HW # 6

Math 121A

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UCB extension

$\frac{2}{2}$ → score changed.

Please Review the grade on this HW.

I have correctly solved 22 problems and just forgot to solve 2 simple problem. for this I lost 50% of the grade?

this is NOT Fair.

Please check the grading policy. I get time/money to only grade 2-3 problems a week carefully. If a student misses 1 of those problems, then it's bad luck maybe. On the other hand, if I grade based on number of problems attempted,

then everyone will get full marks, which defeats the purpose of grading homework. But: 1) 2 points per homework is very little, so I agree that even 1/2 a point is a great %.

2) I have discussed with the instructor, and we shall amend things if necessary.

Thanks for your feedback!

ch 4

$\boxed{9.2}$ what proportions will maximize the volume of a projectile in the form of a circular cylinder with one conical end and one flat end if the surface area is given?

$s^2 = r^2 + h^2$

so $h = \sqrt{s^2 - r^2}$

Volume of cone $= \frac{1}{3}$ base $\times h = \frac{1}{3}(\pi r^2)\sqrt{s^2 - r^2}$

So Total $V =$ Cylinder Volume + Cone Volume

$$\boxed{V = \pi r^2 \ell + \frac{1}{3}(\pi r^2)\sqrt{s^2 - r^2}}$$

now need to find total surface Area. this is the constraint.

Surface area of cylinder $= 2\pi r\ell + \pi r^2$

surface area of cone $= \frac{\text{base circumference}}{2} \times S = \frac{1}{2} 2\pi r S = \pi r s$

So $$\boxed{A = (2\pi r\ell + \pi r^2) + \pi r s}$$

$\to$ lagrange multiplier

So our maximizer function $\boxed{F = V + \lambda A}$ $\to \phi$

$$\boxed{F = \pi r^2 \ell + \frac{1}{3}\pi r^2\sqrt{s^2 - r^2} + \lambda\pi(2r\ell + r^2 + rs)}$$

So F is function of r, ℓ, s.

$\longrightarrow$

$$\frac{\partial F}{\partial r} = 0 = 2r\pi l + \frac{1}{3}\pi r^2\left(\frac{1}{2}\frac{1}{(s^2-r^2)^{1/2}}\cdot(-2r)\right) + \frac{1}{3}2\pi r(s^2-r^2)^{1/2} + 2\lambda\pi l + 2r\lambda\pi + s\lambda\pi$$

$$\frac{\partial F}{\partial r}: 0 = 2r\pi l - \frac{\pi r^2}{3}\frac{r}{\sqrt{s^2-r^2}} + \frac{2\pi r}{3}\sqrt{s^2-r^2} + 2\lambda\pi l + 2r\lambda\pi + s\lambda\pi$$

$$\frac{\partial F}{\partial r}: 0 = \pi\left(2rl - \frac{r^2}{3}\frac{r}{\sqrt{s^2-r^2}} + \frac{2r}{3}\sqrt{s^2-r^2} + 2\lambda l + 2r\lambda + s\lambda\right) \text{———} ①$$

$$\frac{\partial F}{\partial s} = 0 = \frac{1}{3}\pi r^2 \frac{1}{2}\frac{1}{\sqrt{s^2-r^2}}2s + r\lambda\pi$$

$$= \pi\left(\frac{1}{3}r^2\frac{s}{\sqrt{s^2-r^2}} + r\lambda\right) \text{———} ②$$

$$\frac{\partial F}{\partial l} = 0 = \pi r^2 + 2r\pi\lambda = \pi(r^2 + 2r\lambda) \text{———} ③$$

start by eliminating λ.

From ③ $\quad 0 = \pi r^2 + 2\pi r\lambda \Rightarrow \lambda = \frac{-\pi r^2}{2\pi r} = -\frac{r^2}{2r} = \boxed{-\frac{r}{2}}$

Plug this value for λ into ②

$$\Rightarrow 0 = \pi\left(\frac{1}{3}r^2\frac{s}{\sqrt{s^2-r^2}} - \frac{r}{2}r\right) = \pi\left(\frac{r^2 s}{3\sqrt{s^2-r^2}} - \frac{r^2}{2}\right)$$

$$\text{so } 0 = \pi\left(\frac{2r^2 s - 3\sqrt{s^2-r^2}\,r^2}{6\sqrt{s^2-r^2}}\right) \Rightarrow 2r^2 s - 3\sqrt{s^2-r^2}\,r^2 = 0$$

$$\Rightarrow 2s - 3\sqrt{s^2-r^2} = 0 \Rightarrow 2s = 3\sqrt{s^2-r^2} \Rightarrow 4s^2 = 9(s^2-r^2)$$

$$\Rightarrow 4s^2 = 9s^2 - 9r^2 \Rightarrow 9r^2 = 5s^2 \Rightarrow \boxed{\frac{r}{s} = \frac{\sqrt{5}}{3}} \text{ or } \boxed{r = \frac{\sqrt{5}}{3}s}$$

Plug the value for r into ①. also plug value for λ into ①

This leaves ① in terms of l, s only:

$$\lambda = -\frac{r}{2} = \boxed{-\frac{1}{2}\frac{\sqrt{5}}{3}s}$$

so ① becomes ⟶

$$0 = \pi\left(2\left(\frac{\sqrt5}{3}s\right)l - \frac{\left(\frac{\sqrt5}{3}s\right)^2}{3}\frac{\left(\frac{\sqrt5}{3}s\right)}{\sqrt{s^2-\left(\frac{\sqrt5}{3}s\right)^2}} + \frac23\left(\frac{\sqrt5}{3}s\right)\sqrt{s^2-\left(\frac{\sqrt5}{3}s\right)^2} + 2\left(\frac{-\sqrt5}{6}s\right)l\right.$$

$$\left. + 2\left(\frac{\sqrt5}{3}s\right)\left(\frac{-\sqrt5}{6}s\right) + s\left(-\frac{\sqrt5}{6}s\right)\right)$$

divide by $\left(\frac{\sqrt5}{3}s\right) \Rightarrow$

$$0 = \pi\left(2l - \frac{\left(\frac{\sqrt5}{3}s\right)^2}{3}\frac{1}{\sqrt{s^2-\frac59 s^2}} + \frac23\sqrt{s^2-\frac59 s^2} - 2l\frac12 - \frac{\sqrt5}{3}s - \frac12 s\right)$$

$$0 = 2l - \frac{\frac59 s^2}{3}\frac{1}{\sqrt{s^2-\frac59 s^2}} + \frac23\sqrt{s^2-\frac59 s^2} - l - \frac{\sqrt5}{3}s - \frac12 s$$

$$0 = 2l - \frac{5s^2}{9\cdot3}\frac{1}{\sqrt{\frac49 s^2}} + \frac23\sqrt{\frac49 s^2} - l - \frac{\sqrt5}{3}s - \frac12 s$$

$$= 2l - \frac{5s^2}{9\cdot3}\frac{1}{\frac23 s} + \frac23\ \frac23 s - l - \frac{\sqrt5}{3}s - \frac12 s$$

$$= 2l - \frac{5s}{9\cdot2} + \frac49 s - l - \frac{\sqrt5}{3}s - \frac12 s = l + s\left(\frac49 - \frac{\sqrt5}{3} - \frac12 - \frac{5}{18}\right)$$

$$0 = l + s\left(\frac{8-6\sqrt5-9-5}{18}\right) \Rightarrow 0 = l + s\left(\frac{-6-6\sqrt5}{18}\right) \Rightarrow 0 = l - s\left(\frac{1+\sqrt5}{3}\right)$$

$$l = s\left(\frac{1+\sqrt5}{3}\right) \Longrightarrow \boxed{\frac{l}{s} = \frac{1+\sqrt5}{3}}$$

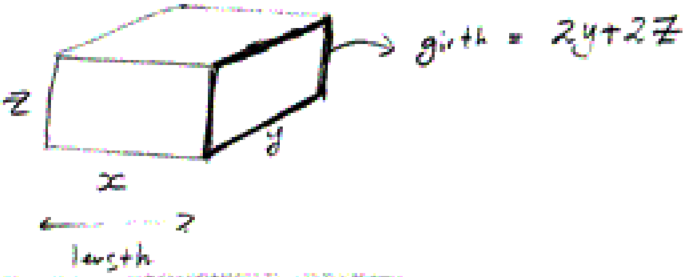
So proportions to maximize volume are

$$\boxed{\frac{r}{s} = \frac{\sqrt5}{3}} \quad \text{and} \quad \boxed{\frac{l}{s} = \frac{1+\sqrt5}{3}}$$

$$\sim \boxed{l:r:s \equiv 1+\sqrt5 : \sqrt5 : 3}$$

Ch 4

$\boxed{9.3}$ Find largest box that can be shipped by parcel post (length plus girth = 84 in)



$$\text{So}\ \boxed{\phi^{(x,y,z)} = \underbrace{2y+2z}_{\text{girth}} + \underbrace{x}_{\text{length}} = 84} \quad \text{———} \quad ①$$

$$V = xyz$$

we want to maximize V subject to $\phi = 84$.

$$\boxed{F = V + \lambda \phi}$$

$$F = xyz + \lambda(2y+2z+x)$$

$$\frac{\partial F}{\partial x} = 0 = yz + \lambda \quad \text{———} \quad ②$$

$$\frac{\partial F}{\partial y} = 0 = xz + 2\lambda \quad \text{———} \quad ③$$

$$\frac{\partial F}{\partial z} = 0 = xy + 2\lambda \quad \text{———} \quad ④$$

equations ①–④ are now solved for λ, x, y, z.

eliminate λ from equations ②–③. this will give 2 new equations in x, y, z. use these 2 new equations with eq ① to solve for x, y, z.

from ④ $\lambda = -\frac{xy}{2}$ plug in ③

$\Rightarrow 0 = xz + 2\left(-\frac{xy}{2}\right) = xz - xy = 0$ ——— ⑤

plug λ in ② $\Rightarrow 0 = yz - \frac{xy}{2}$ ——— ⑥

$\Big\}$ solve ⑤, ⑥ and ① $\Longrightarrow$

$xz - xy = 0$ ——————— ⑤

$yz - \frac{xy}{2} = 0$ ——————— ⑥

$2y + 2z + x = 84$ ——————— ①

From ⑤ $x(z-y) = 0$ so $z = y$

From ⑥ $z = \frac{x}{2}$

$\Rightarrow y = \frac{x}{2}$

so from ① $2\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right) + x = 84$

$x + x + x = 84$

$3x = 84$

$\boxed{x = 28}$

so $\boxed{z = \frac{28}{2} = 14}$

and $\boxed{y = \frac{28}{2} = 14}$

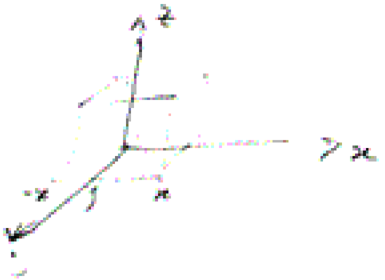
∴ max. Volume = $xyz = 28 \times 14 \times 14$

$= \boxed{5488 \text{ in}^2}$

Ch 4

$\boxed{9.4}$ Find largest box (with Faces parallel to coordinate axes) that can be inscribed in $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$

$$\phi(x,y,z) = \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25} = 1 \quad \text{———} \quad ①$$



$$\text{Volume} = (2x)(2y)(2z) = 6xyz$$

so $\boxed{F = V + \lambda\phi}$

$$F = xyz + \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}\right)$$

$$\frac{\partial F}{\partial x} = 0 = 6yz + \frac{\lambda x}{2} \quad \text{———} \quad ②$$

$$\frac{\partial F}{\partial y} = 0 = 6xz + \frac{2y}{9}\lambda \quad \text{———} \quad ③$$

$$\frac{\partial F}{\partial z} = 0 = 6xy + \frac{2z}{25}\lambda \quad \text{———} \quad ④$$

solve ①,②,③,④ for λ, x, y, z.

First eliminate λ

from ④ $\lambda = -6xy\left(\frac{25}{2z}\right)$

Plug in ③ $\Rightarrow$ $0 = 6xz + \frac{2y}{9}\left(-6xy\,\frac{25}{2z}\right)$

$$0 = xz - \frac{25xy^2}{9z}$$

$$0 = z - \frac{25y^2}{9z} \quad \text{———} \quad ⑤$$

Plug λ into ②

$$0 = 6yz + \frac{x}{2}\left(-6xy\,\frac{25}{2z}\right)$$

$$0 = yz - \frac{x^2y\,25}{4z}$$

$$0 = z - \frac{x^2}{z}\,\frac{25}{4} \quad \text{———} \quad ⑥$$

now use ⑤, ⑥ and ① to find x, y, z $\longrightarrow$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1 \quad \text{———} \quad ①$$

$$0 = z - \frac{25y^2}{9z} \quad \text{———} \quad ⑤$$

$$0 = z - \frac{x^2}{z}\,\frac{25}{4} \quad \text{———} \quad ⑥$$

from ⑤ $9z^2 - 25y^2 = 0 \implies z^2 = \frac{25}{9}y^2$

from ⑥ $4z^2 - 25x^2 = 0 \implies z^2 = \frac{25}{4}x^2$

$\frac{25}{9}y^2 = \frac{25}{4}x^2$

or $4y^2 = 9x^2$

or $y^2 = \frac{9}{4}x^2$

so ① becomes $\frac{x^2}{4} + \frac{1}{9}\left(\frac{9}{4}x^2\right) + \frac{1}{25}\left(\frac{25}{4}x^2\right) = 1$

$$\frac{x^2}{4} + \frac{1}{4}x^2 + \frac{1}{4}x^2 = 1$$

$$\frac{3}{4}x^2 = 1 \implies x^2 = \frac{4}{3} \implies \boxed{x = \frac{2}{\sqrt{3}}}$$

so $y^2 = \frac{9}{4} \times \frac{4}{3} = 3 \implies \boxed{y = \sqrt{3}}$ i.e. $\boxed{2y = \frac{6}{\sqrt{3}}}$ i.e. $\boxed{2x = \frac{4}{\sqrt{3}}}$

so $z^2 = \frac{25}{4} \times \frac{4}{3} = \frac{25}{3} \implies \boxed{z = \frac{5}{\sqrt{3}}}$ i.e. $\boxed{2z = \frac{10}{\sqrt{3}}}$

so largest Box $= 6xyz$

$$= 6\left(\frac{2}{\sqrt{3}} \times \sqrt{3} \times \frac{5}{\sqrt{3}}\right) = \boxed{\frac{60}{\sqrt{3}}} \cong 34.64 \text{ m}^3$$

ch 4

$\boxed{9.5}$ Find the point on $2x + 3y + z - 11 = 0$ for which $4x^2 + y^2 + z^2$ is a min.

$\phi(x,y,z) = 2x + 3y + z = 11$ —— ①

$f(x,y,z) = 4x^2 + y^2 + z^2$

so $\boxed{F = f + \lambda\phi}$ $F = 4x^2 + y^2 + z^2 + \lambda(2x + 3y + z)$

$\frac{\partial F}{\partial x} = 0 = 8x + 2\lambda$ —— ②

$\frac{\partial F}{\partial y} = 0 = 2y + 3\lambda$ —— ③

$\frac{\partial F}{\partial z} = 0 = 2z + \lambda$ —— ④

From ④ $\lambda = -2z$. plug in ③ $\Rightarrow$ $2y = \overset{6z}{\cancel{2z}}$ or $y = 3z$ —— ⑤

plug in ② $\Rightarrow$ $8x = 4z$ or $x = \frac{1}{2}z$ —— ⑥

solve ①, ⑤, ⑥ for x, y, z.

From ⑤, ⑥ $\Rightarrow$ $x = \frac{1}{4}y$. so ① becomes $2x + 3(4x) + 4(x) = 11$

$2x + 12x + 4x = 11 \rightarrow 18x = 11 \Rightarrow \boxed{x = \frac{11}{18}}$

so $z = 4 \times \frac{11}{18} = \frac{44}{18} = \boxed{\frac{22}{9}}$

$\boxed{y = \frac{22}{9}}$

so point is $(x,y,z) = \left(\frac{11}{18}, \frac{22}{9}, \frac{22}{9}\right)$

NOT correct arithmetic error !

ch 4

$\boxed{19.6}$ A box has 3 of its faces in the coordinates planes and one vertex on the plane $2x+3y+4z=6$. Find max Volume.

$\Phi(x,y,z) = 2x+3y+4z=6$ —①

$V(x,y,z) = xyz$

$$\boxed{F = V + \lambda\Phi}$$

$F = xyz + \lambda(2x+3y+4z)$

$\frac{\partial F}{\partial x} = 0 = yz + 2\lambda$ —②

$\frac{\partial F}{\partial y} = 0 = xz + 3\lambda$ —③

$\frac{\partial F}{\partial z} = 0 = xy + 4\lambda$ —④

solve for λ from (1),(2),(3)

From ④ $\lambda = -\frac{xy}{4}$ Plug in ③ $\rightarrow xz + 3\left(-\frac{xy}{4}\right) = 0$

$\Rightarrow xz - \frac{3}{4}xy = 0$ or $z - \frac{3}{4}y = 0$ —⑤

Plug λ in ② $\Rightarrow yz + 2\left(-\frac{xy}{4}\right) = 0 \Rightarrow z - \frac{x}{2} = 0$ —⑥

use ①,⑤,⑥ to solve for x, y, z

$2x+3y+4z=6$ ①

$z - \frac{3}{4}y = 0$ ⑤

$z - \frac{x}{2} = 0$ ⑥

From ⑤,⑥ $\Rightarrow -\frac{3}{4}y = -\frac{x}{2} \Rightarrow \frac{3}{2}y = x \Rightarrow y = \frac{2}{3}x$

From ⑥ $z = \frac{x}{2}$. So ① becomes $2x + 3\left(\frac{2}{3}x\right) + 4\left(\frac{x}{2}\right) = 6$

$\Rightarrow 2x + 2x + 2x = 6 \Rightarrow \boxed{x=1}$. so $\boxed{y = \frac{2}{3}}$ and $\boxed{z = \frac{1}{2}}$

So Max $V = xyz = 1 \times \frac{2}{3} \times \frac{1}{2} = \boxed{\frac{1}{2}}$ m^3

Ch 4

$\boxed{9.7}$ repeat problem 6 if plane is $ax+by+cz=d$.

$\phi(x,y,z) = ax+by+cz = d$ —— ①

$f(x,y,z) = xyz$

$F = f(x,y,z) + \lambda\phi(x,y,z)$

$F = xyz + \lambda(ax+by+cz)$

$\frac{\partial F}{\partial x} = yz + a\lambda = 0$ —— ②

$\frac{\partial F}{\partial y} = xz + b\lambda = 0$ —— ③

$\frac{\partial F}{\partial z} = xy + c\lambda = 0$ —— ④

From ④ $\lambda = -\frac{xy}{c}$. Plug in ③ $\Rightarrow xz + b\left(-\frac{xy}{c}\right) = 0 \Rightarrow z - \frac{by}{c} = 0 \Rightarrow cz - by = 0$ —— ⑤

Plug λ into ② $\Rightarrow yz + a\left(-\frac{xy}{c}\right) = 0 \Rightarrow z - \frac{ax}{c} = 0 \Rightarrow cz - ax = 0$ —— ⑥

Solve ⑤, ⑥ and ① for xyz.

$ax+by+cz = d$ —— ①

$cz - by = 0$ —— ⑤

$cz - ax = 0$ —— ⑥

from ⑤, ⑥ $\Rightarrow -by = -ax \longrightarrow y = \frac{a}{b}x$

from ⑥ $z = \frac{ax}{c}$.

so ① becomes $ax + b\left(\frac{a}{b}\right)x + c\left(\frac{ax}{c}\right) = d$

$ax + ax + ax = d$

$3ax = d \longrightarrow \boxed{x = \frac{d}{3a}}$

so $y = \frac{a}{b}\left(\frac{d}{3a}\right) = \boxed{\frac{d}{3b}}$

$z = \frac{a}{c}\frac{d}{3a} = \boxed{\frac{d}{3c}}$

so max $V = xyz = \frac{d}{3a} \times \frac{d}{3b} \times \frac{d}{3c} = \boxed{\frac{d^3}{27abc}}$

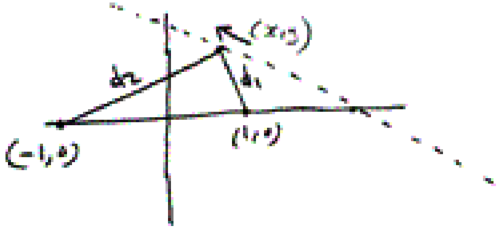
To verify: solve 9.6 using this formula.

in 9.6, $a=2$, $b=3$, $c=4$, $d=6$.

so max $V = \frac{6\times6\times6}{27(2\times3\times4)} = \frac{6\times6}{\underset{9}{27}\times\underset{2}{4}} = \frac{6}{9\times2} = \boxed{\frac{1}{3}}$ which agrees with my solution for 9.6

Ch 4

$\boxed{9.8}$ a point moves in the (x,y) plane on the line $2x+3y-4=0$. Where will it be when the sum of the squares of its distance from $(1,0)$ and $(-1,0)$ is smallest?



when $x=0$ $3y=4 \Rightarrow y=\frac{4}{3}$

when $y=0$ $2x=4 \Rightarrow x=2$

so line is as shown.

let d_1 be distance to $(1,0)$

d_2 be distance to $(-1,0)$.

let point be (x,y). so $d_1^2=(x-1)^2+(y-0)^2$

and $d_2^2=(x+1)^2+(y-0)^2$

i.e. $d_1^2+d_2^2 = (x-1)^2+y^2 + (x+1)^2+y^2 = x^2-2x+1+x^2+2x+1+2y^2$

$f(x,y) = d_1^2+d_2^2 = 2x^2+2y^2+2$

$\phi(x,y) = 2x+3y-4=0$ or $\phi(x,y) = 2x+3y=4$ —— ①

so $\boxed{F=f+\lambda\phi}$

$F = 2x^2+2y^2+2+\lambda(2x+3y)$

$\frac{\partial F}{\partial x} = 4x+2\lambda = 0$ —— ②

$\frac{\partial F}{\partial y} = 4y+3\lambda = 0$ —— ③

eliminate λ. from ③ $\lambda = -\frac{4}{3}y$. plug in ② $\Rightarrow$ $4x+2\left(-\frac{4}{3}y\right)=0$

$\Rightarrow 4x-\frac{8}{3}y=0 \Rightarrow x=\frac{2}{3}y$. Plug into ① $\Rightarrow 2\left(\frac{2}{3}y\right)+3y=4$

$\Rightarrow \frac{4}{3}y+3y=4 \Rightarrow \frac{4+9}{3}y=4 \Rightarrow \frac{13}{3}y=4 \Rightarrow \boxed{y=\frac{12}{13}}$

so $x=\frac{2}{3}\frac{12}{13} = \frac{24}{3\times13} = \boxed{\frac{8}{13}}$

So point will be at $\boxed{\left(\frac{8}{13},\frac{12}{13}\right)}$ when sum of squares is smallest.

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ch 4

$\boxed{10.2}$

Find largest and smallest distance from origin to the conic whose equation is $5x^2-6xy+5y^2-32=0$ and hence determine the length of the semiaxis of conic.

let $d^2=x^2+y^2$. This is our $f(x,y)$ function we want to minimize.

$$\phi(x,y)=5x^2-6xy+5y^2=32 \qquad ①$$

so $F=f+\lambda\phi$

$$F=x^2+y^2+\lambda(5x^2-6xy+5y^2)$$

$$\frac{\partial F}{\partial x}=0=2x+10\lambda x-6\lambda y \qquad ②$$

$$\frac{\partial F}{\partial y}=0=2y+10\lambda y-6\lambda x \qquad ③$$

or

$$\frac{\partial F}{\partial x}=0=x(2+10\lambda)-6\lambda y \text{ ------ } ②$$

$$\frac{\partial F}{\partial y}=0=y(2+10\lambda)-6\lambda x \text{ ------ } ③$$

solve for λ from ② and ③.

from ③ $y=\dfrac{6\lambda x}{2+10\lambda}$ ------ ④

Plug ④ into ② $\Rightarrow$

$$0=x(2+10\lambda)-6\lambda\left(\frac{6\lambda x}{2+10\lambda}\right)$$

i.e. $$0=x(2+10\lambda)-\frac{36\lambda^2x}{2+10\lambda}$$

$$0=x(2+10\lambda)(2+10\lambda)-36\lambda^2x$$

$\longrightarrow$

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$$0 = x\left[\;4 + 100\lambda^2 + 40\lambda - 36\lambda^2\right]$$

either $x = 0$ or $x \neq 0$.

if $x \neq 0$ then $64\lambda^2 + 40\lambda + 4 = 0$

$$\Rightarrow \boxed{\lambda = -\frac{1}{8} \text{ or } -\frac{1}{2}}$$

when $\lambda = -\frac{1}{8}$

from ③ $\Rightarrow$ $y(2 + 10(-\frac{1}{8})) - 6(-\frac{1}{8})x = 0$

$y(2 - \frac{10}{8}) + \frac{6}{8}x = 0$

$y(\frac{16-10}{8}) + \frac{6}{8}x = 0$

$\frac{6}{8}y + \frac{6}{8}x = 0 \Rightarrow \boxed{y = -x}$

from ① $5x^2 - 6x(-x) + 5(-x)^2 = 32$

$5x^2 + 6x^2 + 5x^2 = 32$

$16x^2 = 32$

$x^2 = 2 \Rightarrow \boxed{x = \pm\sqrt{2}}$

so $\boxed{y = \mp\sqrt{2}}$

so for $\lambda = -\frac{1}{8}$, position points are

$\boxed{(\sqrt{2}, -\sqrt{2}) \text{ and } (-\sqrt{2}, \sqrt{2})}$

for $\lambda = -\frac{1}{2}$ from ③ $\Rightarrow$ $y(2 + 10(-\frac{1}{2})) - 6(-\frac{1}{2})x = 0$

ie $y(2 - 5) + 3x = 0$

ie $-3y + 3x = 0 \Rightarrow \boxed{y = x}$

so from ① $5x^2 - 6x(x) + 5(x)^2 = 32$

$5x^2 - 6x^2 + 5x^2 = 32$

$4x^2 = 32 \Rightarrow \boxed{x = \pm 2\sqrt{2}}$

so $y = \pm 2\sqrt{2}$

so points are $(2\sqrt{2}, 2\sqrt{2})$ or $(-2\sqrt{2}, -2\sqrt{2}) \Longrightarrow$

now, all above points were found by assuming $x \neq 0$.

now for the case if $x = 0$.

from ① $\Rightarrow$ $5y^2 = 32$

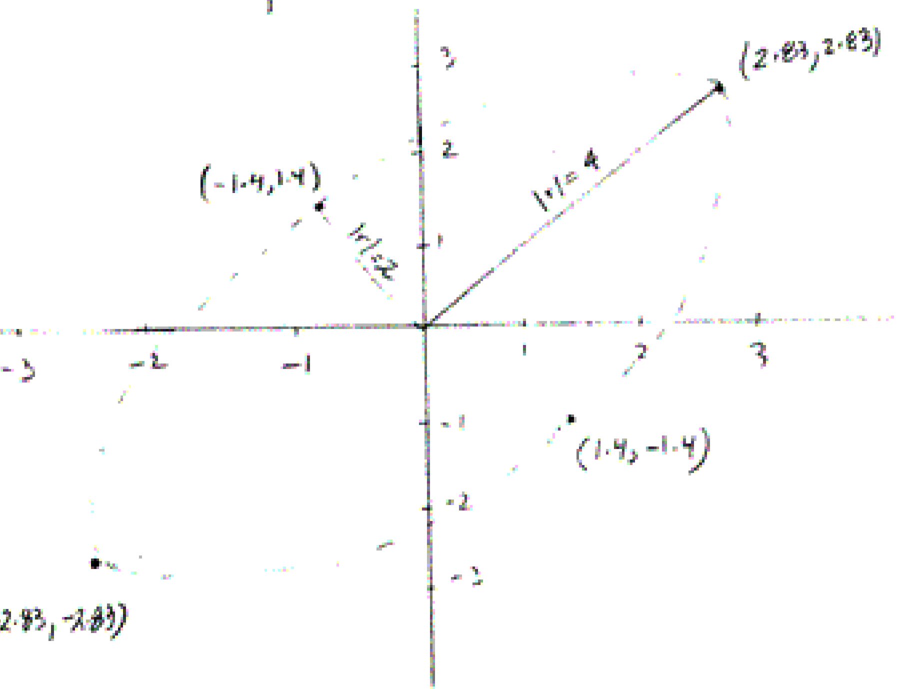
$$\text{or } y^2 = \frac{32}{5} \Rightarrow y = \pm 2.53$$

so points are $(0, 2.53)$ or $(0, -2.53)$.

<u>so in summary</u> I have found 6 points where $f(x,y)$ is either min or max. now for each point need to find the distance from origin to know which is max and which is min.

point	d^2	d	
$(0, \pm 2.53)$	6.4	2.53	
$(\pm 2\sqrt{2}, \pm 2\sqrt{2})$	$8+8=16$	4	→ largest distance
$(\pm\sqrt{2}, \pm\sqrt{2})$	$2+2=4$	2	→ smallest distance

note $(2\sqrt{2}, 2\sqrt{2}) = (2.83, 2.83)$
$(\sqrt{2}, \sqrt{2}) = (1.4, 1.4)$



here $\boxed{\text{major axes length} = 8}$ $\boxed{\text{minor axes length} = 4}$

ch 4

$\boxed{10.7}$

Find the Largest z for which $2x+4y=5$ and $x^2+z^2=2y$.

here the constraint $\phi(x,y) = 2x+4y=5$ —— ①

and $z^2 = 2y - x^2$.

so Largest z is the that will make z^2 Largest as well.

at $f(x,y) = 2y - x^2$.

hence $F = f + \lambda\phi$

$= 2y - x^2 + \lambda(2x+4y)$

$\frac{\partial F}{\partial x} = 0 = -2x + 2\lambda$ — ②

$\frac{\partial F}{\partial y} = 0 = 2 + 4\lambda$ —— ③

from ③, $\lambda = -\frac{1}{2}$

from ②, $0 = -2x + 2(-\frac{1}{2})$

i.e. $0 = -2x - 1$ i.e. $\boxed{x = -\frac{1}{2}}$

so from $2x+4y=5 \Rightarrow 2(-\frac{1}{2})+4y=5 \Rightarrow y = \frac{5+1}{4} = \frac{6}{4} = \boxed{1.5}$

since $z^2 = 2y - x^2$

Then $z^2 = 2(\frac{6}{4}) - (-\frac{1}{2})^2 = 3 - \frac{1}{4} = \frac{12-1}{4} = \frac{11}{4}$

so $z = \pm\sqrt{\frac{11}{4}} = \pm\frac{1}{2}\sqrt{11}$

so Largest z is $\boxed{\frac{1}{2}\sqrt{11}}$

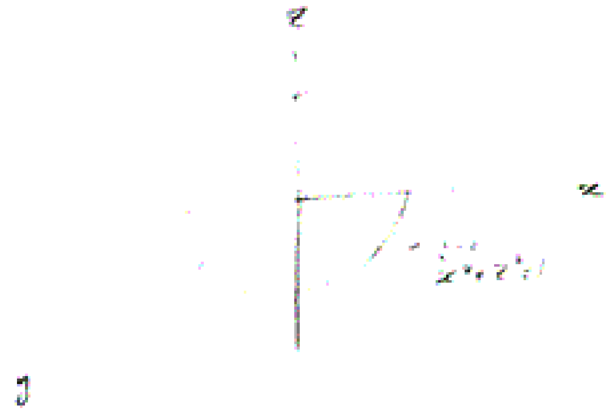
ch 4

$\boxed{10.10}$ the temp at point (x,y,z) in the sphere $x^2+y^2+z^2 \leq 1$ is given by $T = y^2 + xz$. Find largest and smallest values which T takes.

(a) on the circle $y=0$, $x^2+z^2=1$

(b) on the surface $x^2+y^2+z^2=1$

(c) in the whole sphere



(a). here the constraints are $y=0$ and $x^2+z^2=1$ while $f = y^2 + xz$. but $y=0$, hence $\boxed{f = xz}$

so $\phi = x^2+z^2 = 1$ .... ①

$f = xz$

$F = f + \lambda\phi$

$F = xz + \lambda(x^2+z^2)$

$\frac{\partial F}{\partial x} = 0 = z + 2\lambda x$ ——— ②

$\frac{\partial F}{\partial z} = 0 = x + 2\lambda z$ ——— ③

from ③, $\lambda = -\frac{x}{2z}$. sub into ② $\Rightarrow$ $0 = z + 2\cdot\left(\frac{-x}{2z}\right)$

i.e $0 = \frac{2z^2 - 2x^2}{2z}$

z can't be zero, since if $z=0$ then ③ implies $x=0$ also. but then $x^2+z^2=1$ would be a contradiction. so we can divide by z to get $0 = z^2 - x^2$

i.e $z^2 = x^2$ .... ④

from ① and ④ $\Rightarrow$ $2x^2=1$ or $x = \pm\frac{1}{\sqrt{2}}$ and also $z^2=\frac{1}{2} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$

⟶

so $\text{temp} = y^2 + xz$

$= 0 + xz$

$= \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}}$

$$\boxed{T = -\frac{1}{2} \quad \text{or} \quad \frac{1}{2}}$$

max at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and at $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

min.
at $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

(b) on the surface $x^2+y^2+z^2=1$

here $\phi = x^2+y^2+z^2=1$ —— ①

$F = y^2 + xz + \lambda(x^2+y^2+z^2$

$\frac{\partial F}{\partial x} = z + 2\lambda x = 0$ — ②

$\frac{\partial F}{\partial y} = 2y + 2\lambda y = 0$ ③

$\frac{\partial F}{\partial z} = x + 2\lambda z = 0$ ④

Remove λ from 2,3,4. this will result in 2 new equations in x,y,z. which with equation 1, we get 3 equations in x,y,z to solve

from ④ $\lambda = \frac{-x}{2z}$. ⑤

From ⑤ and ③ $\Rightarrow$ $0 = 2y + 2\left(-\frac{x}{2z}y\right)$

$0 = 2y - \frac{xy}{z}$ —— ⑥

From ⑤ and ② $\Rightarrow$ $0 = z + 2\left(\frac{-x}{2z}\right)x$

$0 = z - \frac{x^2}{z}$ —— ⑦

so now we have

$x^2+y^2+z^2 = 1$ —— ①

$2yz - xy = 0$ —— ⑥

$z^2 - x^2 = 0$ —— ⑦

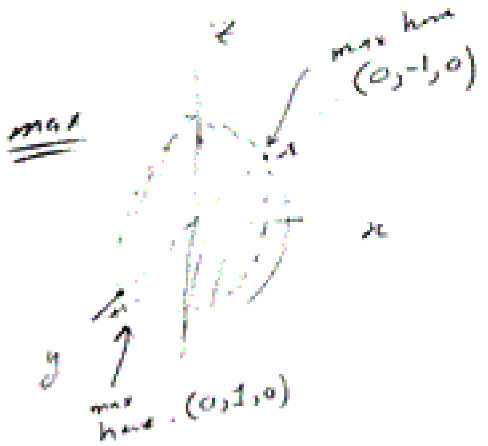
from ⑦ , $x^2 = z^2$
from ⑥ , $2z = x$
$\Big\} \Longrightarrow \boxed{x = 0,\ z = 0}$ only possible solution

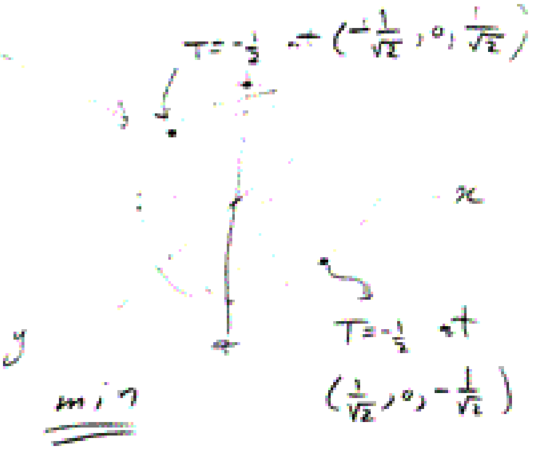
so from ① $y^2 = 1$ or $\boxed{y = \pm 1}$

so $\boxed{T = y^2 + xz = 1}$

so $\boxed{T = 1 \text{ at } (0, \pm 1, 0)}$.

for the minimum, I am not sure how to find it. all what I see is that when $y = 0$, in this case which is part (a) solution.





(c) in the whole sphere:

this means to find T (max, min) inside sphere ————① is the new constraint, in i.e $x^2+y^2+z^2<1$ which I solved for in addition to constraint $x^2+y^2+z^2=1$ and part (b). so only look at $x^2+y^2+z^2<1$ and see what min/max T I get and to compare to min/max T found in part (b) to decide

$\frac{\partial F}{\partial x} = 0 = z + 2\lambda x$ ———— ②

$\frac{\partial F}{\partial y} = 0 = 2y + 2\lambda y$ ———— ③

$\frac{\partial F}{\partial z} = 0 = x + 2\lambda z$ ———— ④ $\longrightarrow$

as from part (b), I set

$$x^2+y^2+z^2 < 1 \quad \text{——} \; ①$$
$$2yz - xy = 0 \quad \text{——} \; ⑥$$
$$z^2 - x^2 = 0 \quad \text{——} \; ⑦$$

so $z^2 = x^2$ from ⑦ and from ⑥ $2z - x = 0$

So as in part (b), $x = 0$, $z = 0$.

so $y^2 < 1$

so $T = y^2 + xz$

$\Rightarrow$ $T < 1$ inside sphere.

so at $y = 0$ $T = 0$, which is at <u>origin</u>

ie at $\boxed{(0,0,0) \quad T = 0}$ which is the min.

for the max, max occurs on surface of sphere as per part (b).

ch 4

$\boxed{11.1}$ in partial diff eq $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$ put $s=y+2x$, $t=y+3x$ and show that eq becomes $\frac{\partial^2 z}{\partial s}\partial t = 0$. following the method of solving 11.6, Solve the equation.

we can think of z as function $z(s,t)$. where $s(x,y)$, $t(x,y)$

so $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}$

so
$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial x^2}} = \frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s\partial x} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x^2} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t\partial x}$$

now find $\frac{\partial^2 z}{\partial x\partial y}$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x\partial y} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s\partial y} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x\partial y} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t\partial y}$$

now find $\frac{\partial^2 z}{\partial y^2}$:

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial y^2} + \frac{\partial s}{\partial y}\frac{\partial^2 z}{\partial s\partial y} + \left(\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial y^2} + \frac{\partial t}{\partial y}\frac{\partial^2 z}{\partial t\partial y}\right)\right.$$

so now plug all these in our PDE ⇒

$$\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s\partial x} + \frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x^2} + \frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t\partial x} - 5\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial x\partial y} - 5\frac{\partial s}{\partial x}\frac{\partial^2 z}{\partial s\partial y} - 5\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial x\partial y} - 5\frac{\partial t}{\partial x}\frac{\partial^2 z}{\partial t\partial y}$$
$$+6\frac{\partial z}{\partial s}\frac{\partial^2 s}{\partial y^2} + 6\frac{\partial s}{\partial y}\frac{\partial^2 z}{\partial s\partial y} + 6\frac{\partial z}{\partial t}\frac{\partial^2 t}{\partial y^2} + 6\frac{\partial t}{\partial y}\frac{\partial^2 z}{\partial t\partial y} = 0 \qquad ①$$

now, $\frac{\partial s}{\partial x} = 2$, $\frac{\partial^2 s}{\partial x^2} = 0$, $\frac{\partial t}{\partial x} = 3$, $\frac{\partial^2 t}{\partial x^2} = 0$, $\frac{\partial s}{\partial y} = 1$, $\frac{\partial^2 s}{\partial y^2} = 0$

$\frac{\partial t}{\partial y} = 1$, $\frac{\partial^2 t}{\partial y^2} = 0$, $\frac{\partial^2 s}{\partial x\partial y} = 0$, $\frac{\partial^2 t}{\partial x\partial y} = 0$ sub into ① ⟶

$$0 + 2\frac{\partial^2 z}{\partial s\partial x} + 0 + 3\frac{\partial^2 z}{\partial t\partial x} - 0 - 5\times 2\frac{\partial^2 z}{\partial s\partial y} - 0 - 5\times 3\frac{\partial^2 z}{\partial t\partial y}$$

$$+ 0 + 6\frac{\partial^2 z}{\partial s\partial y} + 0 + 6\frac{\partial^2 z}{\partial t\partial y} = 0$$

or $$2\frac{\partial^2 z}{\partial s\partial x} + 3\frac{\partial^2 z}{\partial t\partial x} - 10\frac{\partial^2 z}{\partial s\partial y} - 15\frac{\partial^2 z}{\partial t\partial y} + 6\frac{\partial^2 z}{\partial s\partial y} + 6\frac{\partial^2 z}{\partial t\partial y} = 0$$

$$2\frac{\partial^2 z}{\partial s\partial x} + 3\frac{\partial^2 z}{\partial t\partial x} - 4\frac{\partial^2 z}{\partial s\partial y} - 9\frac{\partial^2 z}{\partial t\partial y} = 0 \qquad (1A)$$

now $$\frac{\partial^2 z}{\partial s\partial x} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right) = \frac{\partial}{\partial s}\left(2\frac{\partial z}{\partial s} + 3\frac{\partial z}{\partial t}\right)$$

$$= 2\frac{\partial^2 z}{\partial s^2} + 3\frac{\partial^2 z}{\partial s\partial t} \qquad (2)$$

and $$\frac{\partial^2 z}{\partial t\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial t}\left(2\frac{\partial z}{\partial s} + 3\frac{\partial z}{\partial t}\right) = 2\frac{\partial^2 z}{\partial s\partial t} + 3\frac{\partial^2 z}{\partial t^2} \quad \text{---} \ (3)$$

and $$\frac{\partial^2 z}{\partial s\partial y} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\cdot 1 + \frac{\partial z}{\partial t}\cdot 1\right)$$

$$= \frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial s\partial t} \qquad (4)$$

and $$\frac{\partial^2 z}{\partial t\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial s} + \frac{\partial z}{\partial t}\right) = \frac{\partial^2 z}{\partial t\partial s} + \frac{\partial^2 z}{\partial t^2} \quad \text{------} \ (5)$$

Plug (2), (3), (4), (5) into (1A) $\Rightarrow$

$$2\left(2\frac{\partial^2 z}{\partial s^2} + 3\frac{\partial^2 z}{\partial s\partial t}\right) + 3\left(2\frac{\partial^2 z}{\partial s\partial t} + 3\frac{\partial^2 z}{\partial t^2}\right. - 4\left(\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial s\partial t}\right) - 9\left(\frac{\partial^2 z}{\partial t\partial s} + \frac{\partial^2 z}{\partial t^2}\right) = 0$$

$$4\frac{\partial^2 z}{\partial s^2} + 6\frac{\partial^2 z}{\partial s\partial t} + 6\frac{\partial^2 z}{\partial s\partial t} + 9\frac{\partial^2 z}{\partial t^2} - 4\frac{\partial^2 z}{\partial s^2} - 4\frac{\partial^2 z}{\partial s\partial t} - 9\frac{\partial^2 z}{\partial t\partial s} - 9\frac{\partial^2 z}{\partial t^2} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial s\partial t} = 0}$$

Ch 4

$\boxed{11.3}$ Suppose $\omega = f(x,y)$ satisfies $\frac{\partial^2 \omega}{\partial x^2} - \frac{\partial^2 \omega}{\partial y^2} = 1$

Put $x = u+v$, $y = u-v$ and show that ω satisfies $\frac{\partial^2 \omega}{\partial u \partial v} = 1$. hence solve the equation.

$$\left.\begin{aligned} x(u,v) &= u+v \\ y(u,v) &= u-v \end{aligned}\right\} \Longrightarrow \begin{aligned} &\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial^2 x}{\partial u^2} = 0, \quad \frac{\partial^2 x}{\partial v^2} = 0 \\ &\frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1, \quad \frac{\partial^2 y}{\partial u^2} = 0, \quad \frac{\partial^2 y}{\partial v^2} = 0 \end{aligned}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

so $$\frac{\partial w}{\partial u} = \boxed{\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}} \text{ ——— } ①$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \boxed{\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}} \text{ ------ } ②$$

From ① and ② solve for $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

From ①, $\boxed{\frac{\partial w}{\partial x} = \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial y}\right)}$

sub into ② $\Rightarrow$ $\frac{\partial w}{\partial v} = \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial y}\right) - \frac{\partial w}{\partial y}$

now from above find $\frac{\partial w}{\partial y}$:

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial u} - 2\frac{\partial w}{\partial y} \Longrightarrow \boxed{\frac{\partial w}{\partial y} = \frac{1}{2}\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right)} \text{ ——— } ③$$

So from ① $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} - \frac{1}{2}\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right)$$

$$\boxed{\frac{\partial w}{\partial x} = \frac{1}{2}\frac{\partial w}{\partial u} + \frac{1}{2}\frac{\partial w}{\partial v}} \text{ ——— } ④$$

now need to find second derivatives $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$

introduce $G = \frac{\partial w}{\partial x}$

$H = \frac{\partial w}{\partial y}$

so ③, ④ can be rewritten as

$$H = \frac{1}{2}\frac{\partial w}{\partial u} - \frac{1}{2}\frac{\partial w}{\partial v} \quad \text{———} \quad (3A)$$

$$G = \frac{1}{2}\frac{\partial w}{\partial u} + \frac{1}{2}\frac{\partial w}{\partial v} \quad \text{———} \quad (4A)$$

so $\frac{\partial^2 w}{\partial x^2} = \frac{\partial G}{\partial x}$

$\frac{\partial^2 w}{\partial y^2} = \frac{\partial H}{\partial y}$

so $1 = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \Rightarrow 1 = \frac{\partial G}{\partial x} - \frac{\partial H}{\partial x}$ ——— ⑤

now replace w by H in eq ③ and replace w by G in equation ④ $\Rightarrow$

$$\frac{\partial H}{\partial y} = \frac{1}{2}\frac{\partial H}{\partial u} - \frac{1}{2}\frac{\partial H}{\partial v} \quad \text{———} \quad (3B)$$

$$\frac{\partial G}{\partial x} = \frac{1}{2}\frac{\partial G}{\partial u} + \frac{1}{2}\frac{\partial G}{\partial v} \quad \text{———} \quad (4B)$$

Sub 3B, 4B into equation ⑤

$$1 = \frac{1}{2}\frac{\partial G}{\partial u} + \frac{1}{2}\frac{\partial G}{\partial v} - \frac{1}{2}\frac{\partial H}{\partial u} + \frac{1}{2}\frac{\partial H}{\partial v} \quad \text{———} \quad ⑦$$

now need to find $\frac{\partial G}{\partial u}$, $\frac{\partial G}{\partial v}$, $\frac{\partial H}{\partial u}$, $\frac{\partial H}{\partial v}$. i.e. this from equations (3A) and (4A)

$\Rightarrow$

$$\frac{\partial G}{\partial u} = \frac{1}{2}\frac{\partial^2 w}{\partial u^2} + \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}$$

$$\frac{\partial G}{\partial v} = \frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{2}\frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial H}{\partial u} = \frac{1}{2}\frac{\partial^2 w}{\partial u^2} - \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}$$

$$\frac{\partial H}{\partial v} = \frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 w}{\partial v^2}$$

Plug these into ⑦ to get

$$1 = \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u^2} + \frac{1}{2}\frac{\partial w}{\partial v \partial u}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{2}\frac{\partial^2 w}{\partial v^2}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u^2} - \frac{1}{2}\frac{\partial^2 w}{\partial v \partial u}\right) - \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 w}{\partial v^2}\right)$$

$$1 = \frac{1}{4}w_{uu} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{vv} - \frac{1}{4}w_{uu} + \frac{1}{4}w_{vu} + \frac{1}{4}w_{uv} - \frac{1}{4}w_{vv}$$

$$1 = \frac{1}{4}w_{uv} + \frac{1}{4}w_{uv} + \frac{1}{4}w_{vu} + \frac{1}{4}w_{vu}$$

and since $w_{uv} = w_{vu}$

then $$\boxed{1 = w_{uv} = \frac{\partial^2 w}{\partial u \partial v}}$$

so solution is found from

$$\frac{\partial^2 w}{\partial u \partial v} = 1 \qquad \text{i.e.} \quad \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = 1$$

or $\frac{\partial w}{\partial u} = v + A \Rightarrow \boxed{w = vu + Au + B}$

where A, B are constants

ch 4

$\boxed{11.6}$ reduce the equation $x^2\left(\frac{d^2y}{dx^2}\right)+2x\left(\frac{dy}{dx}\right)-5y=0$ to a differential equation with constant coefficients in $\frac{d^2y}{dz^2}$ $\frac{dy}{dz}$ and y by the change of variable $x=e^z$.

we are given $y(x)$. we need to rewrite the equation so that y is now a function of z instead.

$$\frac{dy}{dz}=\frac{dy}{dx}\,\frac{dx}{dz}$$

$$\frac{dy}{dz}=\frac{dy}{dx}e^{z} \Longrightarrow \boxed{\frac{dy}{dx}=\frac{1}{e^{z}}\frac{dy}{dz}} \quad \text{———} \quad 2)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right)$$

$$=e^{-z}\frac{d^2y}{dz\,dx}+\frac{dy}{dz}\left(\frac{d}{dx}e^{-z}\right)$$

since $e^{-z}=x^{-1}$ given.

$$=e^{-z}\frac{d}{dz}\left(\frac{dy}{dx}\right)+\frac{dy}{dz}\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)$$

$$=e^{-z}\frac{d}{dz}\left(\frac{1}{e^{z}}\frac{dy}{dz}\right)+\frac{dy}{dz}\left(-\frac{1}{x^2}\right.$$

$$\frac{d^2y}{dx^2}=e^{-z}\left(e^{-z}\frac{d^2y}{dz^2}+\frac{dy}{dz}\left(-e^{-z}\right)\right)-\frac{dy}{dz}\frac{1}{x^2}$$, but $\frac{1}{x^2}=e^{-2z}$

so $$\frac{d^2y}{dx^2}=e^{-2z}\frac{d^2y}{dz^2}-e^{-2z}\frac{dy}{dz}-\frac{dy}{dz}e^{-2z} \quad \text{———} \quad ③$$

Plug ② and 3) into ① $\Rightarrow$ and replace x^2 by e^{2z}

$$e^{2z}\left(e^{-2z}\frac{d^2y}{dz^2}-e^{-2z}\frac{dy}{dz}-\frac{dy}{dz}e^{-2z}\right)+2e^{z}\left(e^{-z}\frac{dy}{dz}\right)-5y=0$$

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}-\frac{dy}{dz}+2\frac{dy}{dz}-5y=0 \Longrightarrow \boxed{\frac{d^2y}{dz^2}-5y=0}$$

Ch 4 $\boxed{11.7}$ transfer $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

to $\frac{d^2y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} + 2y = 0$ by using $x = \cos\theta$.

Solution:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$$

$$= \frac{dy}{dx}(-\sin\theta)$$

so $\frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta}$ ——— ①

let $\boxed{G = \frac{dy}{dx}}$!

so $G = \frac{dy}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta}$ ——— (1a)

so our DE is

$$(1-\cos^2\theta)\frac{dG}{dx} - 2\cos\theta\, G + 2y = 0 \quad \text{———} \quad ②$$

① is correct for any function. replacing y by G in ①

$$\frac{dG}{dx} = -\frac{1}{\sin\theta}\frac{dG}{d\theta} \quad \text{———} \quad ③$$

sub ③ into ②

$$(1-\cos^2\theta)\left(-\frac{1}{\sin\theta}\frac{dG}{d\theta}\right) - 2\cos\theta\, G + 2y = 0 \quad \text{———} \quad ④$$

need to find $\frac{dG}{d\theta}$. From (1a)

$$\frac{dG}{d\theta} = \frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right) = -\left(-\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} + \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)$$

sub the above into ④ to get the solution needed →

$$(1-\cos^2\theta)\left(-\frac{1}{\sin\theta}\overbrace{\left(\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}-\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)}^{\frac{dG}{d\theta}}\right)-2\cos\theta\overbrace{\left(-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right)}^{G}+2y=0$$

So above becomes

$$(1-\cos^2\theta)\left(-\frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}+\frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)+2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+2y=0$$

$$-\frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}+\frac{\cos^3\theta}{\sin^3\theta}\frac{dy}{d\theta}-\frac{\cos^2\theta}{\sin^2\theta}\frac{d^2y}{d\theta^2}+2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{1}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta}{\sin^3\theta}+\frac{\cos^3\theta}{\sin^3\theta}+2\frac{\cos\theta}{\sin\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{1-\cos^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos^3\theta+2\cos\,\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(\frac{\sin^2\theta}{\sin^2\theta}\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos\theta(1-\sin^2\theta)+2\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}\left(1\right)+\frac{dy}{d\theta}\left(\frac{-\cos\theta+\cos\theta-\cos\theta\sin^2\theta+2\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\frac{d^2y}{d\theta^2}+\frac{dy}{d\theta}\left(\frac{\cos\theta\sin^2\theta}{\sin^3\theta}\right)+2y=0$$

$$\boxed{\frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}+2y=0}$$

Ch 4

$\boxed{11\cdot 8}$ Change x to $u = 2\sqrt{x}$ in $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (1-x)y = 0$

and show eqn becomes $u^2\frac{d^2y}{du^2} + u\frac{dy}{du} + (u^2-4)y = 0$

$$u = 2\sqrt{x} \Rightarrow x = \left(\frac{u}{2}\right)^2 , \quad x^2 = \left(\frac{u}{2}\right)^4$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{dy}{du}\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d^2y}{du\,dx}$$

So our DE becomes

$$\left(\frac{u}{2}\right)^4\left[\frac{dy}{du}\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d^2y}{du\,dx}\right] + \left(\frac{u}{2}\right)^2\left[\frac{dy}{du}\frac{du}{dx}\right] - \left(1-\left(\frac{u}{2}\right)^2\right)y = 0 \quad \text{—— ①}$$

now $\frac{du}{dx} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{\sqrt{x}}$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(x^{-3/2}\right) = -\frac{1}{2x^{3/2}}$$

i.e. in terms of u, $\boxed{\frac{du}{dx} = \frac{2}{u}}$

and $\frac{d^2u}{dx^2} = -\frac{1}{2}\frac{1}{\left(\frac{u}{2}\right)^3} = -\frac{1}{2}\frac{8}{u^2} = \boxed{-\frac{4}{u^3}}$

so now ① becomes

$$\frac{u^4}{16}\left[\frac{dy}{du}\left(-\frac{4}{u^3}\right) + \left(\frac{2}{u}\right)\frac{d^2y}{du\,dx}\right] + \frac{u^2}{4}\left[\frac{dy}{du}\left(\frac{2}{u}\right)\right] - \left(1-\frac{u^2}{4}\right)y = 0$$

$$-\frac{u}{4}\frac{dy}{du} + \frac{u^3}{8}\frac{d^2y}{du\,dx} + \frac{u}{2}\frac{dy}{du} - \left(1-\frac{u^2}{4}\right)y = 0 \quad \text{—— ②}$$

now what is left is to find $\frac{d^2y}{du\,dx}$.

$$\frac{d^2y}{du\,dx} = \frac{d}{du}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{note: y is function of x here not u}} = \frac{d}{du}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{d}{du}\left(\frac{dy}{du}\frac{1}{\sqrt{x}}\right) = \frac{d}{du}\left(\frac{dy}{du}\frac{2}{u}\right) = -2\frac{1}{[illegible]}\;\frac{1}{u}\frac{dy}{d[illegible]}$$

since u is held constant here when diff. y w.r.t. x. i.e. y(x) only, so $\frac{du}{dx}=0$ here

$$\frac{d^2y}{du\,dx} = 2\left(\frac{1}{u}\frac{d^2y}{du^2} + \frac{dy}{du}\overset{\to =0}{\left(-\frac{1}{u^2}\right)}\right) = \boxed{\frac{2}{u}\frac{d^2y}{du^2}}$$

So now ② becomes:

$$-\frac{u}{4}\frac{dy}{dx} + \frac{u^2}{8}\left[\frac{2}{u}\frac{d^2y}{du^2}\right] + \frac{u}{2}\frac{dy}{dx} - \left(1-\frac{u^2}{4}\right)y = 0$$

$$-\frac{u}{4}\frac{dy}{dx} + \frac{u^2}{4}\frac{d^2y}{du^2} + \frac{u}{2}\frac{dy}{dx} - \left(1-\frac{u^2}{4}\right)y = 0$$

$$\frac{d^2y}{du^2}\left(\frac{u^2}{4}\right) + \frac{dy}{du}\left(\underbrace{-\frac{u}{4}+\frac{u}{2}}_{\frac{u}{4}}\right) + \left(\frac{u^2}{4}-1\right)y = 0$$

×4 →

$$\boxed{u^2\frac{d^2y}{du^2} + u\frac{dy}{du} + (u^2-4)y = 0}$$

O:)

Ch 4

$\boxed{11.9}$ if $x=e^s \cos t$, $y=e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-2s}\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)$$

in $x=e^s \cos t$, here x is function of s and t.

i.e. $x(s,t)=e^s \cos t$.

and $y(s,t)=e^s \sin t$.

$$\frac{\partial x}{\partial s}=e^s \cos t$$

$$\frac{\partial x}{\partial t}=-e^s \sin t$$

$$\frac{\partial^2 x}{\partial s^2}=e^s \cos t$$

$$\frac{\partial^2 x}{\partial t^2}=-e^s \cos t$$

$$\frac{\partial y}{\partial s}=e^s \sin t$$

$$\frac{\partial y}{\partial t}=e^s \cos t$$

$$\frac{\partial^2 y}{\partial s^2}=e^s \sin t$$

$$\frac{\partial^2 y}{\partial t^2}=-e^s \sin t$$

$$\frac{\partial u(x,y)}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\boxed{\frac{\partial u}{\partial x}e^s \cos t+\frac{\partial u}{\partial y}e^s \sin t} \text{ — ①}$$

$$\frac{\partial u(x,y)}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\boxed{-\frac{\partial u}{\partial x}e^s \sin t+\frac{\partial u}{\partial y}e^s \cos t} \text{ — ②}$$

now, solve ①,② for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

From ① $$\frac{\partial u}{\partial y_c} = \frac{\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t}{e^s \sin t}$$

Sub. into ②

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^s \sin t + \left(\frac{\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t}{e^s \sin t} \right) e^s \cos t$$

$$\frac{\partial u}{\partial t} = \frac{-\frac{\partial u}{\partial x} e^{2s} \sin^2 t + \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial x} e^s \cos t \right) e^s \cos t}{e^s \sin t}$$

$$e^s \sin t \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^{2s} \sin^2 t + e^s \cos t \frac{\partial u}{\partial s} - e^{2s} \cos^2 t \frac{\partial u}{\partial x}$$

$$e^s \sin t \frac{\partial u}{\partial t} - e^s \cos t \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \left(-e^{2s} \sin^2 t - e^{2s} \cos^2 t \right)$$

so $$\frac{\partial u}{\partial x} = \frac{e^s \sin t \frac{\partial u}{\partial t} - e^s \cos t \frac{\partial u}{\partial s}}{-e^{2s} \sin^2 t - e^{2s} \cos^2 t} = \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s \sin^2 t - e^s \cos^2 t}$$

$$= \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s \left(\sin^2 t + \cos^2 t \right)} = \frac{\sin t \frac{\partial u}{\partial t} - \cos t \frac{\partial u}{\partial s}}{-e^s}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{e^s} \left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t} \right)} \text{ ——— } ③$$

So, From ②, plus above into ② to find $\frac{\partial u}{\partial y_c}$:

$$\frac{\partial u}{\partial t} = -\frac{1}{e^s} \left(\cos t \frac{\partial u}{\partial s} - \sin t \frac{\partial u}{\partial t} \right) e^s \sin t + \frac{\partial u}{\partial y_c} e^s \cos t$$

$$\frac{\partial u}{\partial y} = \frac{\frac{\partial u}{\partial t} + \frac{1}{e^s}\left(\cos t\,\frac{\partial u}{\partial s} - \sin t\,\frac{\partial u}{\partial t}\right)e^s \sin t}{e^s \cos t}$$

$$= \frac{1}{e^s \cos t}\,\frac{\partial u}{\partial t} + \frac{1}{e^{2s}\cos t}\left(\cos t\,\frac{\partial u}{\partial s} - \sin t\,\frac{\partial u}{\partial t}\right)e^s \sin t$$

$$= \frac{1}{e^s \cos t}\,\frac{\partial u}{\partial t} + \frac{1}{e^{2s}}\,\frac{\partial u}{\partial s}\,e^s \sin t - \frac{e^s \sin^2 t}{e^{2s}\cos t}\,\frac{\partial u}{\partial t}$$

$$= \frac{1}{e^s \cos t}\,\frac{\partial u}{\partial t} + \frac{1}{e^s}\,\frac{\partial u}{\partial s}\,\sin t - \frac{\sin^2 t}{e^s \cos t}\,\frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial t}\left(\frac{1}{e^s \cos t} - \frac{\sin^2 t}{e^s \cos t}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$= \frac{\partial u}{\partial t}\left(\frac{1-\sin^2 t}{e^s \cos t}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\left(\frac{\cos t}{e^s}\right) + \frac{\partial u}{\partial s}\left(\frac{\sin t}{e^s}\right)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{1}{e^s}\left(\cos t\,\frac{\partial u}{\partial t} + \sin t\,\frac{\partial u}{\partial s}\right)} \quad \text{——} \; ④$$

to find second derivations, let $G = \frac{\partial u}{\partial x}$

$H = \frac{\partial u}{\partial y}$

So ③ becomes $G = \frac{1}{e^s}\left(\cos t\,\frac{\partial u}{\partial s} - \sin t\,\frac{\partial u}{\partial t}\right)$ —— ⑤

and ④ becomes $H = \frac{1}{e^s}\left(\cos t\,\frac{\partial u}{\partial t} + \sin t\,\frac{\partial u}{\partial s}\right)$ —— ⑥

now equations 3) and ④ are true for any function,

so replace u by G in ③ and u by H in ④ $\Longrightarrow$

$$\frac{\partial G}{\partial x} = \frac{1}{e^s}\left(\cos t\ \frac{\partial G}{\partial s} - \sin t\ \frac{\partial G}{\partial t}\right) \text{ ——— } ⑦$$

From ③ by replacing u by G

$$\frac{\partial H}{\partial y} = \frac{1}{e^s}\left(\cos t\ \frac{\partial H}{\partial t} + \sin t\ \frac{\partial H}{\partial s}\right) \text{ ——— } ⑧$$

from ④ and replacing u by H.

so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y}$

so, given ⑦ and ⑧, we get

$$\Rightarrow \frac{1}{e^s}\left(\cos t\ \frac{\partial G}{\partial s} - \sin t\ \frac{\partial G}{\partial t}\right) + \frac{1}{e^s}\left(\cos t\ \frac{\partial H}{\partial t} + \sin t\ \frac{\partial H}{\partial s}\right) \text{ ——— } ⑨$$

to find $\frac{\partial G}{\partial s}$, $\frac{\partial G}{\partial t}$, $\frac{\partial H}{\partial t}$, $\frac{\partial H}{\partial s}$, differentiate eqn ⑤, ⑥

From ⑤

$$\frac{\partial G}{\partial s} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s^2}\cos t - \sin t\ \frac{\partial^2 u}{\partial t \partial s}\right) + \left(\cos t\ \frac{\partial u}{\partial s} - \sin t\ \frac{\partial u}{\partial t}\right)\left(-\frac{1}{\partial s}\right)$$

$$\frac{\partial G}{\partial s} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s^2}\cos t - \sin t\ \frac{\partial^2 u}{\partial t \partial s} + \sin t\ \frac{\partial u}{\partial t} - \cos t\ \frac{\partial u}{\partial s}\right) \quad ⑩$$

$$\frac{\partial G}{\partial t} = \frac{1}{e^s}\cos t\ \frac{\partial^2 u}{\partial s \partial t} + \frac{1}{e^s}\frac{\partial u}{\partial s}(-\sin t) - \frac{1}{e^s}\sin t\ \frac{\partial^2 u}{\partial t^2} - \frac{1}{e^s}\ \frac{\partial u}{\partial t}\cos t$$

$$\frac{\partial G}{\partial t} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial s \partial t}\cos t - \frac{\partial u}{\partial s}\sin t - \frac{\partial^2 u}{\partial t^2}\sin t - \frac{\partial u}{\partial t}\cos t\right) \text{ —— } ⑪$$

From ⑥

$$\frac{\partial H}{\partial s} = \frac{1}{e^s}\left(\cos t\ \frac{\partial^2 u}{\partial t \partial s} + \sin t\ \frac{\partial^2 u}{\partial s^2}\right) + \left(\cos t\ \frac{\partial u}{\partial t} + \sin t\ \frac{\partial u}{\partial s}\right)\left(-\frac{1}{e^s}\right)$$

$$= \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial t \partial s}\cos t + \frac{\partial^2 u}{\partial s^2}\sin t - \frac{\partial u}{\partial t}\cos t - \frac{\partial u}{\partial s}\sin t\right) \text{ —— } ⑫$$

$$\frac{\partial H}{\partial t} = \frac{1}{e^s}\cos t\ \frac{\partial^2 u}{\partial t^2} + \frac{1}{e^s}\frac{\partial u}{\partial t}(-\sin t) + \frac{1}{e^s}\sin t\ \frac{\partial^2 u}{\partial s \partial t} + \frac{1}{e^s}\ \frac{\partial u}{\partial s}\cos t$$

$\longrightarrow$

$$\frac{\partial H}{\partial t} = \frac{1}{e^s}\left(\frac{\partial^2 u}{\partial t^2}\cos t - \frac{\partial u}{\partial t}\sin t + \frac{\partial^2 u}{\partial s \partial t}\sin t + \frac{\partial u}{\partial s}\cos t\right) \text{ — } ⑬$$

now sub above equations (10), ⑪, ⑫, ⑬ for $\frac{\partial G}{\partial s}, \frac{\partial G}{\partial t}, \frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}$ into equation ⑨ $\Longrightarrow$ (write C to mean $\cos t$, S to mean $\sin t$ write $u_s, u_{ss}, u_t, u_{tt}, u_{ts}$ to make it easier to do)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{e^s}\left(\cos t\,\frac{\partial G}{\partial s} - \sin t\,\frac{\partial G}{\partial t}\right) + \frac{1}{e^s}\left(\cos t\,\frac{\partial H}{\partial t} + \sin t\,\frac{\partial H}{\partial s}\right)$$

$$= \frac{1}{e^s}\left[\cos t\left(\frac{1}{e^s}(\text{—})\right) - \sin t\left(\frac{1}{e^s}(\text{—})\right)\right] + \frac{1}{e^s}\left[\cos t\left(\frac{1}{e^s}(\text{—})\right) + \sin t\left(\frac{1}{e^s}(\text{—})\right)\right]$$

$$\begin{aligned}
= \frac{1}{e^{2s}}\Big[& C\left(u_{ss}C - u_{st}S + u_t S - u_s C\right) \\
& - S\left(u_{st}C - u_s S - u_{tt}S - u_t C\right) \\
& + C\left(u_{tt}C - u_t S + u_{ts}S + u_s C\right) \\
& + S\left(u_{ts}C + u_{ss}S - u_t C - u_s S\right)\Big]
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{e^{2s}}\Big[& u_{ss}C^2 - u_{st}SC + u_t CS - u_s C^2 \\
& - u_{st}SC + u_s S^2 + u_{tt}S^2 + u_t SC \\
& + u_{tt}C^2 - u_t SC + u_{ts}SC + u_s C^2 \\
& + u_{ts}SC + u_{ss}S^2 - u_t CS - u_s S^2\Big]
\end{aligned}$$

$$= \frac{1}{e^{2s}}\left[u_{ss}(C^2+S^2) + u_{tt}(C^2+S^2)\right]$$

but $C^2+S^2 \equiv \cos^2 + \sin^2 = 1$

so $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{e^{2s}}\left[u_{ss} + u_{tt}\right]}$$

QED

ch 4

$\boxed{12.1}$ Find $\frac{dy}{dx}$ given $y = \int_0^{\sqrt{x}} \sin t^2 \, dt$

$$\frac{dy}{dx} = \sin(\sqrt{x}^2)\,\frac{d}{dx}(\sqrt{x}) - \sin(0^2)$$

$$= \sin(x)\,\frac{d}{dx}\sqrt{x} = \sin(x)\left(\frac{1}{2}\frac{1}{\sqrt{x}}\right) = \boxed{\frac{\sin(x)}{2\sqrt{x}}}$$

✓

$\boxed{12.2}$ if $s = \int_u^v \frac{1-e^t}{t}\,dt$ Find $\frac{\partial s}{\partial v}$, $\frac{\partial s}{\partial u}$ and also their limits as u and v tend to zero.

using Leibniz rule, take differential on the integral:

$$ds = d\left(\int_u^v \frac{1-e^t}{t}\,dt\right)$$

$$ds = \frac{1-e^{(v)}}{v}\,dv - \left(\frac{1-e^u}{u}\right) du$$

$$\frac{ds}{dv} = \frac{1-e^v}{v} - \left(\frac{1-e^u}{u}\right)\frac{du}{dv} = 0$$ assuming u is not a function of v.

so $\boxed{\frac{\partial s}{\partial v} = \frac{1-e^v}{v}}$

similarly $$\frac{ds}{du} = \frac{1-e^v}{v}\,\frac{dv}{du} - \left(\frac{1-e^u}{u}\right)$$

so $\boxed{\frac{\partial s}{\partial u} = -\left(\frac{1-e^u}{u}\right)}$

$$\lim_{v \to 0} \frac{\partial s}{\partial v} = \lim_{v \to 0} \frac{1-e^v}{v}$$

use L'Hopital Rule $\lim_{v \to 0} \frac{f(v)}{g(v)} = \lim_{v \to 0} \frac{f'(v)}{g'(v)}$

so $$\lim_{v \to 0} \frac{1-e^v}{v} = \lim_{v \to 0} \frac{-e^v}{1} = -e^0 = \boxed{-1}$$

and $$\lim_{u \to 0} \frac{e^u - 1}{u} = \lim_{u \to 0} e^u = e^0 = \boxed{1}$$

Ch 4

$\boxed{12.4}$ $\lim_{x\to 2} \frac{1}{x-2}\int_2^x \frac{\sin t}{t}\,dt$.

$$\lim_{x\to 2}\frac{\frac{d}{dx}\int_2^x \frac{\sin t}{t}\,dt}{\frac{d}{dx}(x-2)} = \lim_{x\to 2}\frac{\frac{\sin x}{x}\frac{d}{dx}x - \frac{\sin 2}{2}\overset{=0}{\frac{d}{dx}(2)}}{1}$$

$$= \lim_{x\to 2}\frac{\sin x}{x} = \boxed{\tfrac{1}{2}\sin 2}$$

$\boxed{12.7}$ if $\int_u^v e^{-t^2}\,dt = x$ and $u^v = y$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial y}\right)_x$ and $\left(\frac{\partial y}{\partial v}\right)_u$ at $u=2$, $v=0$.

$\left.\frac{\partial u}{\partial x}\right)_y$ means $u(x,y)$ ———— , next

take differential $d\int_{u(x,y)}^{v(x,y)} e^{-t^2}\,dt$

$$d(x) = d(v(x,y))\,e^{-(v^2)} - d(u(x,y))\,e^{-u^2}$$

$$\text{so}\quad 1 = \left(\frac{dv}{dx}+\frac{dv}{dy}\right)e^{-v^2} - \left(\frac{du}{dx}+\frac{du}{dy}\right)e^{-u^2}$$

$$1 = \frac{dv}{dx}e^{-v^2} + \frac{dv}{dy}e^{-v^2} - \frac{du}{dx}e^{-u^2} - \frac{du}{dy}e^{-u^2}$$

$$\text{so}\quad \frac{du}{dx} = \left(-1 + \frac{dv}{dx}e^{-v^2} + \frac{dv}{dy}e^{-v^2} - \frac{du}{dy}e^{-u^2}\right)\frac{1}{e^{-u^2}}$$

$$\frac{du}{dx} = -1 + \frac{dv}{dx}e^{-v^2+u^2} + \frac{dv}{dy}e^{-v^2+u^2} - \frac{du}{dy}$$

since $u(x,y)$, rewrite above ... →

$$\left(\frac{\partial u}{\partial x}\right)_y = -1 + \left.\frac{\partial v}{\partial x}\right)_y e^{u^2-v^2} + \frac{dv}{dy}e^{u^2-v^2} - \frac{du}{dy}$$

Ch 4

$\boxed{12.7}$

if $\int_u^v e^{-t}\,dt = x$, $u^v = y$, find $\left(\frac{\partial x}{\partial x}\right)_y$, $\frac{\partial u}{\partial y}$, $\frac{\partial y}{\partial x}$ at $u=2$, $v=0$.

we have $u(x,y)$, but v is constant. (since problem does not mention it is function of x or y)

take derivative , using lebniz Rule , we get

$$\frac{d}{dx}(x) = \frac{d}{dx}\int_{u(x,y)}^{v} e^{-t}\,dt$$

$$1 = e^{-v}\overset{=0}{\frac{dv}{dx}} - e^{-u(x,y)}\left(\frac{\partial u}{\partial x}\right)_y$$

so $\boxed{\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{e^{-u(x,y)}}}$ at $u=2 \Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{e^{-2}} = -e^2$

$= \boxed{-7.38}$

to find $\left(\frac{\partial u}{\partial y}\right)_x$, from lebniz Rule :

$$\frac{d}{dy}(x) = \frac{d}{dy}\int_{u(x,y)}^{v} e^{-t}\,dt = e^{-v}\overset{=0}{\frac{dv}{dy}} - e^{-u(x,y)}\left(\frac{\partial u}{\partial y}\right)_x$$

$$\frac{dx}{dy} = -e^{-u(x,y)}\left(\frac{\partial u}{\partial y}\right)_x$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \boxed{-\frac{dx}{dy}\,\frac{1}{e^{-u(x,y)}}}$$ at $u=2 \Rightarrow +\frac{dx}{dy}\cdot 7.38$

from $u^v = y \Rightarrow v\log_e u = \log_e y \Rightarrow \log u = \frac{1}{v}\log y$.

so $d(\log_e u) = \frac{1}{v}\,d(\log_e y) \Rightarrow \frac{1}{u}\,du = \frac{1}{v}\frac{1}{y}\,dy$.

so $\frac{du}{dy} = \frac{u}{v}\frac{1}{y} = \frac{u}{v}\frac{1}{u^v}$

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$\boxed{2.5}$ if $\int_0^x e^{-s^2}\,ds = u$ find $\frac{dx}{du}$

$$\frac{du}{dx} = \frac{d}{dx}\int_0^x e^{-s^2}\,ds$$

$$= e^{-x^2}\frac{d}{dx}x - e^{-0^2}\frac{d}{dx}(0)^{\nearrow =0}$$

$$\frac{du}{dx} = e^{-x^2}\,1$$

so $\frac{du}{dx} = e^{-x^2}$ $\Rightarrow \boxed{\frac{dx}{du} = e^{x^2}}$

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