty+i\pi} e^{-z}dz \\ &= \int_{i\pi}^{\infty+i\pi} e^{-x}e^{-iy}dz \end{aligned}$$

But $dz=dx+idy$, the above becomes

$$\begin{aligned} I &= \int_{i\pi}^{\infty+i\pi} e^{-x}e^{-iy}\left(dx+idy\right) \\ &= \int_{0}^{\infty} e^{-x}e^{-iy}dx + i\int_{i\pi}^{i\pi} e^{-x}e^{-iy}dy \\ &= \int_{0}^{\infty} e^{-x}e^{-iy}dx \end{aligned}$$

But $y=\pi$ over the whole integration. The above simplifies to

$$\begin{aligned} I &= e^{-i\pi}\int_{0}^{\infty} e^{-x}dx \\ &= e^{-i\pi}\left(\frac{e^{-x}}{-1}\right)_{0}^{\infty} \\ &= -e^{-i\pi}\left(0-1\right) \\ &= e^{i\pi} \\ &= -1 \end{aligned}$$

13 chapter 14, problem 3.17

Problem Using Cauchy integral formula to evaluate $\oint\limits_C \frac{\sin z}{2z-\pi}dz$ where (a) C is circle $|z|=1$ and (b) C is circle $|z|=2$
Solution
For part (a), since the pole is at $z=\frac{\pi}{2}$, it is outside the circle $|z|=1$ and $f(z)$ is analytic inside and on C, then by Cauchy theorem $\oint\limits_C \frac{\sin z}{2z-\pi}dz=0$.
For part(b), since now the pole is inside, then

$$\oint\limits_C \frac{\sin z}{2z-\pi}dz = 2\pi i\,\text{Residue}\left(\frac{\pi}{2}\right)$$

But

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)f(z)\\ &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin z}{2z-\pi}\\ &= \sin\left(\frac{\pi}{2}\right)\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)}{2z-\pi}\end{aligned}$$

Applying L'Hopital

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right)\lim_{z\to\frac{\pi}{2}}\frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

Hence

$$\oint_C \frac{\sin z}{2z-\pi}dz = \pi i$$

14 chapter 14, problem 3.18

Problem Integrate $\oint_C \frac{\sin 2z}{6z-\pi}dz$ over circle $|z| = 3$

Solution

The pole is at $z = \frac{\pi}{6}$. This is inside $|z| = 3$. Hence

$$\oint_C \frac{\sin 2z}{6z-\pi}dz = 2\pi i\,\text{Residue}\left(\frac{\pi}{6}\right)$$

But

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{6}\right) &= \lim_{z\to\frac{\pi}{6}}\left(z-\frac{\pi}{6}\right)\frac{\sin 2z}{6z-\pi}\\ &= \sin\left(\frac{\pi}{3}\right)\lim_{z\to\frac{\pi}{6}}\frac{\left(z-\frac{\pi}{6}\right)}{6z-\pi}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{3}\right)\lim_{z\to\frac{\pi}{6}}\frac{1}{6}\\ &= \frac{1}{6}\sin\left(\frac{\pi}{3}\right)\end{aligned}$$

Hence

$$\oint_C \frac{\sin 2z}{6z-\pi}dz = 2\pi i\left(\frac{1}{6}\sin\left(\frac{\pi}{3}\right)\right)$$

But $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and the above simplifies to

$$\begin{aligned}\oint_C \frac{\sin 2z}{6z-\pi}dz &= 2\pi i\left(\frac{1}{6}\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi i}{2\sqrt{3}}\end{aligned}$$

15 chapter 14, problem 3.19

Problem Integrate $\oint\limits_C \frac{e^{3z}}{z-\ln 2} dz$ if C is square with vertices $\pm 1, \pm i$

Solution

The pole is at $z = \ln 2 = 0.693$ so inside C. Hence

$$\oint\limits_C \frac{e^{3z}}{z-\ln 2} dz = 2\pi i \operatorname{Residue}(\ln 2)$$

But

$$\begin{aligned}\operatorname{Residue}(\ln 2) &= \lim_{z\to \ln 2} (z-\ln 2) f(z) \\ &= e^{3\ln 2} \lim_{z\to \ln 2} \frac{z-\ln 2}{z-\ln 2} \\ &= e^{3\ln 2}\end{aligned}$$

Hence

$$\begin{aligned}\oint\limits_C \frac{e^{3z}}{z-\ln 2} dz &= 2\pi i e^{3\ln 2} \\ &= 2\pi i (2)^3 \\ &= 16\pi i\end{aligned}$$

16 chapter 14, problem 3.20

Problem Integrate $\oint\limits_C \frac{\cosh z}{2\ln 2 - z} dz$ if C is (a) circle with $|z| = 1$ and (b) Circle with $|z| = 2$

Solution

Part (a). Pole is at $z = 2\ln 2 = 1.38$. Hence pole is outside C. Therefore $\oint\limits_C \frac{\cosh z}{2\ln 2 - z} dz = 0$ since $f(z)$ is analytic on C

Part(b). Now pole is inside. Hence

$$\oint\limits_C \frac{\cosh z}{2\ln 2 - z} dz = 2\pi i \operatorname{Residue}(2\ln 2)$$

But

$$\begin{aligned}\operatorname{Residue}(2\ln 2) &= \lim_{z\to 2\ln 2} (z-2\ln 2) f(z) \\ &= \lim_{z\to 2\ln 2} (z-2\ln 2) \frac{\cosh z}{2\ln 2 - z} \\ &= \cosh(2\ln 2) \lim_{z\to \ln 2} \frac{z-2\ln 2}{2\ln 2 - z} \\ &= -\cosh(2\ln 2)\end{aligned}$$

Therefore

$$\begin{aligned}\oint\limits_C \frac{\cosh z}{2\ln 2 - z} dz &= -2\pi i \cosh(2\ln 2) \\ &= -4.25\pi i\end{aligned}$$

17 chapter 14, problem 3.23

Problem Integrate $\oint_C \frac{e^{3z}}{(z-\ln 2)^4} dz$ if C is square between $\pm 1, \pm i$

Solution

The pole is at $z = \ln 2 = 0.69$ which is inside the square. The order is 4. Hence

$$\oint_C \frac{e^{3z}}{(z-\ln 2)^4} dz = 2\pi i \operatorname{Residue}(\ln 2)$$

To find Residue (ln 2) we now use different method from earlier, since this is not a simple pole.

$$\begin{aligned}
\operatorname{Residue}(\ln 2) &= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}(z-\ln 2)^4 f(z) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}(z-\ln 2)^4 \left(\frac{e^{3z}}{(z-\ln 2)^4}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^3}{dz^3}\left(e^{3z}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!}\frac{d^2}{dz^2}\left(3e^{3z}\right) \\
&= \lim_{z\to\ln 2} \frac{1}{3!} 9 \frac{d}{dz} e^{3z} \\
&= \lim_{z\to\ln 2} \frac{1}{3!} 27 e^{3z} \\
&= \lim_{z\to\ln 2} \frac{27}{6} e^{3z} \\
&= \frac{27}{6} e^{3\ln 2} \\
&= (27)\left(\frac{8}{6}\right) \\
&= 36
\end{aligned}$$

Hence

$$\begin{aligned}
\oint_C \frac{e^{3z}}{(z-\ln 2)^4} dz &= 2\pi i 36 \\
&= 72\pi i
\end{aligned}$$

18 chapter 14, problem 4.6

Problem Find Laurent series and residue at origin for $f(z) = \frac{1}{z^2(1+z)^2}$

Solution

There is a pole at $z = 0$ and at $z = -1$. We expand around a disk of radius 1 centered at $z = 0$ to find Laurent series around $z = 0$. Hence

$$f(z) = \frac{1}{z^2}\frac{1}{(1+z)^2}$$

For $|z| < 1$ we can now expand $\frac{1}{(1+z)^2}$ using Binomial expansion

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 + (-2) z + (-2)(-3)\frac{z^2}{2!} + (-2)(-3)(-4)\frac{z^3}{3!} + \cdots\right) \\ &= \frac{1}{z^2}\left(1 - 2z + 3z^2 - 4z^3 + \cdots\right) \\ &= \frac{1}{z^2} - \frac{2}{z} + 3 - 4z + \cdots \end{aligned}$$

Hence residue is -2. To find Laurent series outside this disk, we write

$$\begin{aligned} f(z) &= \frac{1}{z^2}\frac{1}{(1+z)^2} \\ &= \frac{1}{z^2}\frac{1}{\left(z\left(1+\frac{1}{z}\right)\right)^2} \\ &= \frac{1}{z^4}\frac{1}{\left(1+\frac{1}{z}\right)^2} \end{aligned}$$

And now we can expand $\frac{1}{\left(1+\frac{1}{z}\right)^2}$ for $\left|\frac{1}{z}\right| < 1$ or $|z| > 1$ using Binomial and obtain

$$\begin{aligned} f(z) &= \frac{1}{z^4}\left(1 + (-2)\frac{1}{z} + \frac{(-2)(-3)}{2!}\left(\frac{1}{z}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{1}{z}\right)^3 + \cdots\right) \\ &= \frac{1}{z^4}\left(1 - \frac{2}{z} + 3\left(\frac{1}{z}\right)^2 - 4\left(\frac{1}{z}\right)^3 + \cdots\right) \\ &= \frac{1}{z^4} - \frac{2}{z^5} + \frac{3}{z^6} - \frac{4}{z^7} + \cdots \end{aligned}$$

We see that outside the disk, the Laurent series contains only the principal part and no analytical part as the case was in the Laurent series inside the disk.

19 chapter 14, problem 4.7

Problem Find Laurent series and residue at origin for $f(z) = \frac{2-z}{1-z^2}$
Solution
There is a pole at $z = \pm 1$. So we need to expand $f(z)$ for $|z| < 1$ around origin. Here there is no pole at origin, hence the series expansion should contain only an analytical part

$$\begin{aligned} f(z) &= \frac{2-z}{1-z^2} \\ &= \frac{2-z}{(1-z)(1+z)} \\ &= \frac{A}{(1-z)} + \frac{B}{(1+z)} \\ &= \frac{1}{2}\frac{1}{(1-z)} + \frac{3}{2}\frac{1}{(1+z)} \\ &= \frac{1}{2}\left(1 + z + z^2 + z^3 + \cdots\right) + \frac{3}{2}\left(1 - z + z^2 - z^3 + z^4 - \cdots\right) \\ &= 2 - z + 2z^2 - z^3 + 2z^4 - z^5 + \cdots \end{aligned}$$

No principal part. Only analytical part, since $f(z)$ is analytical everywhere inside the region. For $|z| > 1$ we write

$$\begin{aligned}
f(z) &= \frac{1}{2}\frac{1}{(1-z)} + \frac{3}{2}\frac{1}{(1+z)}\\
&= \frac{1}{2z}\frac{1}{\left(\frac{1}{z}-1\right)} + \frac{3}{2z}\frac{1}{\left(\frac{1}{z}+1\right)}\\
&= \frac{-1}{2z}\frac{1}{\left(1-\frac{1}{z}\right)} + \frac{3}{2z}\frac{1}{\left(\frac{1}{z}+1\right)}\\
&= \frac{-1}{2z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right) + \frac{3}{2z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4-\cdots\right)\\
&= \frac{1}{z}-\frac{2}{z^2}+\frac{1}{z^3}-\frac{2}{z^4}+\frac{1}{z^5}-\frac{2}{z^6}+\cdots
\end{aligned}$$

We see that outside the disk, the Laurent series contains only the principal part and no analytical part.

20 chapter 14, problem 4.9

Problem Determine the type of singularity at the point given. If it is regular, essential, or pole (and indicate the order if so). (a) $f(z) = \frac{\sin z}{z}, z = 0$ (b) $f(z) = \frac{\cos z}{z^3}, z = 0$, (c) $f(z) = \frac{z^3-1}{(z-1)^3}, z = 1$, (d) $f(z) = \frac{e^z}{z-1}, z = 1$

Solution

(a) There is a singularity at $z = 0$, but we will check if it removable

$$\begin{aligned}
f(z) &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}\\
&= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots
\end{aligned}$$

So the series contain no principal part (since all powers are positive). Hence we have pole of order 1 which is removable. Therefore $z = 0$ is a regular point.

(b) There is a singularity at $z = 0$, but we will check if it removable

$$\begin{aligned}
f(z) &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3}\\
&= \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{4!} - \cdots
\end{aligned}$$

Hence we could not remove the pole. So the the point is a pole of order 3.

(c) There is a singularity at $z = 1$,

$$\begin{aligned}
f(z) &= \frac{z^3 - 1}{(z-1)^3}\\
&= \frac{(z-1)\left(z^2+1+z\right)}{(z-1)^3}\\
&= \frac{\left(z^2+1+z\right)}{(z-1)^2}
\end{aligned}$$

Hence a pole of order 2.

(d)

$$f(z) = \frac{e^z}{z-1}$$

There is no cancellation here. Hence $z = 1$ is a pole or order 1.

21 chapter 14, problem 4.10

Problem Determine the type of singularity at the point given. If it is regular, essential, or pole (and indicate the order if so). (a) $f(z)=\frac{e^z-1}{z^2+4}, z=2i$ (b) $f(z)=\tan^2 z, z=\frac{\pi}{2}$. (c) $f(z)=\frac{1-\cos(z)}{z^4}, z=0$, (d) $f(z)=\cos\left(\frac{\pi}{z-\pi}\right), z=\pi$

Solution

(a) To find if the point is essential or pole or regular, we expand $f(z)$ around the point, and look at the Laurent series. If the number of b_n terms is infinite, then it is essential singularity. If the number of b_n is finite, then it is a pole of order that equal the largest order of the b_n term. If the series contains only analytical part and no principal part (the part which has the b_n terms), then the point is regular.

So we need to expand $\frac{e^z-1}{z^2+4}$ around $z=2i$. For the numerator, this gives

$$e^z = e^{2i} + (z-2i)\,e^{2i} + (z-2i)^2\frac{e^{2i}}{2!} + \cdots$$

For

$$\begin{aligned}\frac{1}{z^2+4} &= \frac{1}{(z-2i)(z+2i)}\\ &= -\frac{i}{4}\frac{1}{(z-2i)} + \frac{1}{16} + \frac{i}{64}(z-2i) - \frac{1}{256}(z-2i)^2 - \cdots\end{aligned}$$

Hence

$$f(z) = \left(1 - e^{2i} + (z-2i)\,e^{2i} + (z-2i)^2\frac{e^{2i}}{2!} + \cdots\right)\left(-\frac{i}{4}\frac{1}{(z-2i)} + \frac{1}{16} + \frac{i}{64}(z-2i) - \frac{1}{256}(z-2i)^2 - \cdots\right)$$

We see that the resulting series will contain infinite number of b_n terms. These are the terms with $\frac{1}{(z-2i)^n}$. Hence the point $z=2i$ is essential singularity.

(b) We need to find the series of $\tan^2 z$ around $z = \frac{\pi}{2}$.

$$
\begin{aligned}
\tan^2\left(z-\frac{\pi}{2}\right) &= \frac{\sin^2\left(z-\frac{\pi}{2}\right)}{\cos^2\left(z-\frac{\pi}{2}\right)} \\
&= \frac{\left(\left(z-\frac{\pi}{2}\right) - \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^5}{5!} - \cdots\right)^2}{\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{4!} - \cdots\right)^2} \\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{5!} - \cdots\right)^2}{\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{4!} - \cdots\right)^2} \\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{5!} - \cdots\right)^2}{\left(\left(z-\frac{\pi}{2}\right)\left(\frac{1}{z-\frac{\pi}{2}} - \frac{\left(z-\frac{\pi}{2}\right)}{2!} + \frac{\left(z-\frac{\pi}{2}\right)^3}{4!} - \cdots\right)\right)^2} \\
&= \frac{\left(z-\frac{\pi}{2}\right)^2\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{5!} - \cdots\right)^2}{\left(z-\frac{\pi}{2}\right)^2\left(\frac{1}{z-\frac{\pi}{2}} - \frac{\left(z-\frac{\pi}{2}\right)}{2!} + \frac{\left(z-\frac{\pi}{2}\right)^3}{4!} - \cdots\right)^2} \\
&= \frac{\left(1 - \frac{\left(z-\frac{\pi}{2}\right)^2}{3!} + \frac{\left(z-\frac{\pi}{2}\right)^4}{5!} - \cdots\right)^2}{\left(\frac{1}{z-\frac{\pi}{2}} - \frac{\left(z-\frac{\pi}{2}\right)}{2!} + \frac{\left(z-\frac{\pi}{2}\right)^3}{4!} - \cdots\right)^2}
\end{aligned}
$$

So we see that the number of b_n terms will be 2 if we simplify the above. We only need to look at the first 2 terms, which will come out as

$$
f(z) = \frac{1}{\left(z-\frac{\pi}{2}\right)^2} - \frac{2}{3} + \frac{1}{15}\left(z-\frac{\pi}{2}\right)^2 + \cdots
$$

Since the order of the b_n is 2, from $\frac{1}{\left(z-\frac{\pi}{2}\right)^2}$, then this is a pole of order 2. If the number of b_n was infinite, this would have been essential singularity.

(c) $f(z) = \frac{1-\cos(z)}{z^4}$, Hence expanding around $z = 0$ gives

$$
\begin{aligned}
f(z) &= \frac{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)}{z^4} \\
&= \frac{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots}{z^4} \\
&= \frac{1}{2}\frac{1}{z^2} - \frac{1}{4!} + \frac{z^2}{6!} + \cdots
\end{aligned}
$$

Since $b_n = \frac{1}{2}\frac{1}{z^2}$ and highest power is 2, then this is pole of order 2.

(d) $f(z) = \cos\left(\frac{\pi}{z-\pi}\right)$. We need to expand $f(z)$ around $z = \pi$ and look at the series. Since $\cos(x)$ expanded around π is

$$
\cos(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \cdots
$$

Replacing $x = \frac{\pi}{z-\pi}$, the above becomes

$$\cos\left(\frac{\pi}{z-\pi}\right) = -1 + \frac{1}{2}\left(\left(\frac{\pi}{z-\pi}\right) - \pi\right)^2 - \frac{1}{24}\left(\left(\frac{\pi}{z-\pi}\right) - \pi\right)^4 + \cdots$$

The series diverges at $z = \pi$ so it is essential singularity at $z = \pi$. One can also see there are infinite number of b_n terms of the form $\frac{1}{(z-\pi)^n}$

22 chapter 14, problem 5.1

Problem If C is circle of radius R about z_0, show that

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution
Since $z = z_0 + Re^{i\theta}$ then $dz = Rie^{i\theta}$ and the integral becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= \int_0^{2\pi} \left(Rie^{i\theta}\right)^{1-n} d\theta \\ &= (R)^{1-n} \int_0^{2\pi} ie^{i\theta(1-n)} d\theta \end{aligned} \tag{1}$$

When $n = 1$ the above becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Rie^{i\theta}\right)^n} d\theta &= \int_0^{2\pi} i d\theta \\ &= 2\pi i \end{aligned}$$

And when $n \neq 1$, then (1) becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= i\,(R)^{1-n} \left[\frac{e^{i\theta(1-n)}}{i(1-n)}\right]_0^{2\pi} \\ &= \frac{R^{1-n}}{1-n} \left[e^{i\theta(1-n)}\right]_0^{2\pi} \\ &= \frac{R^{1-n}}{1-n} \left(e^{i2\pi(1-n)} - 1\right) \end{aligned}$$

But $e^{i2\pi(1-n)} = 1$ since $1 - n$ is integer. Hence the above becomes

$$\begin{aligned} \int_0^{2\pi} \frac{Rie^{i\theta}}{\left(Re^{i\theta}\right)^n} d\theta &= \frac{R^{1-n}}{1-n}(1-1) \\ &= 0 \end{aligned}$$

QED.